Grothendieck toposes as unifying 'bridges' in Mathematics

Research lecture course

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The course will consist of 9 lectures of 2 hours each, for a total of 18 hours. It will be given either in English or in French (according to the preferences of the participants). These lectures are organized by the « Équipe de logique catégorique » and the « Laboratoire Preuves, Programmes et Systèmes » of the Paris 7 University.

Place: Institut de mathématiques de Jussieu
175 rue du CHEVALERET, Paris
Salle 1 C 01

Dates: From Monday 14th of January to Thursday 31st of January 2013
Every Monday, Tuesday and Thursday
From 4pm to 6pm

Abstract:

The course aims to introduce a new view of Grothendieck toposes as unifying spaces which can effectively act as 'bridges' for transferring information, ideas and results across different mathematical theories. The intra-disciplinary methodologies arising from an implementation of such a view have already generated a number of non-trivial applications into distinct mathematical fields, including Algebra, Geometry, Topology, Functional Analysis, Model Theory and Proof Theory, and the potential of this theory has just started to be explored. The course will provide a self-contained introduction to classifying toposes and the 'bridge-building' methodologies, by illustrating the underlying abstract principles and discussing a number of selected examples and applications in a variety of different mathematical contexts.

Lecture 1: **Categorical preliminaries**

Review of the theory of Grothendieck toposes and the relevant categorical background: limits, colimits, exponentials, subobject classifier, associated sheaf functor, subcanonical sites, morphisms of sites, geometric morphisms, flat functors, the Comparison Lemma.

Lecture 2: **The notion of classifying topos**

Introduction to categorical and topos-theoretic semantics: the notion of geometric theory, syntactic categories, construction of classifying toposes for geometric theories via syntactic sites, universal models and representability, theories classified by a presheaf topos and their quotients.

Lecture 3: **The 'bridge-building' method**

General explanation of the view of toposes as unifying spaces yielding 'bridges' between distinct mathematical theories: remarks on the existence of different sites of definition for a given topos and its logical interpretation, analysis of the representation theory of Grothendieck toposes and the problem of finding 'elementary' site characterizations for topos-theoretic invariants.

Lecture 4: **Decks of ‘bridges’: Morita-equivalences**

Introduction to the notion of Morita-equivalence between mathematical theories: methods for establishing and generating Morita-equivalences, the topological interpretation of Morita-equivalences in terms of different sites of definition for the same topos, the link with biinterpretability, Morita-equivalence for rings, topological groups and small categories.

Lecture 5: **Arches of ‘bridges’: site characterizations for invariants**

Discussion of the problem of obtaining site characterization for topos-theoretic invariants, and presentation of two meta-theorems characterizing large classes of invariants admitting bijective site characterizations. Analysis of specific geometric and logical topos-theoretic invariants, notably including the property of a topos to be two-valued, atomic, compact, (connected and) locally connected, equivalent to a presheaf topos, coherent, Boolean, De Morgan.

Lecture 6: **The duality theorem**

Statement and proofs of the theorem providing a duality between the subtoposes of the classifying topos of a geometric theory and the quotients of the theory (considered up to syntactic equivalence). Remarks on the proof-theoretic nature of the notion of Grothendieck topology. Analysis of the lattice structure on the collection of geometric theories over a given language: transfer of topos-theoretic notions across the duality and their logical interpretations. The notions of Booleanization and DeMorganization of a geometric theory and their applications in Algebra.

Lecture 7: **Intra-disciplinary applications I**

Results on theories of presheaf type and their quotients. Topos-theoretic Fraïssé’s construction. Topos-theoretic Gödel’s completeness theorem. Descent and definability.

Lecture 8: **Intra-disciplinary applications II**

Topos-theoretic interpretation and generation of Stone-type and Priestley-type dualities. Construction of analogues of the Zariski spectrum in Algebra and Topology, and applications to Gelfand spectra. Topos-theoretic generation of dualities for compact Hausdorff spaces and C*-algebras through Wallman compactifications.

Lecture 9: **Intra-disciplinary applications III**

Topos-theoretic generalization of Grothendieck’s Galois theory and applications. Galois-type equivalences for graphs, linear orders, Boolean algebras and finite groups.