Topos Theory

Lectures 7-14: Sheaves on a site

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In order to ‘categorify’ the notion of sheaf of a topological space, the first step is to introduce an abstract notion of covering (of an object by a family of arrows to it) in a category.

Definition

- Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a presieve P in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c.
- Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a sieve S in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c such that

$$f \in S \Rightarrow f \circ g \in S$$

  whenever this composition makes sense.
- We say that a sieve S is generated by a presieve P on an object c if it is the smallest sieve containing it, that is if it is the collection of arrows to c which factor through an arrow in P.

If S is a sieve on c and $h : d \to c$ is any arrow to c, then

$$h^*(S) := \{g \mid \mathrm{cod}(g) = d,\ h \circ g \in S\}$$

is a sieve on d.

Grothendieck topologies I

Definition

- A Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each object c of $\mathcal{C}$ a collection $J(c)$ of sieves on c in such a way that
  - (i) (maximality axiom) the maximal sieve $M_c = \{f \mid \mathrm{cod}(f) = c\}$ is in $J(c)$;
  - (ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;
  - (iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

  The sieves S which belong to $J(c)$ for some object c of $\mathcal{C}$ are said to be J-covering.
- A site is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a small category and J is a Grothendieck topology on $\mathcal{C}$.

Notice the following basic properties:

- If $R, S \in J(c)$ then $R \cap S \in J(c)$;
- If R and R' are sieves on an object c such that $R' \supseteq R$ then $R \in J(c)$ implies $R' \in J(c)$.

Grothendieck topologies II

The notion of a Grothendieck topology can be put in the following alternative (but equivalent) form:

Definition

A Grothendieck topology on a category $\mathcal{C}$ is an assignment J sending any object c of $\mathcal{C}$ to a collection $J(c)$ of sieves on c in such a way that

(a) the maximal sieve M_c belongs to $J(c)$;

(b) for each pair of sieves S and T on c such that $T \in J(c)$ and $S \supseteq T$, $S \in J(c)$;

(c) if $R \in J(c)$ then for any arrow $g : d \to c$ there exists a sieve $S \in J(d)$ such that for each arrow f in S, $g \circ f \in R$;

(d) if the sieve S generated by a presieve $\{f_i : c_i \to c \mid i \in I\}$ belongs to $J(c)$ and for each $i \in I$ we have a presieve $\{g_{ij} : d_{ij} \to c_i \mid j \in I_i\}$ such that the sieve T_i generated by it belongs to $J(c_i)$, then the sieve R generated by the family of composites $\{ f_i \circ g_{ij} : d_{ij} \to c \mid i \in I, j \in I_i\}$ belongs to $J(c)$.

The sieve R defined in (d) will be called the composite of the sieve S with the sieves T_i for $i \in I$ and denoted by $S * \{T_i \mid i \in I\}$.

Bases for a Grothendieck topology

Definition

A basis (for a Grothendieck topology) on a category $\mathcal{C}$ with pullbacks is a function K assigning to each object c of $\mathcal{C}$ a collection $K(c)$ of presieves on c in such a way that the following properties hold:

(i) $\{1_c : c \to c\} \in K(c)$

(ii) if $\{f_i : c_i \to c \mid i \in I\} \in K(c)$ then for any arrow $g : d \to c$ in $\mathcal{C}$, the family of pullbacks $\{g^*(f_i) : c_i \times_c d \to d \mid i \in I\}$ lies in $K(d)$.

(iii) if $\{f_i : c_i \to c \mid i \in I\} \in K(c)$ and for each $i \in I$ we have a presieve $\{g_{ij} : d_{ij} \to c_i \mid j \in I_i\} \in K(c_i)$ then the family of composites $\{ f_i \circ g_{ij} : d_{ij} \to c \mid i \in I, j \in I_i\}$ belongs to $K(c)$.

N.B. If $\mathcal{C}$ does not have pullbacks then condition (ii) can be replaced by the following requirement: if $\{f_i : c_i \to c \mid i \in I\} \in K(c)$ then for any arrow $g : d \to c$ in $\mathcal{C}$, there is a presieve $\{h_j : d_j \to d \mid j \in J\} \in K(d)$ such that for each $j \in J$, $g \circ h_j$ factors through some f_i.

Every basis K generates a Grothendieck topology J given by:

$$R \in J(c) \text{ if and only if } R \supseteq S \text{ for some } S \in K(c)$$

Grothendieck topology generated by a coverage

As we shall also see when we talk about sheaves, the axioms for Grothendieck topologies do not have all the same *status*: the most important one is the stability axiom. This motivates the following definition.

Definition

A (sifted) coverage on a category $\mathcal{C}$ is a collection of sieves which is stable under pullback.

Fact

The Grothendieck topology generated by a coverage is the smallest collection of sieves containing it which is closed under maximality and transitivity.

Theorem

Let $\mathcal{C}$ be a small category and D a coverage on D. Then the Grothendieck topology G_D generated by D is given by

$$\begin{aligned} G_D(c) \quad = \quad & \{S \text{ sieve on } c \mid \text{for any arrow } d \xrightarrow{f} c \text{ and sieve } T \text{ on } d, \\ & [(\text{for any arrow } e \xrightarrow{g} d \text{ and sieve } Z \text{ on } e \\ & (Z \in D(e) \text{ and } Z \subseteq g^*(T)) \text{ implies } g \in T) \text{ and} \\ & (f^*(S) \subseteq T)] \text{ implies } T = M_d\} \end{aligned}$$

Examples of Grothendieck topologies I

- For any (small) category $\mathcal{C}$, the trivial topology on $\mathcal{C}$ is the Grothendieck topology in which the only sieve covering an object c is the maximal sieve M_c.
- The dense topology D on a category $\mathcal{C}$ is defined by: for a sieve S,

$$S \in D(c) \quad \text{if and only if} \quad \begin{array}{l} \text{for any } f : d \to c \text{ there exists} \\ g : e \to d \text{ such that } f \circ g \in S\,. \end{array}$$

If $\mathcal{C}$ satisfies the right Ore condition i.e. the property that any two arrows $f : d \to c$ and $g : e \to c$ with a common codomain c can be completed to a commutative square

$$\begin{array}{ccc} \bullet & \dashrightarrow & d \\ \downarrow & & \downarrow f \\ e & \xrightarrow{g} & c \end{array}$$

then the dense topology on $\mathcal{C}$ specializes to the atomic topology on $\mathcal{C}$ i.e. the topology J_{at} defined by: for a sieve S,

$$S \in J_{at}(c) \text{ if and only if } S \neq \emptyset\,.$$

Examples of Grothendieck topologies II

- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathcal{O}(X)}$ on the poset category $\mathcal{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathcal{O}(X))$,

$$S \in J_{\mathcal{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U \, .$$

- More generally, given a frame (or complete Heyting algebra) H, we can define a Grothendieck topology J_H, called the *canonical topology on* H, by:

$$\{a_i \mid i \in I\} \in J_H(a) \text{ if and only if } \bigvee_{i \in I} a_i = a \, .$$

- Given a small category of topological spaces which is closed under finite limits and under taking open subspaces, one may define the open cover topology on it by specifying as basis the collection of open embeddings $\{Y_i \hookrightarrow X \mid i \in I\}$ such that $\bigcup_{i \in I} Y_i = X$.

Topologies with smallest covering sieves

Definition

Let $\mathcal{A}$ be a collection of arrows in a category $\mathcal{C}$ which is closed under composition on the left and which is interpolative in the sense that every arrow in $\mathcal{A}$ can be factored as the composition of two arrows in $\mathcal{A}$. Then there is a Grothendieck topology $J_{\mathcal{A}}$ on $\mathcal{C}$ given by:

$$S \in J_{\mathcal{A}}(c) \quad \text{if and only if} \quad \forall f \in \mathcal{A}, \operatorname{cod}(f) = c \text{ implies } f \in S\,.$$

Example

Given a full subcategory $\mathcal{D}$ of $\mathcal{C}$, one can take $\mathcal{A}$ to be the collection of arrows whose domains lie in $\mathcal{D}$.

Fact

The Grothendieck topologies on $\mathcal{C}$ of the form $J_{\mathcal{A}}$ are precisely as those which have a smallest covering sieve on every object.
N.B. If $\mathcal{A}$ is also closed under composition on the right then it can be recovered from the associated Grothendieck topology as the collection of arrows which belong to a smallest covering sieve.

The Zariski site I

- Given a commutative ring with unit A, we can endow the collection $\text{Spec}(A)$ of its prime ideals with the Zariski topology, whose basis of open sets is given by the subsets
$$\text{Spec}(A)_f := \{P \in \text{Spec}(A) \mid f \notin P\}$$
(for $f \in A$).
- One can prove that $\text{Spec}(A) = \text{Spec}(A)_{f_1} \cup \ldots \cup \text{Spec}(A)_{f_n}$ if and only if $A = (f_1, \ldots, f_n)$.
- We have a structure sheaf $\mathcal{O}$ on $\text{Spec}(A)$ such that $\mathcal{O}(\text{Spec}(A)_f) = A_f$ for each $f \in A$. The fact that it is a sheaf results from the fact that if $A = (f_1, \ldots, f_n)$ then the canonical map
$$A \to \prod_{i \in \{1,\ldots n\}} A_{f_i}$$
is the equalizer of the two canonical maps
$$\prod_{i \in I} A_{f_i} \to \prod_{i,j \in \{1,\ldots,n\}} A_{f_i f_j} .$$
- The stalk $\mathcal{O}_P$ of $\mathcal{O}$ at a prime ideal P is the localization $A_P = \text{colim}_{f \notin P} A_f$.

The Zariski site II

Notice that $\mathrm{Spec}(A)_f$ identifies with $\mathrm{Spec}(A_f)$ under the embedding

$$\mathrm{Spec}(A_f) \hookrightarrow \mathrm{Spec}(A)$$

induced by the canonical homomorphism $A \to A_f$.
This motivates the following definition.

Definition

The Zariski site (over $\mathbb{Z}$) is obtained by equipping the opposite of the category $\mathbf{Rng}_{\text{f.g.}}$ of finitely generated commutative rings with unit with the Grothendieck topology Z given by: for any cosieve S in $\mathbf{Rng}_{\text{f.g.}}$ on an object A, $S \in Z(A)$ if and only if S contains a finite family $\{\xi_i : A \to A_{f_i} \mid 1 \leq i \leq n\}$ of canonical maps $\xi_i : A \to A_{f_i}$ in $\mathbf{Rng}_{\text{f.g.}}$ where $\{f_1, \ldots, f_n\}$ is a set of elements of A which is not contained in any proper ideal of A.

This definition can be generalized to an arbitrary (commutative) base ring k, by considering the category of finitely presented (equivalently, finitely generated) k-algebras and homomorphisms between them. Notice that pushouts exist in this category (whence pullbacks exist in the opposite category) as they are given by tensor products of k-algebras.

Sheaves on a site I

Definition

- A presheaf on a (small) category $\mathcal{C}$ is a functor $P : \mathcal{C}^{\text{op}} \to \textbf{Set}$.
- Let $P : \mathcal{C}^{\text{op}} \to \textbf{Set}$ be a presheaf on $\mathcal{C}$ and S be a sieve on an object c of $\mathcal{C}$.

  A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

  $$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d \,.$$

  An amalgamation for such a family is a single element $x \in P(c)$ such that

  $$P(f)(x) = x_f \quad \text{for all } f \text{ in } S \,.$$

Sheaves on a site II

- Given a site $(\mathcal{C}, J)$, a presheaf on $\mathcal{C}$ is a *J*-sheaf if every matching family for any *J*-covering sieve on any object of $\mathcal{C}$ has a unique amalgamation.
- The *J*-sheaf condition can be expressed as the requirement that for every *J*-covering sieve *S* the canonical arrow

$$P(c) \to \prod_{f \in S} P(\mathrm{dom}(f))$$

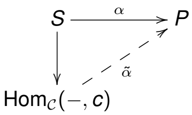
given by $x \to (P(f)(x) \mid f \in S)$ should be the equalizer of the two arrows

$$\prod_{f \in S} P(\mathrm{dom}(f)) \to \prod_{\substack{\text{f,g, } f \in S \\ \mathrm{cod}(g)=\mathrm{dom}(f)}} P(\mathrm{dom}(g))$$

given by $(x_f \to (x_{f \circ g}))$ and $(x_f \to (P(g)(x_f)))$.

The notion of Grothendieck topos

- The J-sheaf condition can also be expressed as the requirement that for every J-covering sieve S (regarded as a subobject of $\mathrm{Hom}_{\mathcal{C}}(-, c)$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$), every natural transformation $\alpha : S \to P$ admits a unique extension $\tilde{\alpha}$ along the embedding $S \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, c)$:



  (notice that a matching family for R of elements of P is precisely a natural transformation $R \to P$)

- It can also be expressed as the condition

$$P(c) = \varprojlim_{f:d \to c \in S} P(d)$$

  for each J-covering sieve S on an object c.

- The category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on the site $(\mathcal{C}, J)$ is the full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.

- A Grothendieck topos is any category equivalent to the category of sheaves on a site.

Examples of toposes

The following examples show that toposes can be naturally attached to mathematical notions as different as (small) categories, topological spaces, or groups. In fact, as we shall see later in the course, toposes can also be naturally attached to many other kinds of mathematical objects.

Examples

- For any (small) category $\mathcal{C}$, $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is the category of sheaves $\mathbf{Sh}(\mathcal{C}, T)$ where T is the trivial topology on $\mathcal{C}$.
- For any topological space X, $\mathbf{Sh}(\mathcal{O}(X), J_{\mathcal{O}(X)})$ is equivalent to the usual category $\mathbf{Sh}(X)$ of sheaves on X.
- For any (topological) group G, the category $BG = \mathbf{Cont}(G)$ of continuous actions of G on discrete sets is a Grothendieck topos (equivalent, as we shall see, to the category $\mathbf{Sh}(\mathbf{Cont}_{\text{t}}(G), J_{\text{at}})$ of sheaves on the full subcategory $\mathbf{Cont}_{\text{t}}(G)$ on the non-empty transitive actions with respect to the atomic topology).

The sheaf condition for presieves

It is sometimes convenient to check the sheaf condition for the sieve generated by a presieve directly in terms of the presieve.

Definition

A presheaf $F : \mathcal{C}^{\text{op}} \to \textbf{Set}$ satisfies the sheaf condition with respect to a presieve $P = \{f_i : c_i \to c \mid i \in I\}$ if for any family of elements $\{x_i \in P(c_i) \mid i \in I\}$ such that for any arrows h and k with $f_i \circ h = f_j \circ k$, $F(h)(x_i) = F(k)(x_j)$ there exists a unique element $x \in P(c)$ such that $F(f_i)(x) = x_i$ for all i.

Clearly, F satisfies the sheaf condition with respect to the presieve P if and only if it satisfies it with respect to the sieve generated by P.

The sheaf condition for the presieve P can be expressed as the requirement that the canonical diagram

$$F(c) \longrightarrow \prod_{i \in I} F(c_i) \rightrightarrows \prod_{\substack{h : e \to c_i, k : e \to c_j \\ f_i \circ h = f_j \circ k}} F(e)$$

is an equalizer.

N.B. If $\mathcal{C}$ has pullbacks then the product on the right-hand side can be simply indexed by the pairs (i, j) ($e = c_i \times_c c_j$ and h and k being equal to the pullback projections).

Some remarks I

The following facts show that the notion of sheaf behaves very naturally with respect to the notions of coverage and of Grothendieck topology:

(i) For any presheaf P, the collection L_P of sieves R such that P satisfies the sheaf axiom with respect to all the pullbacks sieves $f^*(R)$ is a Grothendieck topology, and the largest one for which P is a sheaf.

(ii) By intersecting such topologies, we can deduce that for any given collection of presheaves there is a largest Grothendieck topology for which all of them are sheaves.

(iii) By (i), if a presheaf satisfies the sheaf condition with respect to a coverage then it satisfies the sheaf condition with respect to the Grothendieck topology generated by it.

Some remarks II

Proposition

Let P be a presheaf $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. Then

(i) If P satisfies the sheaf condition with respect to a sieve S and to each of the sieves in a family $\{R_f \mid f \in S\}$ (where R_f is a sieve on $\mathrm{dom}(f)$) and all their pullbacks then P satisfies the sheaf condition with respect to the composite sieve $S * \{R_f \mid f \in S\}$.

(ii) If P satisfies the sheaf condition with respect to all the pullbacks of a sieve S then it satisfies the sheaf condition with respect to each sieve $T \supseteq S$.

(iii) If P satisfies the sheaf condition with respect to all the pullbacks of a sieve S on an object c and all the pullbacks of sieves of the form $g^*(R)$ for a sieve R on c indexed by arrows g in S then it satisfies the sheaf condition with respect to R.

The fact that L_P satisfies the transitivity axiom for Grothendieck topologies follows from (iii) (the sheaf condition for the pullbacks $f^*(R)$ of R follows from (iii) applied to $f^*(R)$ and $f^*(S)$ in place of R and S), which in turn can be proved by using (i) to deduce that P satisfies the sheaf condition for $S * \{g^*(R) \mid g \in S\} \subseteq R$, and by an argument similar to that used for establishing (ii) to deduce from this that P satisfies the sheaf condition also with respect to R.

Subcanonical sites

Definition

A Grothendieck topology J on a (small) category $\mathcal{C}$ is said to be subcanonical if every representable functor $\mathrm{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a J-sheaf.

Fact

For any locally small category $\mathcal{C}$, there exists the largest Grothendieck topology J on $\mathcal{C}$ for which all representables on $\mathcal{C}$ are J-sheaves. It is called the canonical topology on $\mathcal{C}$.

Definition

- A sieve R on an object c of a locally small category $\mathcal{C}$ is said to be effective-epimorphic if it forms a colimit cone under the (large!) diagram consisting of the domains of all the morphisms in R, and all the morphisms over c between them.
- It is said to be universally effective-epimorphic if its pullback along every arrow to c is effective-epimorphic.

The covering sieves for the canonical topology on a locally small category are precisely the universally effective-epimorphic ones. It follows that a Grothendieck topology is subcanonical if and only if it is contained in the canonical topology, that is if and only if all its covering sieves are effective-epimorphic.

Basic properties of Grothendieck toposes

In the next lectures, we shall prove the following result, showing that Grothendieck toposes satisfy all the categorical properties that one might hope for.

Theorem

Let $(\mathcal{C}, J)$ be a site. Then

- *the inclusion* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ *has a left adjoint* $a : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ *(called the associated sheaf functor), which preserves finite limits.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has all (small) limits, which are preserved by the inclusion functor* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$*; in particular, limits are computed pointwise and the terminal object* $1_{\mathbf{Sh}(\mathcal{C}, J)}$ *of* $\mathbf{Sh}(\mathcal{C}, J)$ *is the functor* $T : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ *sending each object* $c \in Ob(\mathcal{C})$ *to the singleton* $\{*\}$.
- *The associated sheaf functor* $a : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ *preserves colimits; in particular,* $\mathbf{Sh}(\mathcal{C}, J)$ *has all (small) colimits.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has exponentials, which are constructed as in the topos* $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$.
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has a subobject classifier.*

Corollary

Every Grothendieck topos is an elementary topos.

The plus construction

Let us start by establishing the following fundamental theorem.

Theorem

For any site $(\mathcal{C}, J)$, the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ has a left adjoint $a : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$, called the associated sheaf functor, which preserves finite limits.

The associated sheaf functor can be constructed as the functor obtained by applying twice the plus construction $P \to P^{+}$. The plus functor is defined as follows:

$$P^{+}(c) = \text{colim}_{R \in J(c)} \text{Match}(R, P)$$

where $\text{Match}(R, P)$ is the set of matching families for R of elements of P (the action of P^{+} on arrows being given by reindexing of the matching family along the pullback sieve).
Notice that this is a filtered (actually, directed) colimit, so the elements of $P^{+}(c)$ are equivalence classes $[\mathbf{x}]$ of matching families $\mathbf{x}$ with respect to the equivalence relation $\sim$ given by equality on a common refinement, that is

$$\mathbf{x} = \{x_f \mid f \in R\} \sim \mathbf{y} = \{y_g \mid g \in S\}$$

if and only if

there exists $T \subseteq R \cap S$ in $J(c)$ such that $x_h = y_h$ for all $h \in T$.

Properties of the plus construction

The following properties of the plus construction will be instrumental for proving that the functor $P \to P^{++}$ satisfies the universal property of the associated sheaf functor.

Notice that we have a natural transformation $\eta_P : P \to P^+$ given by:

$$\eta_P(c)(x) = [\{P(f)(x) \mid f \in M_c\}] .$$

A presheaf is said to be separated if it satisfies the uniqueness (but not necessarily the existence) requirement in the definition of a sheaf.

Theorem

(i) *A presheaf P is separated if and only if $\eta_P : P \to P^+$ is a monomorphism.*

(ii) *A presheaf P is a sheaf if and only if $\eta_P : P \to P^+$ is an isomorphism.*

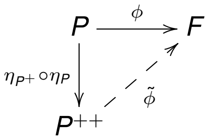
(iii) *Every morphism $P \to F$ of a presheaf P to a sheaf F factors uniquely through $\eta_P : P \to P^+$.*

(iv) *For any presheaf P, P^+ is a separated presheaf.*

(v) *For any separated presheaf P, P^+ is a sheaf.*

The associated sheaf functor

Applying the plus construction just once is in general not enough for building a sheaf starting from a presheaf (unless the presheaf is separated). Nonetheless, by the above theorem, for any presheaf P, P^{++} is a sheaf and the morphism $P \to P^{++}$ given by the composite of $\eta_P : P \to P^{+}$ and $\eta_{P^+} : P^{+} \to P^{++}$ satisfies the universal property of the associated sheaf of the presheaf P; that is, every morphism $\phi : P \to F$ of a presheaf P to a sheaf F factors uniquely through $\eta_{P^+} \circ \eta_P : P \to P^{++}$:



In other words, the associated sheaf functor $a_J : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ is left adjoint to the inclusion functor $i_J : \mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$. This implies in particular that a_J preserves all (small) colimits. On the other hand, the plus construction preserves finite limits and filtered colimits commute with finite limits in **Set**, so a_J also preserves finite limits.

Description in terms of locally matching families

A more compact description of the associated sheaf functor

$$a_J : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$$

is available.

Definition

Let $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ be a presheaf and J a Grothendieck topology on $\mathcal{C}$. Then

- We say that two elements $x, y \in P(c)$ of P are locally equal if there exists a J-covering sieve R on c such that $P(f)(x) = P(f)(y)$ for each $f \in R$.
- Given a sieve S on an object c, a locally matching family for S of elements of P is a function assigning to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that, whenever g is composable with f, $P(g)(x_f)$ and $P(f \circ g)(x)$ are locally equal.

Then $a_J(P)(c)$ consists of equivalence classes of locally matching families for J-covering sieves on c of elements P modulo local equality on a common refinement.

Subobjects in a Grothendieck topos

Since limits in a topos $\mathbf{Sh}(\mathcal{C}, J)$ are computed as in the presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$, a subobject of a sheaf F in $\mathbf{Sh}(\mathcal{C}, J)$ is just a subsheaf, that is a subfunctor which is a sheaf.

Notice that a subfunctor $F' \subseteq F$ is a sheaf if and only if for every J-covering sieve S and every element $x \in F(c)$, $x \in F'(c)$ if and only if $F(f)(x) \in F'(\text{dom}(f))$ for every $f \in S$.

Theorem

- *For any Grothendieck topos $\mathcal{E}$ and any object a of $\mathcal{E}$, the poset $\mathrm{Sub}_{\mathcal{E}}(a)$ of all subobjects of a in $\mathcal{E}$ is a complete Heyting algebra.*
- *For any arrow $f : a \to b$ in a Grothendieck topos $\mathcal{E}$, the pullback functor $f^* : \mathrm{Sub}_{\mathcal{E}}(b) \to \mathrm{Sub}_{\mathcal{E}}(a)$ has both a left adjoint $\exists_f : \mathrm{Sub}_{\mathcal{E}}(a) \to \mathrm{Sub}_{\mathcal{E}}(b)$ and a right adjoint $\forall_f : \mathrm{Sub}_{\mathcal{E}}(a) \to \mathrm{Sub}_{\mathcal{E}}(b)$.*

The Heyting operations on subobjects

Proposition

The collection $\mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(E)$ *of subobjects of an object* E *in* $\mathbf{Sh}(\mathcal{C},J)$ *has the structure of a complete Heyting algebra with respect to the natural ordering* $A \leq B$ *if and only if for every* $c \in \mathcal{C}$, $A(c) \subseteq B(c)$. *We have that*

- $(A \wedge B)(c) = A(c) \cap B(c)$ *for any* $c \in \mathcal{C}$;
- $(A \vee B)(c) = \{x \in E(c) \mid \{f : d \to c \mid E(f)(x) \in A(d) \cup B(d)\} \in J(c)\}$ *for any* $c \in \mathcal{C}$;

  (the infinitary analogue of this holds)
- $(A \Rightarrow B)(c) = \{x \in E(c) \mid$ for every $f : d \to c, E(f)(x) \in A(d)$ implies $E(f)(x) \in B(d)\}$ *for any* $c \in \mathcal{C}$.
- *the bottom subobject* $0 \rightarrowtail E$ *is given by the embedding into* E *of the initial object* 0 *of* $\mathbf{Sh}(\mathcal{C},J)$ *(given by:* $0(c) = \emptyset$ *if* $\emptyset \notin J(c)$ *and* $0(c) = \{*\}$ *if* $\emptyset \in J(c)$*);*
- *the top subobject is the identity one.*

Remark

From the Yoneda Lemma it follows that the subobject classifier Ω *in* $\mathbf{Sh}(\mathcal{C},J)$ *(see below) has the structure of an internal Heyting algebra in* $\mathbf{Sh}(\mathcal{C},J)$.

The interpretation of quantifiers

Let $\phi : E \to F$ be a morphism in $\mathbf{Sh}(\mathcal{C}, J)$.

- The pullback functor
$$\phi^* : \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(F) \to \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(E)$$
is given by: $\phi^*(B)(c) = \phi(c)^{-1}(B(c))$ for any subobject $B \rightarrowtail F$ and any $c \in \mathcal{C}$.
- The left adjoint
$$\exists_\phi : \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(E) \to \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(F)$$
is given by: $\exists_\phi(A)(c) = \{y \in E(c) \mid \{f : d \to c \mid (\exists a \in A(d))(\phi(d)(a) = E(f)(y))\} \in J(c)\}$
for any subobject $A \rightarrowtail E$ and any $c \in \mathcal{C}$.
- The right adjoint
$$\forall_\phi : \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(E) \to \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(F)$$
is given by $\forall_\phi(A)(c) = \{y \in E(c) \mid \text{ for all } f : d \to c, \phi(d)^{-1}(E(f)(y)) \subseteq A(d)\}$
for any subobject $A \rightarrowtail E$ and any $c \in \mathcal{C}$.

Toposes are balanced

Definition

A category is said to be balanced if every arrow which is both a monomorphism and an epimorphism is an isomorphism.

Remark

If in a category a monomorphism is regular (that is, occurs as the equalizer of a pair of arrows) then it is an isomorphism if and only it it is an epimorphism.

Proposition

In a Grothendieck topos $\mathcal{E}$, every monomorphism is regular (that is, it is the equalizer of its cokernel pair). In particular, $\mathcal{E}$ is balanced. In fact, also epimorphisms in $\mathcal{E}$ are all regular.

Recall that regular epimorphisms are stable under pullbacks.

The epi-mono factorization

Definition

The image $\mathrm{Im}(f)$ of an arrow $f : A \to B$ in a category $\mathcal{C}$ is, if it exists, the smallest subobject of B through which f factors.

Remark

Images exist in every Grothendieck topos (and are stable under pullback). In fact, they are obtained from the images calculated in the presheaf topos by applying the associated sheaf functor.

By recalling that a topos is balanced, we can immediately prove the following

Proposition

In every Grothendieck topos, every arrow f can be uniquely (up to a unique isomorphism) factored as an epimorphism followed by a monomorphism; the monic part of the factorization of f is given by its image.

The proposition implies in particular that epimorphisms in $\mathcal{E}$ can be characterized as the arrows whose image is an isomorphism.

The closure operation on subobjects I

The associated sheaf functor $a_J : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ induces a closure operation $c_J(m)$ on subobjects m of $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ (compatible with pullbacks of subobjects), defined by taking the pullback of the image $a_J(m)$ of $m : A' \rightarrowtail A$ under a_J along the unit η_J of the adjunction between i_J and a_J:

$$\begin{array}{ccc} c_J(A') & \longrightarrow & a_J(A') \\ {\scriptstyle c_J(m)}\downarrow & & \downarrow{\scriptstyle a_J(m)} \\ A & \xrightarrow{\eta_J(A)} & a_J(A) \end{array}$$

Concretely, we have

$$c_J(A')(c) = \{x \in A(c) \mid \{f : d \to c \mid A(f)(x) \in A'(d)\} \in J(c)\} \ .$$

Remarks

- *If A is a J-sheaf then $a_J(A')$ is isomorphic to $c_J(A')$.*
- *m is c_J-dense (that is, $c_J(m) = 1_A$) if and only if $a_J(m)$ is an isomorphism.*

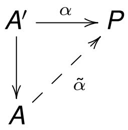
The closure operation on subobjects II

Proposition

Given a sieve S on an object c, regarded as a subobject $m_S : S \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, c)$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, the following conditions are equivalent:

(a) *a_J sends m_S to an isomorphism;*
(b) *the collection of arrows $a_J(y_{\mathcal{C}}(f))$ for $f \in S$ is jointly epimorphic;*
(c) *S is J-covering.*

We have previously remarked that the sheaf condition for a presheaf P with respect to a sieve S could be reformulated as the requirement that every morphism $S \to P$ admits a unique extension along the canonical embedding $S \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, c)$. In fact, for any c_J-dense subobject $A' \rightarrowtail A$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, if P is a J-sheaf then every morphism $\alpha : A' \to P$ admits a unique extension $\tilde{\alpha} : A \to P$ along the embedding $A' \rightarrowtail A$:



Monomorphisms and epimorphisms in $\mathbf{Sh}(\mathcal{C}, J)$

- Since limits in $\mathbf{Sh}(\mathcal{C}, J)$ are computed as in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$, and the latter are computed pointwise, we have that a morphism $\alpha : P \to Q$ in $\mathbf{Sh}(\mathcal{C}, J)$ is a monomorphism if and only if for every $c \in \mathcal{C}$,
$$\alpha(c) : P(c) \to Q(c)$$
is an injective function.

- Since the epimorphisms in $\mathbf{Sh}(\mathcal{C}, J)$ are precisely the morphisms whose image is an isomorphism, we have that a morphism $\alpha : P \to Q$ in $\mathbf{Sh}(\mathcal{C}, J)$ is an epimorphism if and only if it is locally surjective in the sense that for every $c \in \mathcal{C}$ and every $x \in Q(c)$,
$$\{f : d \to c \mid Q(f)(x) \in \text{Im}(\alpha(d))\} \in J(c) .$$

Exponentials in $\mathbf{Sh}(\mathcal{C}, J)$

- We preliminarily remark that *if* exponentials exist in $\mathbf{Sh}(\mathcal{C}, J)$ then they are computed as in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, by using the adjunction between a_J and i_J and the fact that a_J preserves finite products.
- Next, we use the characterization of the J-sheaves on $\mathcal{C}$ as the presheaves P such that for every c_J-dense subobject $A' \rightarrowtail A$, every morphism $A' \to P$ admits a unique extension $A \to P$ along the embedding $A' \rightarrowtail A$ to conclude that if F is a sheaf then F^P is a sheaf for every presheaf P:

$$\begin{array}{ccc} S & \longrightarrow & F^P \\ \downarrow & \nearrow & \\ \mathrm{Hom}_{\mathcal{C}}(-, c) & & \end{array} \qquad \begin{array}{ccc} S \times P & \longrightarrow & F \\ \downarrow & \nearrow & \\ \mathrm{Hom}_{\mathcal{C}}(-, c) \times P & & \end{array}$$

The subobject classifier in $\mathbf{Sh}(\mathcal{C}, J)$

- Given a site $(\mathcal{C}, J)$ and a sieve S in $\mathcal{C}$ on an object c, we say that S is J-closed if for any arrow $f : d \to c$, $f^*(S) \in J(d)$ implies that $f \in S$.
- Let us define $\Omega_J : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ by:
  $\Omega_J(c) = \{R \mid R \text{ is a } J\text{-closed sieve on } c\}$ (for an object $c \in \mathcal{C}$),
  $\Omega_J(f) = f^*(-)$ (for an arrow f in $\mathcal{C}$),
  where $f^*(-)$ denotes the operation of pullback of sieves in $\mathcal{C}$ along f.
  Then the arrow $true : 1_{\mathbf{Sh}(\mathcal{C},J)} \to \Omega_J$ defined by:
  $true(*)(c) = M_c$ for each $c \in Ob(\mathcal{C})$
  is a subobject classifier for $\mathbf{Sh}(\mathcal{C}, J)$.
- The classifying arrow $\chi_{A'} : A \to \Omega_J$ of a subobject $A' \subseteq A$ in $\mathbf{Sh}(\mathcal{C}, J)$ is given by:

$$\chi_{A'}(c)(x) = \{f : d \to c \mid A(f)(x) \in A'(d)\}$$

  where $c \in Ob(\mathcal{C})$ and $x \in A(c)$.

Closure operations

Definition

(a) A closure operation c on a partially ordered set $(A, \leq)$ is a function $c : A \to A$ satisfying the following properties:

- (extensivity) $a \leq c(a)$ for any $a \in A$;
- (order preservation) if $a \leq b$ then $c(a) \leq c(b)$;
- (idempotency) $c(c(a)) = c(a)$ for any $a \in A$.

(b) A closure operation c on subobjects in a topos $\mathcal{E}$ is said to be universal if it commutes with pullback, that is if $c(f^*(m)) = f^*(c(m))$ for any subobject $m : A' \to A$ and any arrow $f : B \to A$ in $\mathcal{E}$.

Proposition

Every universal closure operation c on subobjects in an elementary topos preserves finite intersections of subobjects; that is, $c(m \cap n) = c(m) \cap c(n)$.

Remark

Given a Grothendieck topology J on a small category $\mathcal{C}$, the operation c_J on subobjects in $[\mathcal{C}^{\text{op}}, \textbf{Set}]$ induced by the associated sheaf functor a_J (as described above) is a universal closure operation in the sense of this definition.

The concept of local operator

Definition

Let $\mathcal{E}$ be a topos, with subobject classifier $\top : 1 \to \Omega$. A *local operator* (or *Lawvere-Tierney topology*) on $\mathcal{E}$ is an arrow $j : \Omega \to \Omega$ in $\mathcal{E}$ such that the diagrams

$$
\begin{array}{ccc}
1 & & \\
{\scriptstyle \top}\downarrow & \searrow^{\top} & \\
\Omega & \xrightarrow[j]{} & \Omega
\end{array}
\qquad
\begin{array}{ccc}
\Omega & & \\
{\scriptstyle j}\downarrow & \searrow^{j} & \\
\Omega & \xrightarrow[j]{} & \Omega
\end{array}
\qquad
\begin{array}{ccc}
\Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\
{\scriptstyle j \times j}\downarrow & & \downarrow{\scriptstyle j} \\
\Omega \times \Omega & \xrightarrow[\wedge]{} & \Omega
\end{array}
$$

commute (where $\wedge : \Omega \times \Omega \to \Omega$ is the meet operation of the internal Heyting algebra Ω).

Theorem

For any elementary topos $\mathcal{E}$, there is a bijection between universal closure operations on $\mathcal{E}$ and local operators on $\mathcal{E}$.

Sketch of proof.

The bijection sends a universal closure operation c on $\mathcal{E}$ to the local operator $j_c : \Omega \to \Omega$ given by the classifying arrow of the subobject $c(1 \stackrel{\top}{\rightarrowtail} \top)$, and a local operator j to the closure operation c_j induced by composing classifying arrows with j. □

Abstract sheaves

Definition

Let c be a universal closure operation on an elementary topos $\mathcal{E}$.

- A subobject $m : a' \to a$ in $\mathcal{E}$ is said to be c-dense if $c(m) = id_a$, and c-closed if $c(m) = m$.
- An object a of $\mathcal{E}$ is said to be a c-sheaf if whenever we have a diagram

$$\begin{array}{ccc} b' & \xrightarrow{f'} & a \\ {\scriptstyle m}\downarrow & & \\ b & & \end{array}$$

  where m is a c-dense subobject, there exists exactly one arrow $f : b \to a$ such that $f \circ m = f'$.
- The full subcategory of $\mathcal{E}$ on the objects which are c-sheaves will be denoted by $\mathbf{sh}_c(\mathcal{E})$.

Fact

A subobject of a c-sheaf is c-closed if and only if its domain is a c-sheaf.

The dense-closed factorization

- For any universal closure operation c, we have the following orthogonality property: for any commutative square

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ \Big\downarrow m & & \Big\downarrow n \\ A & \xrightarrow[f]{} & B \end{array}$$

  where m is c-dense and n is c-closed, there exists a unique arrow $g : A \to B$ such that $n \circ g = f$ and $g \circ m = f'$.

- The factorization of a monomorphism $m : A' \to A$ as the canonical monomorphism $A' \to c(A')$ followed by the subobject $c(m)$ is the unique factorization (up to isomorphism) of m as a c-dense subobject followed by a c-closed one.

Cartesian reflectors

Definition

(a) A subcategory $\mathcal{F}$ of a category $\mathcal{E}$ is said to reflective it is replete (that is, every object isomorphic in $\mathcal{E}$ to an object of $\mathcal{F}$ also lies in $\mathcal{F}$) and the inclusion functor $\mathcal{F} \hookrightarrow \mathcal{E}$ is full and has a left adjoint.

(b) A reflective subcategory of a cartesian category is said to be a localization if the left adjoint to the inclusion functor preserves finite limits.

(c) The reflector associated with a localization of a topos $\mathcal{C}$ is the functor $\mathcal{E} \to \mathcal{E}$ given by the composite of the inclusion functor with its left adjoint. (Notice that such a functor is always cartesian.)

Proposition

Every (cartesian) reflector $L : \mathcal{E} \to \mathcal{E}$ associated with a localization of $\mathcal{E}$ induces a universal closure operation c_L on $\mathcal{C}$ defined as follows: for any subobject $m : A' \rightarrowtail A$ in $\mathcal{E}$, $c_L(m)$ is the subobject of A obtained by taking the pullback of the image $L(m)$ of $m : A' \rightarrowtail A$ under L along the unit η of the localization:

$$\begin{array}{ccc} c_L(A') & \longrightarrow & L(A') \\ {\scriptstyle c_L(m)}\downarrow & & \downarrow{\scriptstyle L(m)} \\ A & \xrightarrow{\eta(A)} & L(A) \end{array}$$

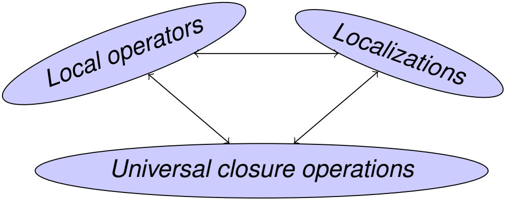
Three equivalent points of view I

Theorem

(i) *For any local operator j on an elementary (resp. Grothendieck) topos $\mathcal{E}$, $\mathbf{sh}_{c_j}(\mathcal{E})$ is an elementary (resp. Grothendieck) topos, and the inclusion $\mathbf{sh}_{c_j}(\mathcal{E}) \hookrightarrow \mathcal{E}$ has a left adjoint $a_j : \mathcal{E} \to \mathbf{sh}_{c_j}(\mathcal{E})$ which preserves finite limits (and satisfies the property that the monomorphisms which it sends to isomorphisms are precisely the c_j-dense ones).*

(ii) *Conversely, a localization of $\mathcal{E}$ defines, as specified above, a universal closure operation on $\mathcal{E}$ and hence a local operator on $\mathcal{E}$.*

(iii) *In fact, these assignments define a bijection between the localizations of $\mathcal{E}$ and the local operators (equivalently, the universal closure operations) on $\mathcal{E}$:*



Three equivalent points of view II

- The proof of the fact that a localization can be recovered from the corresponding closure operation relies on the following result: for any localization $\mathcal{L}$ of $\mathcal{E}$ with associated reflector L and closure operation c_L, the following conditions are equivalent for an object A of $\mathcal{C}$:
  - (i) A is c_L-separated;
  - (ii) $\eta_A : A \to LA$ is a monomorphism.

  Also, the following conditions are equivalent:
  - (i) A is a c_L-sheaf;
  - (ii) $\eta_A : A \to LA$ is an isomorphism;
  - (iii) A lies in $\mathcal{L}$.

- The fact that a closure operation c can be recovered from the associated localization follows from the fact that, for a monomorphism $m : A' \to A$, m factors as the canonical monomophism $A' \to \text{dom}(\eta_A^*(Lm))$, which is c-dense (since it is sent by L to an isomorphism), followed by the monomorphism $\eta_A^*(Lm)$, which is c-closed (as it is the pullback of Lm, which is closed).

Local operators and Grothendieck topologies

Theorem

If $\mathcal{C}$ is a small category, the Grothendieck topologies J on $\mathcal{C}$ correspond exactly to the local operators on the presheaf topos $[\mathcal{C}^{op}, \mathbf{Set}]$. (More generally, the Grothendieck topologies J' which contain a given Grothendieck topology J on $\mathcal{C}$ correspond exactly to the local operators on the topos $\mathbf{Sh}(\mathcal{C}, J)$.)

Sketch of proof.

The correspondence sends a local operator $j : \Omega \to \Omega$ to the subobject $J \rightarrowtail \Omega$ which it classifies, that is to the Grothendieck topology J on $\mathcal{C}$ defined by:

$$S \in J(c) \text{ if and only if } j(c)(S) = M_c$$

Conversely, it sends a Grothendieck topology J, regarded as a subobject $J \rightarrowtail \Omega$, to the arrow $j : \Omega \to \Omega$ that classifies it. □

In fact, if J is the Grothendieck topology corresponding to a local operator j, an object of $[\mathcal{C}^{op}, \mathbf{Set}]$ is a J-sheaf (in the sense of Grothendieck toposes) if and only if it is a c_j-sheaf (in the sense of universal closure operations).

For further reading

O. Caramello.
Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'
Oxford University Press, 2017.

P. T. Johnstone.
Sketches of an Elephant: a topos theory compendium, vols. 1 and 2
Oxford University Press, 2002.

S. Mac Lane and I. Moerdijk.
Sheaves in geometry and logic: a first introduction to topos theory
Springer-Verlag, 1992.