Topos Theory

Classifying toposes and the ‘bridge’ technique

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Toposes as mathematical universes

- Recall that every Grothendieck topos $\mathscr{E}$ is an elementary topos. Thus, given the fact that arbitrary colimits exist in $\mathscr{E}$, we can consider models of any kind of first-order (even infinitary) theory in $\mathscr{E}$. In particular, we can consider models of geometric theories in $\mathscr{E}$.
- Inverse image functors of geometric morphisms of toposes preserve finite limits (by definition) and arbitrary colimits (having a right adjoint); in particular, they are geometric functors and hence they preserve the interpretation of (arbitrary) geometric formulae. In general, they are *not* Heyting functors, which explains why the next definition only makes sense for geometric theories.

The notion of classifying topos

Definition

Let $\mathbb{T}$ be a geometric theory over a given signature. A classifying topos of $\mathbb{T}$ is a Grothendieck topos $\mathbf{Set}[\mathbb{T}]$ such that for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathbf{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

natural in $\mathscr{E}$.

Naturality means that for any geometric morphism $f : \mathscr{E} \to \mathscr{F}$, we have a commutative square

$$\begin{tikzcd}
\mathbf{Geom}(\mathscr{F}, \mathbf{Set}[\mathbb{T}]) \arrow[r, "\simeq"] \arrow[d, "- \circ f"] & \mathbb{T}\text{-mod}(\mathscr{F}) \arrow[d, "\mathbb{T}\text{-mod}(f^*)"] \\
\mathbf{Geom}(\mathscr{E}, \mathbf{Set}[\mathbb{T}]) \arrow[r, "\simeq"] & \mathbb{T}\text{-mod}(\mathscr{E})
\end{tikzcd}$$

Theorem

Every geometric theory (over a given signature) has a classifying topos.

Representability of the $\mathbb{T}$-model functor

Remark

- *The classifying topos of a geometric theory $\mathbb{T}$ can be seen as a representing object for the (pseudo-)functor*

$$\mathbb{T}\text{-mod} : \mathfrak{BTop}^{\mathrm{op}} \to \boldsymbol{Cat}$$

which assigns

  - *to a topos $\mathscr{E}$ the category $\mathbb{T}\text{-mod}(\mathscr{E})$ of models of $\mathbb{T}$ in $\mathscr{E}$ and*
  - *to a geometric morphism $f : \mathscr{E} \to \mathscr{F}$ the functor $\mathbb{T}\text{-mod}(f^*) : \mathbb{T}\text{-mod}(\mathscr{F}) \to \mathbb{T}\text{-mod}(\mathscr{E})$ sending a model $M \in \mathbb{T}\text{-mod}(\mathscr{F})$ to its image $f^*(M)$ under the functor f^*.*

- *In particular, classifying toposes are unique up to categorical equivalence.*

Universal models

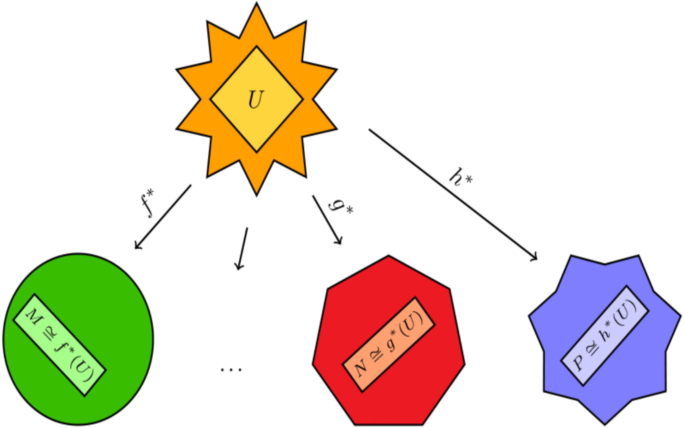
Definition

Let $\mathbb{T}$ be a geometric theory. A universal model of a geometric theory $\mathbb{T}$ is a model $U_{\mathbb{T}}$ of $\mathbb{T}$ in a Grothendieck topos $\mathscr{G}$ such that for any $\mathbb{T}$-model M in a Grothendieck topos $\mathscr{F}$ there exists a unique (up to isomorphism) geometric morphism $f_M : \mathscr{F} \to \mathscr{G}$ such that $f_M^*(U_{\mathbb{T}}) \cong M$.

Remark

- *By the (2-dimensional) Yoneda Lemma, if a topos $\mathscr{G}$ contains a universal model of a geometric theory $\mathbb{T}$ then $\mathscr{G}$ satisfies the universal property of the classifying topos of $\mathbb{T}$. Conversely, if a topos $\mathscr{E}$ classifies a geometric theory $\mathbb{T}$ then $\mathscr{E}$ contains a universal model of $\mathbb{T}$.*
- *In particular classifying toposes, and hence universal models, are unique up to equivalence. In fact, if M and N are universal models of a geometric theory $\mathbb{T}$ lying respectively in toposes $\mathscr{F}$ and $\mathscr{G}$ then there exists a unique (up to isomorphism) geometric equivalence between $\mathscr{F}$ and $\mathscr{G}$ such that its inverse image functors send M and N to each other (up to isomorphism).*

Classifying toposes



Classifying topos

The Morleyization of a first-order theory

It is a fact that most of the first-order theories naturally arising in Mathematics have a geometric axiomatization. Anyway, if a finitary first-order theory $\mathbb{T}$ is not geometric, we can canonically construct a coherent theory over a larger signature, called the Morleyization of $\mathbb{T}$ whose models in **Set** (more generally, in any Boolean coherent category) can be identified with those of $\mathbb{T}$.

Definition

A homomorphism of **Set**-models of a first-order theory $\mathbb{T}$ is an elementary embedding if it is compatible with respect to the interpretation of all first-order formulae in the signature of $\mathbb{T}$. The category of $\mathbb{T}$-models in **Set** and elementary embeddings between them will be denoted by $\mathbb{T}\text{-mod}_e(\mathbf{Set})$.

Theorem

Let $\mathbb{T}$ be a finitary first-order theory over a signature Σ. Then there is a signature Σ' containing Σ, and a coherent theory $\mathbb{T}'$ over Σ', called the Morleyization of $\mathbb{T}$, such that we have

$$\mathbb{T}\text{-mod}_e(\mathbf{Set}) \simeq \mathbb{T}'\text{-mod}(\mathbf{Set})$$

Theories of continuous flat functors

Theorem

Every Grothendieck topos is classified by some geometric theory.

In fact, any topos $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a (small) site $(\mathscr{C}, J)$ is classified by the theory $\mathbb{T}^{\mathscr{C}}_{J}$ of J-continuous flat functors on $\mathscr{C}$ defined as follows: the signature of $\mathbb{T}^{\mathscr{C}}_{J}$ has one sort $\ulcorner a \urcorner$ for each object a of $\mathscr{C}$ and one function symbol $\ulcorner f \urcorner : \ulcorner a \urcorner \to \ulcorner b \urcorner$ for each arrow $f : a \to b$ in $\mathscr{C}$, and the axioms of $\mathbb{T}^{\mathscr{C}}_{J}$ are the following (to indicate that a variable x has sort $\ulcorner a \urcorner$ we write x^a):

$$(\top \vdash_x \ulcorner f \urcorner(x) = x)$$

for any identity arrow f in $\mathscr{C}$;

$$(\top \vdash_x \ulcorner f \urcorner(x) = \ulcorner h \urcorner(\ulcorner g \urcorner(x)))$$

for any triple of arrows f, g, h of $\mathscr{C}$ such that $f = h \circ g$;

$$\left(\top \vdash_{[]} \bigvee_{a \in Ob(\mathscr{C})} (\exists x^a)\top\right);$$

$$\left(\top \vdash_{x^a, y^b} \bigvee_{a \xleftarrow{f} c \xrightarrow{g} b} (\exists z^c)(\ulcorner f \urcorner(z^c) = x^a \wedge \ulcorner g \urcorner(z^c) = y^b)\right)$$

for any objects a, b of $\mathscr{C}$;

$$\left(\ulcorner f \urcorner(x^a) = \ulcorner g \urcorner(x^a) \vdash_{x^a} \bigvee_{\substack{c \xrightarrow{h} a \mid \\ f \circ h = g \circ h}} (\exists z^c)(\ulcorner h \urcorner(z^c) = x^a)\right)$$

for any pair of arrows $f, g : a \to b$ in $\mathscr{C}$ with common domain and codomain;

$$\left(\top \vdash_{x^a} \bigvee_{i \in I} (\exists y_i^{b_i})(\ulcorner f_i \urcorner(y_i^{b_i}) = x^a)\right)$$

for each J-covering family $\{f_i : b_i \to a \mid i \in I\}$.

Syntactic categories I

Definition

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. The syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the 'renaming'-equivalence classes of geometric formulae-in-context $\{\vec{x} \,.\, \phi\}$ over Σ and as arrows $\{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, \psi\}$ (where the contexts $\vec{x}$ and $\vec{y}$ are disjoint) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\vec{x}, \vec{y})$ which are $\mathbb{T}$-provably functional i.e. such that the sequents

$$(\phi \vdash_{\vec{x}} (\exists \vec{y})\theta),$$
$$(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi), \text{ and}$$
$$((\theta \wedge \theta[\vec{z}/\vec{y}]) \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{y} = \vec{z}))$$

are provable in $\mathbb{T}$.

Syntactic categories II

- The composite of two arrows

$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\theta]} \{\vec{y} \,.\, \psi\} \xrightarrow{[\gamma]} \{\vec{z} \,.\, \chi\}$$

is defined as the $\mathbb{T}$-provable-equivalence class of the formula $(\exists \vec{y})\theta \wedge \gamma$.

- The identity arrow on an object $\{\vec{x} \,.\, \phi\}$ is the arrow

$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\phi \wedge \vec{x'} = \vec{x}]} \{\vec{x'} \,.\, \phi[\vec{x'}/\vec{x}]\}$$

- For a regular (resp. coherent, first-order) theory $\mathbb{T}$ one can define the regular (resp. coherent, first-order) syntactic category $\mathscr{C}_{\mathbb{T}}^{\text{reg}}$ (resp. $\mathscr{C}_{\mathbb{T}}^{\text{coh}}$, $\mathscr{C}_{\mathbb{T}}^{\text{fo}}$) of $\mathbb{T}$ by replacing the word 'geometric' with 'regular' (resp. 'coherent', 'first-order') in the definition above. If $\mathbb{T}$ is a Horn theory then one can construct the cartesian syntactic category $\mathscr{C}_{\mathbb{T}}^{\text{cart}}$ by allowing as objects and arrows of the category those formulae which can be built from atomic formulae by binary conjunction, truth and 'unique-existential' quantifications (relative to $\mathbb{T}$).

Properties of syntactic categories

Theorem

(i) *For any Horn theory* $\mathbb{T}$, $\mathscr{C}_{\mathbb{T}}^{\text{cart}}$ *is a cartesian category.*
(ii) *For any regular theory* $\mathbb{T}$, $\mathscr{C}_{\mathbb{T}}^{\text{reg}}$ *is a regular category.*
(iii) *For any coherent theory* $\mathbb{T}$, $\mathscr{C}_{\mathbb{T}}^{\text{coh}}$ *is a coherent category.*
(iv) *For any first-order theory* $\mathbb{T}$, $\mathscr{C}_{\mathbb{T}}^{\text{fo}}$ *is a Heyting category.*
(v) *For any geometric theory* $\mathbb{T}$, $\mathscr{C}_{\mathbb{T}}$ *is a geometric category.*

Conversely, any regular (resp. coherent, geometric) category is, up to categorical equivalence, the regular (resp. coherent, geometric) syntactic category of some regular (resp. coherent, geometric) theory.

Lemma

Any subobject of $\{\vec{x} . \phi\}$ *in* $\mathscr{C}_{\mathbb{T}}$ *is isomorphic to one of the form*

$$\{\vec{x'} . \psi[\vec{x'}/\vec{x}]\} \xrightarrow{[\psi \wedge \vec{x'}=\vec{x}]} \{\vec{x} . \phi\}$$

where ψ *is a formula such that the sequent* $\psi \vdash_{\vec{x}} \phi$ *is provable in* $\mathbb{T}$*. We will denote this subobject simply by* $[\psi]$*.*
Moreover, for two such subobjects $[\psi]$ *and* $[\chi]$*, we have* $[\psi] \leq [\chi]$ *in* $\mathrm{Sub}_{\mathscr{C}_{\mathbb{T}}}(\{\vec{x} . \phi\})$ *if and only if the sequent* $\psi \vdash_{\vec{x}} \chi$ *is provable in* $\mathbb{T}$*.*

The universal model in $\mathscr{C}_{\mathbb{T}}$

Definition

Let $\mathbb{T}$ be a geometric theory over a signature Σ. The universal model of $\mathbb{T}$ in $\mathscr{C}_{\mathbb{T}}$ is defined as the structure $M_{\mathbb{T}}$ which assigns

- to a sort A the object $\{x^A . \top\}$ where x^A is a variable of sort A,
- to a function symbol $f : A_1 \cdots A_n \to B$ the morphism

$$\{x_1^{A_1}, \ldots, x_n^{A_n} . \top\} \xrightarrow{[f(x_1^{A_1},\ldots,x_n^{A_n})=y^B]} \{y^B . \top\}$$

  and
- to a relation symbol $R \rightarrowtail A_1 \cdots A_n$ the subobject

$$\{x_1^{A_1}, \ldots, x_n^{A_n} . R(x_1^{A_1}, \ldots, x_n^{A_n})\} \xrightarrow{[R(x_1^{A_1},\ldots,x_n^{A_n})]} \{x_1^{A_1}, \ldots, x_n^{A_n} . \top\}$$

Theorem

- *For any geometric formula-in-context $\{\vec{x} . \phi\}$ over Σ, the interpretation $[[\vec{x} . \phi]]_{M_{\mathbb{T}}}$ in $M_{\mathbb{T}}$ is the subobject $[\phi] : \{\vec{x} . \phi\} \rightarrowtail \{\vec{x} . \top\}$.*
- *A geometric sequent $(\phi \vdash_{\vec{x}} \psi)$ is satisfied in $M_{\mathbb{T}}$ if and only if it is provable in $\mathbb{T}$.*

Logical topologies I

- In a regular category, every arrow $f : a \to b$ factors uniquely through its image $\mathrm{Im}(f) \rightarrowtail b$ as the composite $a \to \mathrm{Im}(f) \to b$ of $\mathrm{Im}(f) \rightarrowtail b$ with an arrow $c(f) : a \to \mathrm{Im}(f)$; arrows of the form $c(f)$ for some f are called covers. In fact, every arrow in a regular category can be factored uniquely as a cover followed by a monomorphism, and covers are precisely the arrows g such that $\mathrm{Im}(g) = 1_{cod(g)}$.
- In a coherent (resp. geometric) category, a finite (resp. small) covering family is a family of arrows such that the union of their images is the maximal subobject.

Definition

- For a regular theory $\mathbb{T}$, the regular topology is the Grothendieck topology $J_{\mathbb{T}}^{reg}$ on $\mathscr{C}_{\mathbb{T}}^{reg}$ whose covering sieves are those which contain a cover.
- For a coherent theory $\mathbb{T}$, the coherent topology is the Grothendieck topology $J_{\mathbb{T}}^{coh}$ on $\mathscr{C}_{\mathbb{T}}^{coh}$ whose covering sieves are those which contain finite covering families.
- For a geometric theory $\mathbb{T}$, the geometric topology is the Grothendieck topology $J_{\mathbb{T}}$ on $\mathscr{C}_{\mathbb{T}}$ whose covering sieves are those which contain small covering families.

Logical topologies II

Notation: we denote by **Reg**$(\mathscr{C}_{\mathbb{T}}^{\text{reg}}, \mathscr{D})$ (resp. **Coh**$(\mathscr{C}_{\mathbb{T}}^{\text{coh}}, \mathscr{D})$, **Geom**$(\mathscr{C}_{\mathbb{T}}, \mathscr{D})$) the categories of regular (resp. coherent, geometric) functors from $\mathscr{C}_{\mathbb{T}}^{\text{reg}}$ (resp. $\mathscr{C}_{\mathbb{T}}^{\text{coh}}$, $\mathscr{C}_{\mathbb{T}}$) to a regular (resp. coherent, geometric) category $\mathscr{D}$ and natural transformations between them.

Fact

A cartesian functor $\mathscr{C}_{\mathbb{T}}^{\text{reg}} \to \mathscr{D}$ (resp. $\mathscr{C}_{\mathbb{T}}^{\text{coh}} \to \mathscr{D}$, $\mathscr{C}_{\mathbb{T}} \to \mathscr{D}$) is regular (resp. coherent, geometric) if and only it sends $J_{\mathbb{T}}^{\text{reg}}$-covering (resp. $J_{\mathbb{T}}^{\text{coh}}$-covering, $J_{\mathbb{T}}$-covering) sieves to covering families.

Models as functors

Theorem

(i) *For any Horn theory $\mathbb{T}$ and cartesian category $\mathscr{D}$, we have an equivalence of categories* $\mathbf{Cart}(\mathscr{C}_{\mathbb{T}}^{\text{cart}}, \mathscr{D}) \simeq \mathbb{T}\text{-mod}(\mathscr{D})$ *natural in $\mathscr{D}$.*

(ii) *For any regular theory $\mathbb{T}$ and regular category $\mathscr{D}$, we have an equivalence of categories* $\mathbf{Reg}(\mathscr{C}_{\mathbb{T}}^{\text{reg}}, \mathscr{D}) \simeq \mathbb{T}\text{-mod}(\mathscr{D})$ *natural in $\mathscr{D}$.*

(iii) *For any coherent theory $\mathbb{T}$ and coherent category $\mathscr{D}$, we have an equivalence of categories* $\mathbf{Coh}(\mathscr{C}_{\mathbb{T}}^{\text{coh}}, \mathscr{D}) \simeq \mathbb{T}\text{-mod}(\mathscr{D})$ *natural in $\mathscr{D}$.*

(iv) *For any geometric theory $\mathbb{T}$ and geometric category $\mathscr{D}$, we have an equivalence of categories* $\mathbf{Geom}(\mathscr{C}_{\mathbb{T}}, \mathscr{D}) \simeq \mathbb{T}\text{-mod}(\mathscr{D})$ *natural in $\mathscr{D}$.*

Sketch of proof.

- One half of the equivalence sends a model $M \in \mathbb{T}\text{-mod}(\mathscr{E})$ to the functor $F_M : \mathscr{C}_{\mathbb{T}} \to \mathscr{E}$ assigning to a formula $\{\vec{x} . \phi\}$ (the domain of) its interpretation $[[\vec{x} . \phi]]_M$ in M.
- The other half of the equivalence sends a functor $F : \mathscr{C}_{\mathbb{T}} \to \mathscr{D}$ to the image $F(M_{\mathbb{T}})$ of the universal model $M_{\mathbb{T}}$ under F.

Classifying toposes via syntactic sites

Corollary

- *For any Horn theory* $\mathbb{T}$*, the topos* $[(\mathscr{C}_{\mathbb{T}}^{\text{cart}})^{\text{op}}, \mathbf{Set}]$ *classifies* $\mathbb{T}$*.*
- *For any regular theory* $\mathbb{T}$*, the topos* $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}^{\text{reg}}, J_{\mathbb{T}}^{\text{reg}})$ *classifies* $\mathbb{T}$*.*
- *For any coherent theory* $\mathbb{T}$*, the topos* $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathbb{T}}^{\text{coh}})$ *classifies* $\mathbb{T}$*.*
- *For any geometric theory* $\mathbb{T}$*, the topos* $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ *classifies* $\mathbb{T}$*.*

Classifying toposes for Horn theories I

Definition

Let $\mathbb{T}$ be a Horn theory over a signature Σ. We say that a $\mathbb{T}$-model M in **Set** is finitely presented by a Horn formula $\phi(\vec{x})$, where $A_1 \cdots A_n$ is the string of sorts associated to $\vec{x}$, if there exists a string of elements $(\xi_1, \ldots, \xi_n) \in MA_1 \times \ldots \times MA_n$, called the generators of M, such that for any $\mathbb{T}$-model N in **Set** and string of elements $\vec{b} = (b_1, \ldots, b_n) \in MA_1 \times \ldots \times MA_n$ such that $(b_1, \ldots, b_n) \in [[\vec{x} \,.\, \phi]]_N$, there exists a unique arrow $f^{\vec{b}} : M \to N$ in $\mathbb{T}$-mod(**Set**) such that $(f^{\vec{b}}_{A_1} \times \ldots \times f^{\vec{b}}_{A_n})((\xi_1, \ldots, \xi_n)) = (b_1, \ldots, b_n)$. We denote by f.p.$\mathbb{T}$-mod(**Set**) the full subcategory of $\mathbb{T}$-mod(**Set**) on the finitely presented models.

Classifying toposes for Horn theories II

Theorem

For any Horn theory $\mathbb{T}$, we have an equivalence of categories

$$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}) \simeq (\mathscr{C}_{\mathbb{T}}^{\text{cart}})^{\text{op}}$$

In particular, $\mathbb{T}$ is classified by the topos $[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$.

Examples

- The theory of Boolean algebras is classified by the topos $[\mathbf{Bool}_{fin}, \mathbf{Set}]$, where $\mathbf{Bool}_{fin}$ is the category of finite Boolean algebras.
- The theory of commutative rings with unit is classified by the topos $[\mathbf{Rng}_{f.g.}, \mathbf{Set}]$, where $\mathbf{Rng}_{f.g.}$ is the category of finitely generated rings.

Classifying toposes for propositional theories I

Definition

- A propositional theory is a geometric theory over a signature Σ which has no sorts.
- A localic topos is any topos of the form $\mathbf{Sh}(L)$ for a locale L.

Theorem

Localic toposes are precisely the classifying toposes of propositional theories.

Classifying toposes for propositional theories II

Specifically, given a locale L, we can consider the propositional theory $\mathbb{P}_L$ of completely prime filters in L, defined as follows. We take one atomic proposition F_a (to be thought of as the assertion that a is in the filter) for each $a \in L$; the axioms are

$$(\top \vdash F_1),$$

all the sequents of the form

$$(F_a \vdash F_b)$$

for any $a \leq b$ in L,

all the sequents of the form

$$(F_a \wedge F_b \vdash F_{a \wedge b}),$$

for any $a, b \in L$, and all the sequents of the form

$$(F_a \vdash \bigvee_{i \in I} F_{a_i})$$

whenever $\bigvee_{i \in I} a_i = a$ in L.

In fact, for any locale L, the topos $\mathbf{Sh}(L)$ classifies $\mathbb{P}_L$.

Toposes as *bridges*

- In the topos-theoretic study of theories, the latter are represented by sites (of definition of their classifying topos or of some other topos naturally attached to them), and the existence of theories which are Morita-equivalent to each other translates into the existence of different sites of definition for the same Grothendieck topos.
- Grothendieck toposes can be effectively used as 'bridges' for transferring notions, properties and results across different Morita-equivalent theories:

$$
\begin{tikzcd}
 & \mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'} \arrow[dr, dashed] & \\
\mathbb{T} \arrow[ur, dashed] & & \mathbb{T}'
\end{tikzcd}
$$

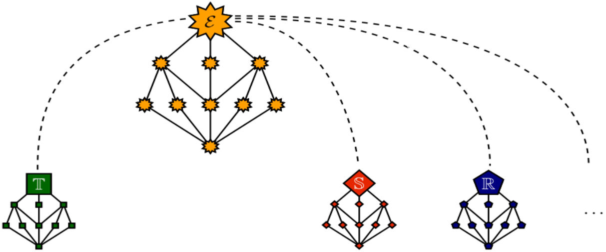
- The transfer of information takes place by expressing topos-theoretic invariants in terms of the different sites of definition (or, more generally, presentations) for the given topos.

Toposes as *bridges*

- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.
- This methodology is technically effective because the relationship between a topos and its representations is often very natural, enabling us to easily transfer invariants across different representations (and hence, between different theories).
- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories and constructions. Indeed, many important invariants of mathematical structures are actually invariants of toposes (think for instance of cohomology or homotopy groups) and topos-theoretic invariants considered on the classifying topos $\mathscr{E}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ often translate into interesting logical (i.e. syntactic or semantic) properties of $\mathbb{T}$.

Toposes as *bridges*

- The fact that topos-theoretic invariants specialize to important properties or constructions of natural mathematical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications in Mathematics as a whole: for instance

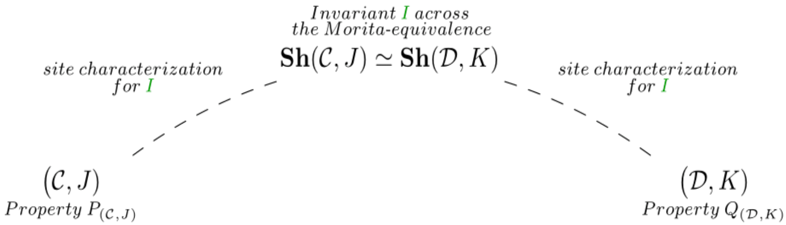


Lattices of theories

This picture represents the lattice structure on the collection of the subtoposes of a topos $\mathscr{E}$ inducing lattice structures on the collection of 'quotients' of geometric theories $\mathbb{T}$, $\mathbb{S}$, $\mathbb{R}$ classified by it.

The ‘bridge-building’ technique

- Decks of ‘bridges’: Morita-equivalences (or more generally morphisms or other kinds of relations between toposes)
- Arches of ‘bridges’: Site characterizations (or more generally ‘unravelings’ of topos-theoretic invariants in terms of concrete representations of the relevant topos)



The ‘bridge’ yields a logical equivalence (or an implication) between the ‘concrete’ properties $P_{(\mathscr{C},J)}$ and $Q_{(\mathscr{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

A duality theorem I

Definition

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. A quotient of $\mathbb{T}$ is a geometric theory $\mathbb{T}'$ over Σ such that every axiom of $\mathbb{T}$ is provable in $\mathbb{T}'$.
- Let $\mathbb{T}$ and $\mathbb{T}'$ be geometric theories over a signature Σ. We say that $\mathbb{T}$ and $\mathbb{T}'$ are syntactically equivalent, and we write $\mathbb{T} \equiv_s \mathbb{T}'$, if for every geometric sequent σ over Σ, σ is provable in $\mathbb{T}$ if and only if σ is provable in $\mathbb{T}'$.

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the $\equiv_s$-equivalence classes of quotients of $\mathbb{T}$ and the subtoposes of the classifying topos **Set**$[\mathbb{T}]$ *of $\mathbb{T}$.*

A duality theorem II

If $i_J : \mathscr{E} \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J) \hookrightarrow \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the subtopos of $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ corresponding to a quotient $\mathbb{T}'$ of $\mathbb{T}$ via the theorem, we have a commutative (up to natural isomorphism) diagram in **Cat** (where i is the obvious inclusion)

$$\begin{tikzcd} \mathbb{T}'\text{-mod}(\mathscr{E}) \arrow[r, "\simeq"] \arrow[d, "i"] & \mathbf{Geom}(\mathscr{E}, \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J)) \arrow[d, "i_J \circ -"] \\ \mathbb{T}\text{-mod}(\mathscr{E}) \arrow[r, "\simeq"] & \mathbf{Geom}(\mathscr{E}, \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})) \end{tikzcd}$$

naturally in $\mathscr{E} \in \mathfrak{BTop}$.

Sketch of proof.

We establish a bijection between the Grothendieck topologies J on $\mathscr{C}_{\mathbb{T}}$ which contain the topology $J_{\mathbb{T}}$ and the (syntactic-equivalence classes) of quotients $\mathbb{T}'$ of $\mathbb{T}$:

- One half of the bijection sends a quotient $\mathbb{T}'$ to the Grothendieck topology generated by the principal sieves generated by the arrows $[\phi \wedge \psi] : \{\vec{x} . \phi \wedge \psi\} \rightarrowtail \{\vec{x} . \phi\}$ where $(\phi \vdash_{\vec{x}} \psi)$ is provable in $\mathbb{T}'$.
- The other half of the bijection sends a Grothendieck topology $J \supseteq J_{\mathbb{T}}$ to the quotient of $\mathbb{T}$ consisting of the sequents of the form $(\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x}_i)\theta_i)$ for a small J-covering family $\{[\theta_i] : \{\vec{x}_i . \phi_i\} \to \{\vec{y} . \psi\}\}$.

A simple example

Suppose to have a Morita equivalence between two geometric theories $\mathbb{T}$ and $\mathbb{S}$.

Question: If $\mathbb{T}'$ is a quotient of $\mathbb{T}$, is there a quotient $\mathbb{S}'$ of $\mathbb{S}$ such that the given duality restricts to a duality between $\mathbb{T}'$ and $\mathbb{S}'$?

The duality theorem gives a straight positive answer to this question. In fact, both quotients of $\mathbb{T}$ and quotients of $\mathbb{S}$ correspond bijectively with subtoposes of the classifying topos $\mathbf{Set}[\mathbb{T}] = \mathbf{Set}[\mathbb{S}]$.

Note the role of the classifying topos as a 'bridge' between the two theories!

A simple example

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Theories of presheaf type

Definition

- A geometric theory is said to be of presheaf type if it is classified by a presheaf topos.
- A set-based model M of a geometric theory $\mathbb{T}$ is said to be finitely presentable if the functor $\mathrm{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(M, -) : \mathbb{T}\text{-mod}(\mathbf{Set}) \to \mathbf{Set}$ preserves filtered colimits.

Theories of presheaf type are very important in that they constitute the basic 'building blocks' from which every geometric theory can be built. Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a 'quotient' of a theory of presheaf type.

These theories are the logical counterpart of small categories, in the sense that:

- For any theory of presheaf type $\mathbb{T}$, its category $\mathbb{T}\text{-mod}(\mathbf{Set})$ of (set-based) models is equivalent to the ind-completion of the full subcategory $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ on the finitely presentable models.
- Any small category $\mathscr{C}$ is, up to idempotent splitting completion, equivalent to the category $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ for some theory of presheaf type $\mathbb{T}$.

Theories of presheaf type

Every finitary algebraic (or, more generally, cartesian) theory is of presheaf type, but this class contains many other interesting mathematical theories including

- the theory of linear orders (classified by the simplicial topos)
- the theory of algebraic extensions of a given field
- the theory of flat modules over a ring
- the theory of lattice-ordered abelian groups with strong unit
- the 'cyclic theories' (classified by the cyclic topos, the epicyclic topos and the arithmetic topos)
- the theory of perfect MV-algebras (or more generally of local MV-algebras in a proper variety of MV-algebras)
- the geometric theory of finite sets

Any theory of presheaf type $\mathbb{T}$ gives rise to two different representations of its classifying topos, which can be used to build 'bridges' connecting its syntax and semantics:

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

$$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \qquad\qquad\qquad\qquad (\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

Irreducible formulae and finitely presentable models

Definition

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then a geometric formula $\phi(\vec{x})$ over Σ is said to be $\mathbb{T}$-irreducible if, regarded as an object of the syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$, it does not admit any non-trivial $J_{\mathbb{T}}$-covering sieves.

Theorem

Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ. Then

(i) *Any finitely presentable $\mathbb{T}$-model in* **Set** *is presented by a $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ;*

(ii) *Conversely, any $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ presents a $\mathbb{T}$-model.*

In fact, the category f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ *is equivalent to the full subcategory $\mathscr{C}_{\mathbb{T}}^{\text{irr}}$ of $\mathscr{C}_{\mathbb{T}}$ on the $\mathbb{T}$-irreducible formulae.*

Irreducible object

$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$

Every object

$(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

$\mathbb{T}$-irreducible formula

A definability theorem

Theorem

Let $\mathbb{T}$ be a theory of preshef type and suppose that we are given, for every finitely presentable **Set***-model $\mathscr{M}$ of $\mathbb{T}$, a subset $R_{\mathscr{M}}$ of $\mathscr{M}^n$ in such a way that every $\mathbb{T}$-model homomorphism $h : \mathscr{M} \to \mathscr{N}$ maps $R_{\mathscr{M}}$ into $R_{\mathscr{N}}$. Then there exists a geometric formula-in-context $\phi(x_1, \ldots, x_n)$ such that $R_{\mathscr{M}} = [[\vec{x}\ .\ \phi]]_{\mathscr{M}}$ for each finitely presentable $\mathbb{T}$-model $\mathscr{M}$.*

Subobject of $UA_1 \times \cdots \times UA_n$

$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{\text{op}}$

Functorial assignment

$M \to R_M \subseteq MA_1 \times \cdots \times MA_n$

$(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

Geometric formula

$\phi(x_1^{A_1}, \ldots, x_n^{A_n})$

Characterization theorems

Theorem

A geometric theory $\mathbb{T}$ over a signature Σ is of presheaf type if and only if every geometric formula $\phi(\vec{x})$ over Σ, when regarded as an object of $\mathscr{C}_{\mathbb{T}}$, is $J_{\mathbb{T}}$-covered by $\mathbb{T}$-irreducible formulae over Σ.

Theorem

A geometric theory $\mathbb{T}$ over a signature Σ is of presheaf type if and only if the following conditions are satisfied:

(i) *Every finitely presentable model is presented by a geometric formula over Σ.*

(ii) *Every property of finite tuples of elements of a finitely presentable $\mathbb{T}$-model which is preserved by $\mathbb{T}$-model homomorphisms is definable (in finitely presentable $\mathbb{T}$-models) by a geometric formula over Σ.*

(iii) *The finitely presentable $\mathbb{T}$-models are jointly conservative for $\mathbb{T}$.*

My book also contains a characterization theorem providing necessary and sufficient semantic conditions for a theory to be of presheaf type.

‘Bridges’ between quotients and topologies

The duality theorem between subtoposes and quotients allows one in particular to establish ‘bridges’ of the following form:

Subtopos of
$\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Set}[\mathbb{T}]$

Grothendieck topology on $\mathscr{C}$ containing J

Quotient of $\mathbb{T}$

That is, if the classifying topos of a geometric theory $\mathbb{T}$ can be represented as the category $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a (small) site $(\mathscr{C}, J)$ then we have a natural, order-preserving bijection

$$\begin{array}{c}\text{quotients of } \mathbb{T} \\ \Vert\! \\ \text{Grothendieck topologies on } \mathscr{C} \text{ which contain } J\end{array}$$

This is relevant for instance in connection with the calculation of classifying toposes of quotients of theories which are classified by a presheaf topos.

Two notable cases

This result can be applied in particular in the following two cases:

(1) $(\mathscr{C}, J)$ is the syntactic site $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$

(2)
- $\mathbb{T}$ is a theory of presheaf type,
- $\mathscr{C}$ is the opposite of its category f.p.$\mathbb{T}$-mod(**Set**) of finitely presentable models, and
- J is the trivial topology on it.

In the first case, we obtain an order-preserving bijective correspondence between the quotients of $\mathbb{T}$ and the Grothendieck topologies on $\mathscr{C}_{\mathbb{T}}$ which contain $J_{\mathbb{T}}$.

In the second case, we obtain an order-preserving bijective correspondence between the quotients of $\mathbb{T}$ and the Grothendieck topologies on f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$.

In both cases, these correspondences can be naturally interpreted as proof-theoretic equivalences between the classical proof system of geometric logic over $\mathbb{T}$ and new proof systems for sieves whose inference rules correspond to the axioms of Grothendieck topologies.

Quotients of a theory of presheaf type I

The Grothendieck topology J on f.p.$\mathbb{T}$-mod($\mathbf{Set}$)$^{\text{op}}$ associated with a quotient $\mathbb{T}'$ of a theory of presheaf type $\mathbb{T}$ can be explicitly described as follows.

- By using the fact that every geometric formula over Σ can be $J_{\mathbb{T}}$-covered in $\mathscr{C}_{\mathbb{T}}$ by $\mathbb{T}$-irreducible formulae, one can show that every geometric sequent over Σ is provably equivalent in $\mathbb{T}$ to a collection of sequents σ of the form $(\phi \vdash_{\vec{x}} \bigvee\limits_{i \in I} (\exists \vec{y_i})\theta_i)$ where, for each $i \in I$, $[\theta_i] : \{\vec{y_i} \,.\, \psi_i\} \to \{\vec{x} \,.\, \phi\}$ is an arrow in $\mathscr{C}_{\mathbb{T}}$ and $\phi(\vec{x})$, $\psi(\vec{y_i})$ are geometric formulae over Σ presenting respectively $\mathbb{T}$-models $M_{\{\vec{x}\,.\,\phi\}}$ and $M_{\{\vec{y_i}\,.\,\psi_i\}}$.
- To such a sequent σ, we can associate the cosieve S_σ on $M_{\{\vec{x}\,.\,\phi\}}$ in f.p.$\mathbb{T}$-mod($\mathbf{Set}$) generated by the arrows s_i defined as follows. For each $i \in I$, $[[\theta_i]]_{M_{\{\vec{y_i}\,.\,\psi_i\}}}$ is the graph of a morphism $[[\vec{y_i} \,.\, \psi_i]]_{M_{\{\vec{y_i}\,.\,\psi_i\}}} \to [[\vec{x} \,.\, \phi]]_{M_{\{\vec{y_i}\,.\,\psi_i\}}}$; then the image of the generators of $M_{\{\vec{y_i}\,.\,\psi_i\}}$ via this morphism is an element of $[[\vec{x} \,.\, \phi]]_{M_{\{\vec{y_i}\,.\,\psi_i\}}}$ and this in turn determines, by definition of $M_{\{\vec{x}\,.\,\phi\}}$, a unique arrow $s_i : M_{\{\vec{x}\,.\,\phi\}} \to M_{\{\vec{y_i}\,.\,\psi_i\}}$ in $\mathbb{T}$-mod($\mathbf{Set}$).
- Conversely, by the equivalence f.p.$\mathbb{T}$-mod($\mathbf{Set}$)$^{\text{op}} \simeq \mathscr{C}_{\mathbb{T}}^{\text{irr}}$, every sieve in f.p.$\mathbb{T}$-mod($\mathbf{Set}$)$^{\text{op}}$ is of the form S_σ for such a sequent σ.

Quotients of a theory of presheaf type II

The Grothendieck topology J on f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{\text{op}}$ associated with a quotient $\mathbb{T}'$ of $\mathbb{T}$ is generated by the sieves S_σ, where σ varies among the sequents of the required form which are equivalent to the axioms of $\mathbb{T}'$.

The equivalence

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

of classifying toposes for $\mathbb{T}$ restricts to an equivalence

$$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$$

of classifying toposes for $\mathbb{T}'$.
In particular, for any σ of the above form, σ is provable in $\mathbb{T}'$ if and only if S_σ belongs to J.

These equivalences are useful in that they allow us to study (the proof theory of) geometric theories through the associated Grothendieck topologies: the condition of provability of a sequent in a geometric theory gets transformed in the requirement for a sieve (or a family of sieves) to belong to a certain Grothendieck topology, something which is often much easier to investigate.

The Zariski topos

Let Σ be the one-sorted signature for the theory $\mathbb{T}$ of commutative rings with unit i.e. the signature consisting of two binary function symbols $+$ and $\cdot$, one unary function symbol $-$ and two constants 0 and 1.
The coherent theory of local rings is obtained from $\mathbb{T}$ by adding the sequents

$$((0 = 1) \vdash_{[]} \bot)$$

and

$$((\exists z)((x + y) \cdot z = 1) \vdash_{x,y} ((\exists z)(x \cdot z = 1) \vee (\exists z)(y \cdot z = 1))),$$

Definition

The Zariski topos is the topos $\mathbf{Sh}(\mathbf{Rng}_{f.g.}^{\mathrm{op}}, J)$ of sheaves on the opposite of the category $\mathbf{Rng}_{f.g.}$ of finitely generated rings with respect to the topology J on $\mathbf{Rng}_{f.g.}^{\mathrm{op}}$ defined by: given a cosieve S in $\mathbf{Rng}_{f.g.}$ on an object A, $S \in J(A)$ if and only if S contains a finite family $\{\xi_i : A \to A[{s_i}^{-1}] \mid 1 \leq i \leq n\}$ of canonical inclusions $\xi_i : A \to A[{s_i}^{-1}]$ in $\mathbf{Rng}_{f.g.}$ where $\{s_1, \ldots, s_n\}$ is any set of elements of A which is not contained in any proper ideal of A.

Fact

The (coherent) theory of local rings is classified by the Zariski topos.

The classifying topos for integral domains

The theory of integral domains is the theory obtained from the theory of commutative rings with unit by adding the axioms

$$((0 = 1) \vdash_{[]} \bot)$$

$$((x \cdot y = 0) \ \vdash_{x,y} \ ((x = 0) \vee (y = 0))) .$$

Fact

The theory of integral domains is classified by the topos $\mathbf{Sh}(\mathbf{Rng}_{f.g.}^{op}, J)$ *of sheaves on the opposite of the category* $\mathbf{Rng}_{f.g.}$ *of finitely generated rings with respect to the topology* J *on* $\mathbf{Rng}_{f.g.}^{op}$ *defined by: given a cosieve* S *in* $\mathbf{Rng}_{f.g.}$ *on an object* A*,* $S \in J_2(A)$ *if and only if*

- *either* A *is the zero ring and* S *is the empty sieve on it or*
- S *contains a non-empty finite family* $\{\pi_{a_i} : A \to A/(a_i) \mid 1 \leq i \leq n\}$ *of canonical projections* $\pi_{a_i} : A \to A/(a_i)$ *in* $\mathbf{Rng}_{f.g.}$ *where* $\{a_1, \ldots, a_n\}$ *is any set of elements of* A *such that* $a_1 \cdot \ldots \cdot a_n = 0$*.*

Topos-theoretic Fraïssé theorem

The following result, which generalizes Fraïssé's theorem in classical model theory, arises from a triple 'bridge'.

Definition

A set-based model M of a geometric theory $\mathbb{T}$ is said to be homogeneous if for any arrow $y : c \to M$ in $\mathbb{T}$-mod(**Set**) and any arrow f in f.p.$\mathbb{T}$-mod(**Set**) there exists an arrow u in $\mathbb{T}$-mod(**Set**) such that $u \circ f = y$:

$$\begin{tikzcd} c \arrow[r, "y"] \arrow[d, "f"'] & M \\ d \arrow[ur, dashed, "u"'] & \end{tikzcd}$$

Theorem

Let $\mathbb{T}$ be a theory of presheaf type such that the category f.p.$\mathbb{T}$-mod(**Set**) *is non-empty and has AP and JEP. Then the theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models is complete and atomic.*

Topos-theoretic Fraïssé theorem

Atomic topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
Atomic site i.e.
AP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Atomicity of $\mathbb{T}'$

Two-valued topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
JEP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Completeness of $\mathbb{T}'$

Point of

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
homogeneous $\mathbb{T}$-model in **Set**

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
$\mathbb{T}'$-model in **Set**

For further reading

O. Caramello.
Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'
Oxford University Press, 2017.

P. T. Johnstone.
Sketches of an Elephant: a topos theory compendium, vols. 1 and 2
Oxford University Press, 2002.