Topos Theory

Lectures 15-18

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Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism. The natural notion of morphism of geometric morphisms if that of geometric transformation.

Definition

(i) Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of a pair of functors $f_* : \mathcal{E} \to \mathcal{F}$ (the direct image of f) and $f^* : \mathcal{F} \to \mathcal{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathcal{E} \to \mathcal{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

- Grothendieck toposes and geometric morphisms between them form a category, denoted by $\mathfrak{BTop}$.
- Given two toposes $\mathcal{E}$ and $\mathcal{F}$, geometric morphisms from $\mathcal{E}$ to $\mathcal{F}$ and geometric transformations between them form a category, denoted by $\mathbf{Geom}(\mathcal{E}, \mathcal{F})$.

Examples of geometric morphisms

- A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The direct image $\mathbf{Sh}(f)_*$ sends a sheaf $F \in Ob(\mathbf{Sh}(X))$ to the sheaf $\mathbf{Sh}(f)_*(F)$ defined by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for any open subset V of Y. The inverse image $\mathbf{Sh}(f)^*$ acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.
- Every Grothendieck topos $\mathcal{E}$ has a unique geometric morphism $\mathcal{E} \to \mathbf{Set}$. The direct image is the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$, sending an object $e \in \mathcal{E}$ to the set $\mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, e)$, while the inverse image functor $\Delta : \mathbf{Set} \to \mathcal{E}$ sends a set S to the coproduct $\coprod_{s \in S} 1_{\mathcal{E}}$.
- For any site $(\mathcal{C}, J)$, the pair of functors formed by the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and the associated sheaf functor $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ yields a geometric morphism $i : \mathbf{Sh}(\mathcal{C}, J) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

Slice toposes

The notion of Grothendieck topos is stable with respect to the slice construction:

Proposition

(i) *For any Grothendieck topos $\mathcal{E}$ and any object P of $\mathcal{E}$, the slice category $\mathcal{E}/P$ is also a Grothendieck topos; more precisely, if $\mathcal{E} = \mathbf{Sh}(\mathcal{C}, J)$ then $\mathcal{E}/P \simeq \mathbf{Sh}(\int P, J_P)$, where J_P is the Grothendieck topology on $\int P$ whose covering sieves are precisely the sieves whose image under the canonical projection functor $\pi_P : \int P \to \mathcal{C}$ is J-covering.*

(ii) *For any Grothendieck topos $\mathcal{E}$ and any morphism $f : P \to Q$ in $\mathcal{E}$, the pullback functor $f^* : \mathcal{E}/Q \to \mathcal{E}/P$ has both a left adjoint (namely, the functor Σ_f given by composition with f) and a right adjoint π_f. It is therefore the inverse image of a geometric morphism $\mathcal{E}/P \to \mathcal{E}/Q$.*

The notion of locale

To better understand the relationship between topological spaces and the associated toposes, it is convenient to introduce the notion of a locale.

Definition

- A frame is a complete lattice A satisfying the infinite distributive law
$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$
- A frame homomorphism $h : A \to B$ is a mapping preserving finite meets and arbitrary joins.
- We write **Frm** for the category of frames and frame homomorphisms.

Fact

A poset is a frame if and only if it is a complete Heyting algebra.

Note that we have a functor $\mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$ which sends a topological space X to its lattice $\mathcal{O}(X)$ of open sets and a continuous function $f : X \to Y$ to the function $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$ sending an open subset V of Y to the open subset $f^{-}(V)$ of X. This motivates the following

Definition

The category **Loc** of locales is the dual $\mathbf{Frm}^{\mathrm{op}}$ of the category of frames (a locale is an object of the category **Loc**).

Pointless topology

Pointless topology is an attempt to do Topology without making reference to the points of topological spaces but rather entirely in terms of their open subsets and of the inclusion relation between them. For example, notions such as connectedness or compactness of a topological space can be entirely reformulated as properties of its lattice of open subsets:

- A space X is connected if and only if for any $a, b \in \mathcal{O}(X)$ such that $a \wedge b = 0$, $a \vee b = 1$ implies either $a = 1$ or $b = 1$;
- A space X is compact if and only if whenever $1 = \bigvee_{i \in I} a_i$ in $\mathcal{O}(X)$, there exist a finite subset $I' \subseteq I$ such that $1 = \bigvee_{i \in I'} a_i$.

Pointless topology thus provides tools for working with locales *as they were* lattices of open subsets of a topological space (even though not all of them are of this form). On the other hand, a locale, being a complete Heyting algebra, can also be studied by using an algebraic or logical intuition.

The dual nature of the concept of locale

This interplay of topological and logical aspects in the theory of locales is very interesting and fruitful; in fact, important 'topological' properties of locales translate into natural logical properties, via the identification of locales with complete Heyting algebras:

Example

Locales	Complete Heyting algebras
Extremally disconnected locales	Complete De Morgan algebras
Almost discrete locales	Complete Boolean algebras

The adjunction between locales and spaces

- For any topological space X, the lattice $\mathcal{O}(X)$ of its open sets is a locale.
- Conversely, with any locale F one can associate a topological space X_F, whose points are the frame homomorphism $F \to \{0,1\}$ and whose open sets are the subsets of frame homomorphisms $F \to \{0,1\}$ which send a given element $f \in F$ to 1.
- In fact, the assignments

$$X \to \mathcal{O}(X)$$

and

$$F \to X_F$$

lift to an adjunction $\mathcal{O} \dashv X_{\_}$ between the category **Top** of topological spaces and continuous maps and the category $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$ of locales.
- The topological spaces X such that the unit $\eta_X : X \to X_{\mathcal{O}(X)}$ is a homeomorphism are precisely the sober spaces, while the locales F such that the counit $\mathcal{O}(X_F) \to F$ is an isomorphism are the spatial locales.
- The adjunction thus restricts to an equivalence between the full subcategories on the sober spaces and on the spatial locales.

Sheaves on a locale

Definition

Given a locale L, the topos $\mathbf{Sh}(L)$ of sheaves on L is defined as $\mathbf{Sh}(L, J_L)$, where J_L is the Grothendieck topology on L (regarded as a poset category) given by:

$$\{a_i \mid i \in I\} \in J_L(a) \text{ if and only if } \bigvee_{i \in I} a_i = a\,.$$

Theorem

- *For any locale L, there is a Heyting algebra isomorphism $L \cong \mathrm{Sub}_{\mathbf{Sh}(L)}(1_{\mathbf{Sh}(L)})$.*
- *The assignment $L \to \mathbf{Sh}(L)$ is the object-map of a full and faithful (pseudo-)functor from the category **Loc** of locales to the category $\mathfrak{BTop}$ of Grothendieck toposes.*

The assignment $L \to \mathbf{Sh}(L)$ indeed brings (pointless) Topology into the world of Grothendieck toposes; in fact, important topological properties of locales can be expressed as topos-theoretic invariants (i.e. properties of toposes which are stable under categorical equivalence) of the corresponding toposes of sheaves. These invariants can in turn be used to give definitions of topological properties for Grothendieck toposes.

A general hom-tensor adjunction I

Theorem

Let $\mathcal{C}$ be a small category, $\mathcal{E}$ be a locally small cocomplete category and $A : \mathcal{C} \to \mathcal{E}$ a functor. Then we have an adjunction

$$L_A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \rightleftarrows \mathcal{E} : R_A$$

where the right adjoint $R_A : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is defined for each $e \in Ob(\mathcal{E})$ and $c \in Ob(\mathcal{C})$ by:

$$R_A(e)(c) = \text{Hom}_{\mathcal{E}}(A(c), e)$$

and the left adjoint $L_A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}$ is defined by

$$L_A(P) = \text{colim}(A \circ \pi_P),$$

where π_P is the canonical projection functor $\int P \to \mathcal{C}$ from the category of elements $\int P$ of P to $\mathcal{C}$.

A general hom-tensor adjunction II

Remarks

- *The functor L_A can be considered as a generalized tensor product, since, by the construction of colimits in terms of coproducts and coequalizers, we have the following coequalizer diagram:*

$$\coprod_{\substack{c\in\mathcal{C}, p\in P(c)\\ u:c'\to c}} A(c') \overset{\theta}{\underset{\tau}{\rightrightarrows}} \coprod_{c\in\mathcal{C},\, p\in P(c)} A(c) \xrightarrow{\phi} L_A(P),$$

where

$$\theta(c, p, u, x) = (c', P(u)(p), x)$$

and

$$\tau(c, p, u, x) = (c, p, A(u)(x))\ .$$

For this reason, we shall also denote L_A by

$$-\otimes_{\mathcal{C}} A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}\ .$$

- *We can rewrite the above coequalizer as follows:*

$$\coprod_{c,c'\in\mathcal{C}} P(c)\times \mathrm{Hom}_{\mathcal{C}}(c', c)\times A(c') \overset{\theta}{\underset{\tau}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} P(c)\times A(c) \xrightarrow{\phi} P\otimes_{\mathcal{C}} A\ .$$

From this we see that this definition is symmetric in P and A, that is

$$P\otimes_{\mathcal{C}} A \cong A\otimes_{\mathcal{C}^{\text{op}}} P\ .$$

A couple of corollaries

Corollary

Every presheaf is a colimit of representables. More precisely, for any presheaf $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$, we have

$$P \cong \text{colim}(y_{\mathcal{C}} \circ \pi_P),$$

where $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is a Yoneda embedding and π_P is the canonical projection $\int P \to \mathcal{C}$.

Corollary

For any small category $\mathcal{C}$, the topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is the free cocompletion of $\mathcal{C}$ (via the Yoneda embedding $y_{\mathcal{C}}$); that is, any functor $A : \mathcal{C} \to \mathcal{E}$ to a cocomplete category $\mathcal{E}$ extends, uniquely up to isomorphism, to a colimit-preserving functor $[\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}$ along $y_{\mathcal{C}}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "A"] \arrow[d, "y_{\mathcal{C}}"'] & \mathcal{E} \\ {[\mathcal{C}^{\text{op}}, \mathbf{Set}]} \arrow[ur, dashed] & \end{tikzcd}$$

Separating sets of objects

Definition

A separating set of objects for a Grothendieck topos $\mathcal{E}$ is a set C of objects of $\mathcal{E}$ such that for any object A of $\mathcal{E}$, the collection of arrows from objects in C to A is epimorphic.

Proposition

For any site $(\mathcal{C}, J)$, the collection of objects of the form $l_J(c)$ (for $c \in \mathcal{C}$), where

$$l_J : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$$

is the composite of the Yoneda embedding $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ with the associated sheaf functor $a_J : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$, is a separating set of objects for the topos $\mathbf{Sh}(\mathcal{C}, J)$.

The following theorem, which we shall prove below, provides a sort of converse to this proposition.

Theorem

For any set of objects $\mathcal{C}$ of $\mathcal{E}$ which is separating, we have an equivalence

$$\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}^{\text{can}}|_{\mathcal{C}})$$

where $J_{\mathcal{E}}^{\text{can}}|_{\mathcal{C}}$ is the Grothendieck topology induced on $\mathcal{C}$ (regarded as a full subcategory of $\mathcal{E}$).

Geometric morphisms as flat functors I

Definition

- A functor $A : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a locally small topos $\mathcal{E}$ with small colimits is said to be flat if the functor $- \otimes_{\mathcal{C}} A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}$ preserves finite limits.
- The full subcategory of $[\mathcal{C}, \mathcal{E}]$ on the flat functors will be denoted by $\mathbf{Flat}(\mathcal{C}, \mathcal{E})$.

Theorem

Let $\mathcal{C}$ be a small category and $\mathcal{E}$ be a Grothendieck topos. Then we have an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, [\mathcal{C}^{\text{op}}, \mathbf{Set}]) \simeq \mathbf{Flat}(\mathcal{C}, \mathcal{E})$$

(natural in $\mathcal{E}$), which sends

- *a flat functor $A : \mathcal{C} \to \mathcal{E}$ to the geometric morphism $\mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ determined by the functors R_A and $- \otimes_{\mathcal{C}} A$, and*
- *a geometric morphism $f : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ to the flat functor given by the composite $f^* \circ y_{\mathcal{C}}$ of $f^* : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}$ with the Yoneda embedding $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$.*

Flat = filtering

Definition

A functor $F : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a Grothendieck topos $\mathcal{E}$ is said to be filtering if it satisfies the following conditions:

(a) For any object E of $\mathcal{E}$ there exist an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an object b_i of $\mathcal{C}$ and a generalized element $E_i \to F(b_i)$ in $\mathcal{E}$.

(b) For any two objects c and d in $\mathcal{C}$ and any generalized element $\langle x, y \rangle : E \to F(c) \times F(d)$ in $\mathcal{E}$ there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an object b_i of $\mathcal{C}$ with arrows $u_i : b_i \to c$ and $v_i : b_i \to d$ in $\mathcal{C}$ and a generalized element $z_i : E_i \to F(b_i)$ in $\mathcal{E}$ such that $\langle F(u_i), F(v_i) \rangle \circ z_i = \langle x, y \rangle \circ e_i$ for all $i \in I$.

(c) For any two parallel arrows $u, v : d \to c$ in $\mathcal{C}$ and any generalized element $x : E \to F(d)$ in $\mathcal{E}$ for which $F(u) \circ x = F(v) \circ x$, there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an arrow $w_i : b_i \to d$ and a generalized element $y_i : E_i \to F(b_i)$ such that $u \circ w_i = v \circ w_i$ and $F(w_i) \circ y_i = x \circ e_i$ for all $i \in I$.

Theorem

A functor $F : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a Grothendieck topos $\mathcal{E}$ is flat if and only if it is filtering.

Remarks

- *For any small category $\mathcal{C}$, a functor $P : \mathcal{C} \to$ **Set** is filtering if and only if its category of elements $\int P$ is a filtered category (equivalently, if it is a filtered colimit of representables).*
- *For any small cartesian category $\mathcal{C}$, a functor $\mathcal{C} \to \mathcal{E}$ is flat if and only if it preserves finite limits.*

Geometric morphisms to $\mathbf{Sh}(\mathcal{C}, J)$ I

We shall characterize the geometric morphisms to $\mathbf{Sh}(\mathcal{C}, J)$ by identifying them with the geometric morphisms to $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ which factor through the canonical geometric morphism $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$.

Definition

A subtopos of a topos $\mathcal{E}$ is a geometric morphism of the form

$$\mathbf{sh}_j(\mathcal{E}) \hookrightarrow \mathcal{E}$$

for a local operator j on $\mathcal{E}$.

Recall that we proved that the subtoposes of a topos $\mathbf{Sh}(\mathcal{C}, J)$ are in bijective correspondence with the Grothendieck topologies on $\mathcal{C}$ which contain J.

Proposition

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism and j a local operator on $\mathcal{E}$. Then the following conditions are equivalent:

(i) *f factors through $\mathbf{sh}_j(\mathcal{E}) \hookrightarrow \mathcal{E}$*

(ii) *f_* sends all objects of $\mathcal{F}$ to j-sheaves in $\mathcal{E}$.*

(iii) *f^* maps c_j-dense subobjects in $\mathcal{E}$ to isomorphisms in $\mathcal{F}$.*

Corollary

Let $f : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ be a geometric morphism. The the following conditions are equivalent:

(i) *f factors through $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$.*

(ii) *$f^* \circ y_{\mathcal{C}}$ maps all J-covering sieves to epimorphic families in $\mathcal{E}$.*

Geometric morphisms to $\mathbf{Sh}(\mathcal{C}, J)$ I

Definition

If $(\mathcal{C}, J)$ is a site, a functor $F : \mathcal{C} \to \mathcal{E}$ to a Grothendieck topos is said to be *J-continuous* if it sends J-covering sieves to epimorphic families.

The full subcategory of $\mathbf{Flat}(\mathcal{C}, \mathcal{E})$ on the J-continuous flat functors will be denoted by $\mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$.

Theorem

For any site $(\mathcal{C}, J)$ and Grothendieck topos $\mathcal{E}$, the above-mentioned equivalence between geometric morphisms and flat functors restricts to an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$$

natural in $\mathcal{E}$.

Sketch of proof.

Appeal to the previous theorem

- identifying the geometric morphisms $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ with the geometric morphisms $\mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ which factor through the canonical geometric inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, and
- using the characterization of such morphisms as the geometric morphisms $f : \mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ such that the composite $f^* \circ y$ of the inverse image functor f^* of f with the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ sends J-covering sieves to epimorphic families in $\mathcal{E}$.

Further properties of Grothendieck toposes I

Let $\mathcal{E}$ be a Grothendieck topos. Then

(i) $\mathcal{E}$ is locally small.

(ii) $\mathcal{E}$ is well-powered.

(iii) Every equivalence relation in $\mathcal{E}$ is *effective* (in the sense that it is the kernel pair of its own coequalizer), and every epimorphism is the coequalizer of its kernel pair. In particular, $\mathcal{E}$ is co-well-powered.

(iv) Finite limits commute with filtered colimits.

(v) The 'change of base' functor $f^* : \mathcal{E}/A \to \mathcal{E}/B$ along any arrow $f : B \to A$ in $\mathcal{E}$ preserves arbitrary colimits.

(vi) Coproducts are disjoint.

(vii) $\mathcal{E}$ has a separating set of objects (if $\mathcal{E} = \mathbf{Sh}(\mathcal{C}, J)$ then the object of the form $l_J(c)$ for $c \in \mathcal{C}$, where l_J is the composite of the Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ with the associated sheaf functor $a_J : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, form a separating set).

(viii) $\mathcal{E}$ has a coseparating set of objects (if $\mathcal{E} = \mathbf{Sh}(\mathcal{C}, J)$ then the objects of the form $\Omega^{l_J(c)}$ for $c \in \mathcal{C}$ form a coseparating set).

Further properties of Grothendieck toposes II

Let us recall the following result:

Theorem

Let $\mathcal{C}$ be a locally small and complete (resp. cocomplete) category. If $\mathcal{C}$ is well-powered (resp. co-well-powered) and admits a coseparating (resp. separating) set of objects then it has an initial (resp. a terminal) object.

This can be notably applied to categories of the form $\int F$ where F is a contravariant functor $\mathcal{C}^{\text{op}} \to \mathbf{Set}$ defined on a locally small, complete and well-powered category $\mathcal{C}$ with a coseparating set of objects which sends colimits in $\mathcal{C}$ to limits to deduce that F is representable (and the dual result). Also, this latter result can be used to show the Special Adjoint Functor Theorem: any functor $F : \mathcal{C} \to \mathcal{D}$ defined on a locally small, complete (resp. cocomplete) and well-powered category $\mathcal{C}$ with a coseparating (resp. separating) set of objects with values in a locally small category $\mathcal{D}$ preserves all small limits (resp. colimits) if and only if it has a left (resp. right) adjoint.

From the above properties of Grothendieck toposes it thus follows that:

- A covariant (resp. contravariant) functor on a Grothendieck topos with values in **Set** is representable if and only if it preserves arbitrary (small) limits (resp. sends colimits to limits).
- A functor between Grothendieck toposes admits a left adjoint (resp. a right adjoint) if and only if it preserves arbitrary (small) limits (resp. colimits).

The canonical topology on a Grothendieck topos

Recall that the canonical topology $J_{\mathcal{C}}^{\text{can}}$ on a category $\mathcal{C}$ is the largest one for which all representables in $\mathcal{C}$ are sheaves. Its covering sieves are precisely the universally effective-epimorphic ones.

Proposition

Let $\mathcal{E}$ be a Grothendieck topos. Then a sieve is $J_{\mathcal{E}}^{\text{can}}$-covering if and only if it contains a small epimorphic family.

Sketch of proof.

- Every small epimorphic family generates a sieve which is universally effective-epimorphic, and every sieve which contains a universally effective-epimorphic one is universally effective-epimorphic.
- On the other hand, $\mathcal{E}$ being well-powered, every epimorphic sieve contains a small epimorphic family (notice that if two arrows f, g with common codomain A have the same image then for any arrows $\xi, \chi : A \to B$, we have $\xi \circ f = \chi \circ f$ if and only if $\xi \circ g = \chi \circ g$).

Giraud's theorem

It is possible to axiomatically characterize Grothendieck toposes, thanks to a theorem due to J. Giraud:

Theorem

The following conditions are equivalent:

(i) *$\mathcal{E}$ is a Grothendieck topos.*

(ii) *$\mathcal{E}$ is a locally small category with finite limits in which colimits exist and are preserved by the change of base functors, coproducts are disjoint, equivalence relations are effective and epimorphisms are the coequalizers of their kernel pairs, and there is a separating set of objects.*

Sketch of proof.

The implication $(i) \Rightarrow (ii)$ was established above. For the converse, we shall prove that if $\mathcal{C}$ is a separating set of objects for $\mathcal{E}$, regarded as a full subcategory of $\mathcal{E}$, then we have an equivalence $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}^{\text{can}}|_{\mathcal{C}})$. This equivalence will be given by the geometric morphism $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}^{\text{can}}|_{\mathcal{C}})$ corresponding to the inclusion functor A of $\mathcal{C}$ into $\mathcal{E}$ (notice that this functor is flat since it is filtering, and it is filtering since $\mathcal{C}$ is separating for $\mathcal{E}$).

Sketch of proof I

In order to prove that the functors $R_A = \mathrm{Hom}_{\mathcal{E}}(A(-), -) : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}^{\mathrm{can}}|_{\mathcal{C}})$ and $L_A : \mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}^{\mathrm{can}}|_{\mathcal{C}}) \to \mathcal{E}$ are quasi-inverses to each other, we verify that

- The counit $\epsilon_e : L_A(R_A(e)) = \varinjlim_{c \to e} c$ is an isomorphism for every $e \in \mathcal{E}$ (this follows from the fact that $\mathcal{C}$ is separating in light of the exactness properties needed to establish the fact that the covering sieves for the canonical topology on a Grothendieck topos are precisely those which contain small epimorphic families).
- The unit $\eta_P : P \to R_A(L_A(P))$ is an isomorphism for every $P \in \mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}^{\mathrm{can}}|_{\mathcal{C}})$. For this, since η_P is clearly an isomorphism for R a representable, it clearly suffices to prove that $R_A : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}^{\mathrm{can}}|_{\mathcal{C}})$ preserves arbitrary colimits.

Sketch of proof II

To prove that R_A preserves colimits, one can resort to the following criterion for a cocone $\{D(i) \to C \mid i \in \mathcal{I}\}$ in $\mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}^{\text{can}}|_{\mathcal{C}})$ on a diagram $D : \mathcal{I} \to \mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}^{\text{can}}|_{\mathcal{C}})$ to be colimiting in terms of the canonical arrow $\xi : \text{colim}_p(D) \to C$ in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ from the colimit $\text{colim}_p(D)$ of D in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$: this is the case if and only if the arrow $a_{J_{\mathcal{E}}^{\text{can}}|_{\mathcal{C}}}(\xi)$ is an isomorphism, that is if and only if in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$

(1) $\text{Im}(\xi)$ is $c_{J_{\mathcal{E}}^{\text{can}}|_{\mathcal{C}}}$-dense, and

(2) the diagonal monomorphism $\text{colim}_p(D) \to K_\xi$ to the domain K_ξ of the kernel pair of ξ is locally surjective.

In order to verify (2), one can in turn use the following characterization of colimits in $\mathcal{E}$: if $E = \text{colim}_{i \in \mathcal{I}}(E_i)$ then, denoting by ξ_i the canonical coproduct arrows $E_i \to \coprod_{i \in \mathcal{I}} E_i$ and by $\pi : \coprod_{i \in \mathcal{I}} E_i \to E$ the canonical arrow, for any arrows $x : F \to E_i$ and $y : F \to E_j$, we have that $\pi \circ \xi_i \circ x = \pi \circ \xi_j \circ y$ if and only if there exists an epimorphic family $\{e_k : F_k \to F \mid k \in K\}$ in $\mathcal{E}$ such that $(\xi_i \circ x \circ e_k, \xi_j \circ y \circ e_k)$ belongs to the equivalence relation on $\text{Hom}_{\mathcal{E}}(F_k, \coprod_{i \in \mathcal{I}} E_i)$ generated by the pairs $(\xi_i \circ a, \xi_j \circ E_u \circ a)$ for an arrow $u : i \to j$ in $\mathcal{I}$ and a generalized element $a : F_k \to E_i$.

Essential geometric morphisms I

Definition

A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is said to be essential if the inverse image functor $f^* : \mathcal{F} \to \mathcal{E}$ has a left adjoint.

Theorem

- *Every functor $f : \mathcal{C} \to \mathcal{D}$ induces an essential geometric morphism*

$$E(f) : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to [\mathcal{D}^{\text{op}}, \mathbf{Set}],$$

  whose inverse image functor f^* is given by composition with f^{op}.

- *In fact, the inverse image f^* is simultaneously a 'tensor product' functor L_{A_1} and a 'hom functor' R_{A_2}, where*

$$A_1 = \mathrm{Hom}_{\mathcal{D}}(f(-), -) : \mathcal{D} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$$

  and

$$A_2 = \mathrm{Hom}_{\mathcal{D}}(-, f(-)) : \mathcal{C} \to [\mathcal{D}^{\text{op}}, \mathbf{Set}],$$

  so it has a left adjoint L_{A_2} and a right adjoint R_{A_1}.

Essential geometric morphisms II

Remarks

- *The direct image of the geometric morphism $E(f)$, namely the right adjoint to f^*, is the right Kan extension $\varprojlim_{f^{op}}$ along f^{op}, given by the following formula:*

$$\varprojlim_{f^{op}}(F)(b) = \varprojlim_{\phi: fa \to b} F(a),$$

where the limit is taken over the opposite of the comma category $(f \downarrow b)$.

- *The left adjoint to f^* is the left Kan extension $\varinjlim_{f^{op}}$ along f^{op}, given by the following formula:*

$$\varinjlim_{f^{op}}(F)(b) = \varinjlim_{\phi: b \to fa} F(a),$$

where the colimit is taken over the opposite of the comma category $(b \downarrow f)$.

Proposition

If $\mathcal{C}$ and $\mathcal{D}$ are Cauchy-complete categories, a geometric morphism $[\mathcal{C}^{op}, \mathbf{Set}] \to [\mathcal{D}^{op}, \mathbf{Set}]$ is of the form $E(f)$ for some functor $f : \mathcal{C} \to \mathcal{D}$ if and only if it is essential; in this case, f can be recovered from $E(f)$ (up to isomorphism) as the restriction to the full subcategories of representables of the left adjoint to the inverse image of $E(f)$.

Morphisms of sites I

By using the characterization of filtering functors with values in a Grothendieck topos as functors which send certain families to epimorphic families and the fact that the image under the associated sheaf functor of a family of natural transformations with common codomain is epimorphic if and only if the family is locally jointly surjective, we obtain the following result:

Corollary

Let $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ be essentially small sites, and $l : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$, $l' : \mathcal{C}' \to \mathbf{Sh}(\mathcal{C}', J')$ be the canonical functors (given by the composite of the relevant Yoneda embedding with the associated sheaf functor). Then, given a functor $F : \mathcal{C} \to \mathcal{C}'$, the following conditions are equivalent:

(i) *A induces a geometric morphism $u : \mathbf{Sh}(\mathcal{C}', J') \to \mathbf{Sh}(\mathcal{C}, J)$ making the following square commutative:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "l"'] & \mathcal{C}' \arrow[d, "l'"] \\ \mathbf{Sh}(\mathcal{C}, J) \arrow[r, "u^*"] & \mathbf{Sh}(\mathcal{C}', J'); \end{tikzcd}$$

(ii) *The functor F is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{C}', J')$ in the sense that it satisfies the following properties:*

(1) *A sends every J-covering family in $\mathcal{C}$ into a J'-covering family in $\mathcal{C}'$.*

(2) *Every object c' of $\mathcal{C}'$ admits a J'-covering family*

$$c'_i \longrightarrow c' \,, \quad i \in I \,,$$

by objects c'_i of $\mathcal{C}'$ which have morphisms

$$c'_i \longrightarrow F(c_i)$$

to the images under A of objects c_i of $\mathcal{C}$.

Morphisms of sites II

(3) For any objects c_1, c_2 of $\mathcal{C}$ and any pair of morphisms of $\mathcal{C}'$

$$f'_1 : c' \longrightarrow F(c_1)\,, \quad f'_2 : c' \longrightarrow F(c_2)\,,$$

there exists a J'-covering family

$$g'_i : c'_i \longrightarrow c'\,, \quad i \in I\,,$$

and a family of pairs of morphisms of $\mathcal{C}$

$$f^i_1 : b_i \longrightarrow c_1\,, \quad f^i_2 : b_i \to c_2\,, \quad i \in I\,,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : c'_i \longrightarrow F(b_i)\,, \quad i \in I\,,$$

making the following squares commutative:

$$\begin{tikzcd} c'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & c' \arrow[d, "f'_1"] \\ F(b_i) \arrow[r, "F(f^i_1)"] & F(c_1) \end{tikzcd} \qquad \begin{tikzcd} c'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & c' \arrow[d, "f'_2"] \\ F(b_i) \arrow[r, "F(f^i_2)"] & F(c_2) \end{tikzcd}$$

Morphisms of sites III

(4) For any pair of arrows $f_1, f_2 : c \rightrightarrows d$ of $\mathcal{C}$ and any arrow of $\mathcal{C}'$

$$f' : b' \longrightarrow F(c)$$

satisfying

$$F(f_1) \circ f' = F(f_2) \circ f' \, ,$$

there exist a J'-covering family

$$g'_i : b'_i \longrightarrow b' \, , \quad i \in I \, ,$$

and a family of morphisms of $\mathcal{C}$

$$h_i : b_i \longrightarrow c \, , \quad i \in I \, ,$$

satisfying

$$f_1 \circ h_i = f_2 \circ h_i \, , \quad \forall \, i \in I \, ,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : b'_i \longrightarrow F(b_i) \, , \quad i \in I \, ,$$

making commutative the following squares:

$$\begin{tikzcd} b'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & b' \arrow[d, "f'"] \\ F(b_i) \arrow[r, "F(h_i)"] & F(c) \end{tikzcd}$$

Morphisms of sites IV

If F is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$, we denote by $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ the geometric morphism which it induces.

Remarks

- *If $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ are cartesian sites (that is, $\mathcal{C}$ and $\mathcal{D}$ are cartesian categories) then a functor $\mathcal{C} \to \mathcal{D}$ which is cartesian and sends J-covering sieves to K-covering sieves is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$.*
- *If J and K are subcanonical then a geometric morphism $g : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ is of the form $\mathbf{Sh}(f)$ for some f if and only if the inverse image functor g^* sends representables to representables; if this is the case then f is isomorphic to the restriction of g^* to the full subcategories of representables.*

Corollary

The assignment $L \to \mathbf{Sh}(L)$ from locales to Grothendieck toposes is a full and faithful 2-functor.

The Comparison Lemma I

Definition

Let $\mathcal{D}$ be a full subcategory of a small category $\mathcal{C}$, and let J be a Grothendieck topology on $\mathcal{C}$. Then $\mathcal{D}$ is said to be J-dense if for every object $c \in \mathcal{C}$ there exists a sieve $S \in J(c)$ generated by a family of arrows whose domains lie in $\mathcal{D}$.

Theorem (The Comparison Lemma)

Let $(\mathcal{C}, J)$ be a site and $\mathcal{D}$ be a J-dense subcategory of $\mathcal{C}$. Then the sieves in $\mathcal{D}$ of the form $R \cap arr(\mathcal{D})$ for a J-covering sieve R in $\mathcal{C}$ form a Grothendieck topology $J|_{\mathcal{D}}$ on $\mathcal{D}$, called the induced topology, and, denoted by $i : \mathcal{D} \to \mathcal{C}$ the canonical inclusion functor, the geometric morphism

$$E(i) : [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}],$$

restricts to an equivalence of categories

$$E(i)| : \mathbf{Sh}(\mathcal{D}, J|_{\mathcal{D}}) \simeq \mathbf{Sh}(\mathcal{C}, J) .$$

The Comparison Lemma II

Corollary

- *Let B be a basis of a frame L, i.e. a subset $B \subset L$ such that every element in L can be written as a join of elements in B; then we have an equivalence of categories*

$$\mathbf{Sh}(L) \simeq \mathbf{Sh}(B, J^L|_B),$$

where J^L is the canonical topology on L.

- *Let $\mathcal{C}$ be a preorder and J be a subcanonical topology on $\mathcal{C}$. Then we have an equivalence of categories*

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(Id_J(\mathcal{C})),$$

where $Id_J(\mathcal{C})$ is the frame of J-ideals on $\mathcal{C}$ (regarded as a locale). In fact, this result holds also without the subcanonicity assumption.

Comorphisms of sites

Definition

A comorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ is a functor $\mathcal{D} \to \mathcal{C}$ which is cover-reflecting (in the sense that for any $d \in \mathcal{D}$ and any J-covering sieve S on $\pi(d)$ there is a K-covering sieve R on d such that $\pi(R) \subseteq S$).

Proposition

Every comorphism of sites $\pi : \mathcal{D} \to \mathcal{C}$ induces a flat and J-continuous functor $A_\pi : \mathcal{C} \to \mathbf{Sh}(\mathcal{D}, K)$ given by

$$A_\pi(c) = a_K(\mathrm{Hom}_{\mathcal{C}}(\pi(-), c))$$

and hence a geometric morphism

$$f : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

with inverse image $f^*(F) \cong a_K(F \circ \pi)$ for any J-sheaf F on $\mathcal{C}$.

Points of toposes

Definition

By a point of a topos $\mathcal{E}$, we mean a geometric morphism $\mathbf{Set} \to \mathcal{E}$.

Examples

- For any site $(\mathcal{C}, J)$, the points of the topos $\mathbf{Sh}(\mathcal{C}, J)$ correspond precisely to the J-continuous flat functors $\mathcal{C} \to \mathbf{Set}$;
- For any locale L, the points of the topos $\mathbf{Sh}(L)$ correspond precisely to the frame homomorphisms $L \to \{0, 1\}$;
- For any small category $\mathcal{C}$ and any object c of $\mathcal{C}$, we have a point $e_c : \mathbf{Set} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ of the topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$, whose inverse image is the evaluation functor at c.

Fact

Any set of points P of a Grothendieck topos $\mathcal{E}$ indexed by a set X via a function $\xi : X \to P$ can be identified with a geometric morphism $\tilde{\xi} : [X, \mathbf{Set}] \to \mathcal{E}$.

Separating sets of points

Definition

- Let $\mathcal{E}$ be a topos and P be a collection of points of $\mathcal{E}$ indexed by a set X via a function $\xi : X \to P$. We say that P is separating for $\mathcal{E}$ if the points in P are jointly surjective, i.e. if the inverse image functors of the geometric morphisms in P jointly reflect isomorphisms (equivalently, if the geometric morphism $\tilde{\xi} : [X, \mathbf{Set}] \to \mathcal{E}$ is surjective, that is its inverse image reflects isomorphisms).
- A topos is said to have enough points if the collection of all the points of $\mathcal{E}$ is separating for $\mathcal{E}$.

Fact

A Grothendieck topos has enough points if and only if there exists a set of points of $\mathcal{E}$ which is separating for $\mathcal{E}$.

The subterminal topology

The following notion provides a way for endowing a given set of points of a topos with a natural topology.

Definition

Let $\xi : X \to \mathrm{P}$ be an indexing of a set P of points of a Grothendieck topos $\mathcal{E}$ by a set X. We define the subterminal topology $\tau_{\xi}^{\mathcal{E}}$ as the image of the function $\phi_{\mathcal{E}} : \mathrm{Sub}_{\mathcal{E}}(1) \to \mathscr{P}(X)$ given by

$$\phi_{\mathcal{E}}(u) = \{x \in X \mid \xi(x)^*(u) \cong 1_{\mathbf{Set}}\} .$$

We denote the space X endowed with the topology $\tau_{\xi}^{\mathcal{E}}$ by $X_{\tau_{\xi}^{\mathcal{E}}}$.

The interest of this notion lies in its level of generality, as well as in its formulation as a topos-theoretic invariant admitting a 'natural behaviour' with respect to sites.

Fact

If P is a separating set of points for $\mathcal{E}$ then the frame $\mathcal{O}(X_{\tau_{\xi}^{\mathcal{E}}})$ of open sets of $X_{\tau_{\xi}^{\mathcal{E}}}$ is isomorphic to $\mathrm{Sub}_{\mathcal{E}}(1)$ (via $\phi_{\mathcal{E}}$).

Categories of toposes paired with points

The construction of the subterminal topology can be made functorial.

Definition

The category $\mathfrak{Top}_p$ toposes paired with points has as objects the pairs $(\mathcal{E}, \xi)$, where $\mathcal{E}$ is a Grothendieck topos and $\xi : X \to P$ is an indexing of a set of points P of $\mathcal{E}$, and whose arrows $(\mathcal{E}, \xi) \to (\mathcal{F}, \xi')$, where $\xi : X \to P$ and $\xi' : Y \to Q$, are the pairs (f, l) where $f : \mathcal{E} \to \mathcal{F}$ is a geometric morphism and $l : X \to Y$ is a function such that the diagram

$$\begin{tikzcd} {[X, \mathbf{Set}]} \arrow[r, "E(l)"] \arrow[d, "\tilde{\xi}"] & {[Y, \mathbf{Set}]} \arrow[d, "\tilde{\xi}'"] \\ \mathcal{E} \arrow[r, "f"] & \mathcal{F} \end{tikzcd}$$

commutes (up to isomorphism).

Theorem

We have a functor $\mathfrak{Top}_p \to \mathbf{Top}$ (where $\mathbf{Top}$ is the category of topological spaces) sending an object $(\mathcal{E}, \xi)$ of $\mathfrak{Top}_p$ to the space $X_{\tau_\xi^{\mathcal{E}}}$ and an arrow $(f, l) : (\mathcal{E}, \xi) \to (\mathcal{F}, \xi')$ in $\mathfrak{Top}_p$ to the continuous function $l : X_{\tau_\xi^{\mathcal{E}}} \to X_{\tau_{\xi'}^{\mathcal{F}}}$.

Examples of subterminal topologies I

Definition

Let $(\mathcal{C}, \leq)$ be a preorder category. A J-prime filter on $\mathcal{C}$ is a subset $F \subseteq ob(\mathcal{C})$ such that F is non-empty, $a \in F$ implies $b \in F$ whenever $a \leq b$, for any $a, b \in F$ there exists $c \in F$ such that $c \leq a$ and $c \leq b$, and for any J-covering sieve $\{a_i \to a \mid i \in I\}$ in $\mathcal{C}$ if $a \in F$ then there exists $i \in I$ such that $a_i \in F$.

Theorem

Let $\mathcal{C}$ be a preorder and J be a Grothendieck topology on it. Then the space $X_{\tau_{\mathbf{Sh}(\mathcal{C},J)}}$ has as set of points the collection $\mathcal{F}^J_{\mathcal{C}}$ of the J-prime filters on $\mathcal{C}$ and as open sets the sets the form

$$\mathcal{F}_I = \{F \in \mathcal{F}^J_{\mathcal{C}} \mid F \cap I \neq \emptyset\},$$

where I ranges among the J-ideals on $\mathcal{C}$. In particular, a sub-basis for this topology is given by the sets

$$\mathcal{F}_c = \{F \in \mathcal{F}^J_{\mathcal{C}} \mid c \in F\},$$

where c varies among the objects of $\mathcal{C}$.

Examples of subterminal topologies II

- The Alexandrov topology ($\mathcal{E} = [\mathcal{P}, \mathbf{Set}]$, where $\mathcal{P}$ is a preorder and ξ is the indexing of the set of points of $\mathcal{E}$ corresponding to the elements of $\mathcal{P}$)
- The Stone topology for distributive lattices ($\mathcal{E} = \mathbf{Sh}(\mathcal{D}, J_{coh})$ and ξ is an indexing of the set of all the points of $\mathcal{E}$, where $\mathcal{D}$ is a distributive lattice and J_{coh} is the coherent topology on it)
- A topology for meet-semilattices ($\mathcal{E} = [\mathcal{M}^{\text{op}}, \mathbf{Set}]$ and ξ is an indexing of the set of all the points of $\mathcal{E}$, where $\mathcal{M}$ is a meet-semilattice)
- The space of points of a locale ($\mathcal{E} = \mathbf{Sh}(L)$ for a locale L and ξ is an indexing of the set of all the points of $\mathcal{E}$)
- A logical topology ($\mathcal{E} = \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the classiying topos of a geometric theory $\mathbb{T}$ and ξ is any indexing of the set of all the points of $\mathcal{E}$ i.e. models of $\mathbb{T}$)
- The Zariski topology

...

The topos $\mathbf{Cont}(G)$

Proposition

Given a topological group G, the category $\mathbf{Cont}(G)$ of continuous actions of G on discrete sets and G-equivariant maps between them is a Grothendieck topos, equivalent to the topos $\mathbf{Sh}(\mathbf{Cont}_t(G), J_{\mathrm{at}})$ of sheaves with respect to the atomic topology J_{at} on the full subcategory $\mathbf{Cont}_t(G)$ on the non-empty transitive actions.

Remark

In fact, the same result holds by replacing $\mathbf{Cont}_t(G)$ with the full subcategory $\mathbf{Cont}_{\mathcal{U}}(G)$ of $\mathbf{Cont}(G)$ on the actions of the form G/U where $U \in \mathcal{C}$ for a cofinal system $\mathcal{U}$ of open subgroups of G (that is, a collection of subgroups such that every open subgroup contains a member of $\mathcal{U}$).

Recall that classical topological Galois theory provides, given a Galois extension $F \subseteq L$, a bijective correspondence between the intermediate field extensions (resp. finite field extensions) $F \subseteq K \subseteq L$ and the closed (resp. open) subgroups of the Galois group $Aut_F(L)$. This admits the following categorical reformulation: the functor $K \to \mathrm{Hom}(K, L)$ defines an equivalence of categories

$$(\mathcal{L}_F^L)^{\mathrm{op}} \simeq \mathbf{Cont}_t(Aut_F(L)),$$

which extends to an equivalence of toposes

$$\mathbf{Sh}({\mathcal{L}_F^L}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_F(L))\ .$$

A natural question thus arises: can we characterize the categories $\mathcal{C}$ whose dual is equivalent to (or fully embeddable into) the category of (non-empty transitive) actions of a topological automorphism group?

Topos-theoretic Galois theory

Theorem

Let $\mathcal{C}$ be a small category satisfying AP and JEP, and let u be a $\mathcal{C}$-universal et $\mathcal{C}$-ultrahomogeneous object of the ind-completion Ind-$\mathcal{C}$ *of $\mathcal{C}$. Then there is an equivalence of toposes*

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u)),$$

where $Aut(u)$ is endowed with the topology in which a basis of open neighbourhoods of the identity is given by the subgroups of the form $I_\chi = \{\alpha \in Aut(u) \mid \alpha \circ \chi = \chi\}$ for $\chi : c \to u$ an arrow in Ind-$\mathcal{C}$ *from an object c of $\mathcal{C}$.*
This equivalence is induced by the functor

$$F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}(Aut(u))$$

which sends any object c of $\mathcal{C}$ to the set $\mathrm{Hom}_{\text{Ind-}\mathcal{C}}(c, u)$ (endowed with the obvious action of $Aut(u)$) and any arrow $f : c \to d$ in $\mathcal{C}$ to the $Aut(u)$-equivariant map

$$- \circ f : \mathrm{Hom}_{\text{Ind-}\mathcal{C}}(d, u) \to \mathrm{Hom}_{\text{Ind-}\mathcal{C}}(c, u)\ .$$

Restricting to sites

The following result arises from two 'bridges', respectively obtained by considering the invariant notions of atom and of arrow between atoms.

$$\begin{tikzcd}[column sep=small] & \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u)) \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \mathcal{C}^{\mathrm{op}} & & \mathbf{Cont}_t(Aut(u)) \end{tikzcd}$$

Theorem

Under the hypotheses of the last theorem, the functor F is full and faithful if and only if every arrow of $\mathcal{C}$ is a strict monomorphism, and it is an equivalence on the full subcategory $\mathbf{Cont}_t(Aut(u))$ of $\mathbf{Cont}(Aut(u))$ on the non-empty transitive actions if $\mathcal{C}$ is moreover atomically complete.

Applications

- A natural source of ultrahomogenenous and universal objects is provided by Fraïssé's construction in Model Theory and its categorical generalizations.

  For instance, if the category $\mathcal{C}$ is countable and all its arrows are monomorphisms then there always exists a $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous object in Ind-$\mathcal{C}$.

- This theorem generalizes Grothendieck's theory of Galois categories (which corresponds to the particular case when the fundamental group is profinite).

- It can be applied for generating Galois-type theories in different fields of Mathematics, which do *not* fit in the formalism of Galois categories.

Examples

Natural categories with monic arrows: $\mathcal{C}$ equal to the category of

- Finite sets and injections
- Finite graphs and embeddings
- Finite groups and injective homomorphisms

Natural categories with epic arrows: $\mathcal{C}^{\text{op}}$ equal to the category of

- Finite sets and surjections
- Finite groups and surjective homomorphisms
- Finite graphs and homomorphisms which are surjective both at the level of vertices and at the level of edges.

Categories of 'imaginaries'

- If a category $\mathcal{C}$ satisfies the first but not the second condition of our last theorem, our topos-theoretic approach gives us a fully explicit way to complete it, by means of the addition of 'imaginaries', so that also the second condition gets satisfied.
- This is the case for instance for the categories considered above; so we get notions of 'imaginary finite set', 'imaginary finite group' etc.
- The objects of the atomic completion admit an explicit description in terms of equivalence relations in the topos $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$ on objects coming from the site $\mathcal{C}^{\text{op}}$.
- In a joint work with L. Lafforgue we give an alternative concrete description of the atomic completion.

For further reading

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