# Topos Theory

## Lecture 1: Overview of the course

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# The "unifying notion" of topos

*"It is the topos theme which is this "bed" or "deep river" where come to be married geometry and algebra, topology and arithmetic, mathematical logic and category theory, the world of the "continuous" and that of "discontinuous" or discrete structures. It is what I have conceived of most broad to perceive with finesse, by the same language rich of geometric resonances, an "essence" which is common to situations most distant from each other coming from one region or another of the vast universe of mathematical things".*

*A. Grothendieck*

Topos theory can be regarded as a unifying subject in Mathematics, with great relevance as a framework for systematically investigating the relationships between different mathematical theories and studying them by means of a multiplicity of different points of view. Its methods are transversal to the various fields and complementary to their own specialized techniques. In spite of their generality, the topos-theoretic techniques are liable to generate insights which would be hardly attainable otherwise and to establish deep connections that allow effective transfers of knowledge between different contexts.

# The multifaceted nature of toposes

The role of toposes as unifying spaces is intimately tied to their multifaceted nature; for instance, a topos can be seen as:

- a generalized space
- a mathematical universe
- a theory (modulo a certain notion of equivalence).

The course will start by presenting the relevant categorical and logical background and extensively treat these different perspectives on the notion of topos, with the final aim of providing the student with tools and methods to study mathematical theories from a topos-theoretic perspective, extract new information about correspondences, dualities or equivalences, and establish new and fruitful connections between distinct fields.

# Prerequisites and exam

## Prerequisites:

A Bachelor's degree in Mathematics (or equivalent mathematical maturity). Although some familiarity with the language of category theory and the basics of first-order logic would be desirable, no previous knowledge of these subjects is required. Indeed, the course will present all the relevant preliminaries as they are needed.

## Exam:

The student will be able to choose between two alternative exam modes:

- Solutions (prepared at home) to exercises assigned by the lecturer and oral presentation on a suitable topic (chosen in agreement with the lecturer) not treated in the course.
- Taking two partial written exams, one administered half way through the course and the other immediately after the end of the course.

# A bit of history

- Toposes were originally introduced by Alexander Grothendieck in the early 1960s, in order to provide a mathematical underpinning for the 'exotic' cohomology theories needed in algebraic geometry. Every topological space gives rise to a topos and every topos in Grothendieck's sense can be considered as a 'generalized space'.
- At the end of the same decade, William Lawvere and Myles Tierney realized that the concept of Grothendieck topos also yielded an abstract notion of mathematical universe within which one could carry out most familiar set-theoretic constructions, but which also, thanks to the inherent 'flexibility' of the notion of topos, could be profitably exploited to construct 'new mathematical worlds' having particular properties.
- A few years later, the theory of classifying toposes added a further fundamental viewpoint to the above-mentioned ones: a topos can be seen not only as a generalized space or as a mathematical universe, but also as a suitable kind of first-order theory (considered up to a general notion of equivalence of theories).

# Toposes as generalized spaces

- The notion of topos was introduced in the early sixties by A. Grothendieck with the aim of bringing a topological or geometric intuition also in areas where actual topological spaces do not occur.
- Grothendieck realized that many important properties of topological spaces $X$ can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.
- He then defined toposes as more general categories of sheaves of sets, by replacing the topological space $X$ by a pair $(\mathscr{C}, J)$ consisting of a (small) category $\mathscr{C}$ and a ‘generalized notion of covering’ $J$ on it, and taking sheaves (in a generalized sense) over the pair:

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, squiggly] & \mathbf{Sh}(X) \arrow[d, squiggly] \\ (\mathscr{C}, J) \arrow[r, dashed] & \mathbf{Sh}(\mathscr{C}, J) \end{tikzcd}$$

# Topos-theoretic invariants

## Definition

By a topos-theoretic invariant we mean a property of (or a construction involving) toposes which is stable under categorical equivalence.

- The notion of a geometric morphism of toposes has notably allowed to build general comology theories starting from the categories of internal abelian groups or modules in toposes. The cohomological invariants have had a tremendous impact on the development of modern Algebraic Geometry and beyond.
- On the other hand, homotopy-theoretic invariants such as the fundamental group and the higher homotopy groups can be defined as invarants of toposes.
- Still, these are by no means the only invariants that one can consider on toposes: indeed, there are infinitely many invariants of toposes (of algebraic, logical, geometric or whatever nature), the notion of identity for toposes being simply categorical equivalence.

# Toposes as mathematical universes

A decade later, W. Lawvere and M. Tierney discovered that a topos could not only be seen as a generalized space, but also as a mathematical universe in which one can do mathematics similarly to how one does it in the classical context of sets (with the only important exception that one must argue constructively).

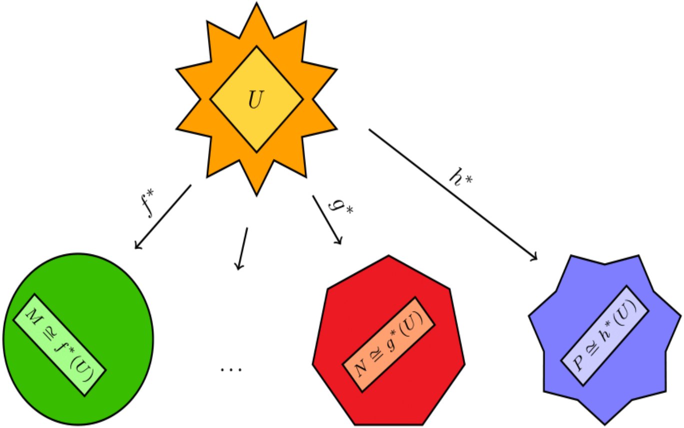
Amongst other things, this discovery made it possible to:

- Exploit the inherent ‘flexibility’ of the notion of topos to construct ‘new mathematical worlds’ having particular properties.
- Consider models of any kind of (first-order) mathematical theory not just in the classical set-theoretic setting, but inside every topos, and hence ‘relativise’ Mathematics.

# Classifying toposes

- It was realized in the seventies (thanks to the work of several people, notably including W. Lawvere, A. Joyal, G. Reyes and M. Makkai) that to any mathematical theory $\mathbb{T}$ (of a general specified form) one can canonically associate a topos $\mathscr{E}_{\mathbb{T}}$, called the classifying topos of the theory, which represents its 'semantical core', a sort of 'DNA' of the theory.
- Conversely, every Grothendieck topos is the classifying topos of some theory.
- Hence, a Grothendieck topos can be seen as a *canonical representative* of equivalence classes of geometric theories modulo Morita-equivalence.

# Classifying toposes




Classifying topos

# The idea of *bridge*

- We can think of a *bridge* object connecting two objects $a$ and $b$ as an object $u$ which can be 'built' from any of the two objects and admits two different representations $f(a)$ and $g(b)$ related by some kind of equivalence $\simeq$, the former being in terms of the object $a$ and the latter in terms of the object $b$:

$$\begin{tikzcd} & f(a) \simeq u \simeq g(b) \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ a & & b \end{tikzcd}$$

- The transfer of information arises from the process of 'translating' invariant (with respect to $\simeq$) properties of (resp. constructions on) the 'bridge object' $u$ into properties of (resp. constructions on) the two objects $a$ and $b$ by using the two different representations $f(a)$ and $g(b)$ of the bridge object.

# *Toposes as bridges I*

- Toposes can effectively act as 'bridge objects' across Morita-equivalent theories (or, more generally, between theories to which one can attach equivalent toposes).
- The notion of Morita-equivalence is ubiquitous in Mathematics; indeed, it formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods'.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted in terms of Morita-equivalences.
- Any two theories which are bi-interpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.

# *Toposes as bridges* II

- The fact that different mathematical theories have equivalent classifying toposes translates, at the technical level, into the existence of different representations of one topos.
- So Topos Theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.
- We can think of a topos as embodying the 'common features' of mathematical theories which are Morita-equivalent to each other.
- The underlying intuition is that a given mathematical property can manifest itself in different forms in the context of mathematical theories which have a common 'semantical core' but a different linguistic presentation.

# *Toposes as bridges* III

- The essential features of Morita-equivalences are all ‘hidden’ inside toposes, and can be revealed by using their different representations.
- For example, imagine starting with a property, say geometrical, of a certain mathematical object, and being able to find a topos and a property of it which is (logically) equivalent to the given property of our object; then one can use e.g. a logical representation for the topos to convert this property of the topos into a logical statement of a certain kind; as a result, one obtains the equivalence of our initial geometrical property with a logical one.

# *Toposes as bridges* IV

- If a property of a mathematical object is formulated as a topos-theoretic invariant on some topos then the expression of it in terms of the different theories classified by that topos is determined to a great extent by the technical relationship between the topos and the different representations of it.

- Topos-theoretic invariants can thus be used to transfer information from one theory to another:

$$\begin{tikzcd} & \mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'} \arrow[dr, dashed] & \\ \mathbb{T} \arrow[ur, dashed] & & \mathbb{T}' \end{tikzcd}$$

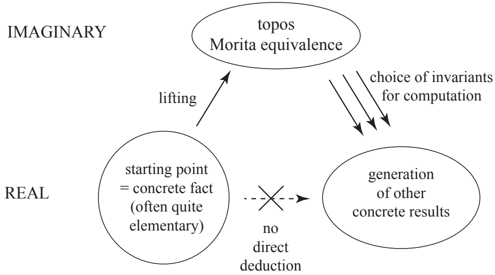
- The transfer of information takes place by expressing a given invariant in terms of the different representations of the topos.

# *Toposes as bridges V*

- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories.
- The fact that topos-theoretic invariants specialize to important properties or constructions of natural mathematical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications into Mathematics as a whole.

# The duality between 'real' and 'imaginary'

- The passage from a site (or a theory) to the associated topos can be regarded as a sort of 'completion' by the addition of 'imaginaries' (in the model-theoretic sense), which materializes the potential contained in the site (or theory).
- The duality between the (relatively) unstructured world of presentations of theories and the maximally strucured world of toposes is of great relevance as, on the one hand, the 'simplicity' and concreteness of theories or sites makes it easy to manipulate them, while, on the other hand, computations are much easier in the 'imaginary' world of toposes thanks to their very rich internal structure and the fact that invariants live at this level.



# A mathematical morphogenesis

- The essential ambiguity given by the fact that any topos is associated in general with an infinite number of theories or different sites allows to study the relations between different theories, and hence the theories themselves, by using toposes as ’bridges’ between these different presentations.
- Every topos-theoretic invariant generates a veritable mathematical morphogenesis resuting from its expression in terms of different representations of toposes, which gives rise in general to connections between properties or notions that are completely different and apparently unrelated from each other

# A new way of doing Mathematics I

- Taking these methodologies seriously entails a deep paradigmatic shift as they suggest a new way of doing Mathematics which is 'upside-down' in that the mathematical exploration is guided by Morita-equivalences and topos-theoretic invariants rather than by 'concrete' considerations.
- These are the objects at the 'center of the stage', and it is from them that one proceeds to extract concrete information on the theories that one wishes to study.
- The 'working mathematician' would greatly profit attempting to formulate his or her properties of interest in terms of topos-theoretic invariants, and derive equivalent versions of them by using alternative representations of the relevant toposes.

# A new way of doing Mathematics II

- There is an strong element of automatism in these techniques; by means of them, one can generate a great number of new mathematical results without really making any creative effort.
- The results generated in this way are in general non-trivial; in some cases they can be rather 'weird' according to the usual mathematical standards (although they might still be quite deep) but, with a careful choice of Morita-equivalences and invariants, one can obtain interesting and natural mathematical results.
- In fact, a lot of information that is not visible with the usual 'glasses' is revealed by the application of this machinery.
- On the other hand, the range of applicability of these methods is extremely broad within Mathematics, by the very generality of the notion of topos.