Cohomology of toposes

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Chapter III:

De Rham cohomology and its key properties: De Rham's theorem, Poincaré duality, Lefschetz fixed points formula

The function-theoretic definition of manifolds

Definition:

(i) A ringed space is a topological space X endowed with a sheaf of rings

$$\mathcal{O}_X : \underset{\substack{\| \\ \textit{open subset of } X}}{U} \longmapsto \underset{\substack{\| \\ \textit{ring of ``coordinate''} \\ \textit{functions on } U}}{\mathcal{O}_X(U)} .$$

(ii) A morphism of ringed space

$$(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

consists in a continuous map

$$f : X \longrightarrow Y$$

and a morphism of sheaves of rings

$$\begin{array}{ccc} \mathcal{O}_Y & \longrightarrow & f_* \mathcal{O}_X \,, \\ \underset{\substack{\| \\ \textit{coordinate function on} \\ \textit{an open subset } V \subset Y}}{(\varphi \in \mathcal{O}_Y(V))} & \longmapsto & \underset{\substack{\| \\ \textit{pull-back of } \varphi \\ \textit{on } f^{-1}(V)}}{(f_* \varphi \in \mathcal{O}_X(f^{-1}(V)))} \end{array}$$

or, equivalently,

$$f^* \mathcal{O}_Y \longrightarrow \mathcal{O}_X \,.$$

Remarks:

(i) Ringed spaces make up a category.

(ii) If $(X, \mathcal{O}_X)$ is a ringed space,
an $\mathcal{O}_X$-Module is a sheaf of abelian groups $\mathcal{M}$ endowed with a sheaf morphism

$$\mathcal{O}_X \times \mathcal{M} \longrightarrow \mathcal{M}$$

such that

$$\begin{cases} \bullet \quad \text{for any open subset } U \subset X,\ \mathcal{M}(U) \text{ is a module on the ring } \mathcal{O}_X(U), \\ \bullet \quad \text{for any open subsets } U_1 \subset U_2 \subset X, \\ \quad\ \ \text{the restriction map } \mathcal{M}(U_2) \to \mathcal{M}(U_1) \\ \quad\ \ \text{is a morphism of } \mathcal{O}_X(U_2)\text{-modules} \\ \quad\ \ \text{via the restriction map } \mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1). \end{cases}$$

$\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ make up a (linear) category.

Definition:

(i) A differential [resp. analytic] manifold is a ringed space $(X, \mathcal{O}_X)$ such that any point $x \in X$ has a open neighborhood

$$(U, \mathcal{O}_U = \text{restriction of } \mathcal{O}_X \text{ to } U)$$

which is isomorphic to an open subset

$$U' \text{ of some } \mathbb{R}^n \quad [\text{resp. } \mathbb{C}^n]$$

endowed with the sheaf

$$\mathcal{O}_{U'} : (V' \subset U') \longmapsto \underset{\substack{\text{ring of } C^\infty \ [\text{resp. holomorphic}] \\ \text{functions } V' \to \mathbb{R} \ [\text{resp. } V' \to \mathbb{C}]}}{\underset{\|}{\mathcal{O}_{U'}(V')}} .$$

(ii) A morphism of differential [resp. analytic] manifolds

$$(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is a morphism of ringed spaces

$$(X \xrightarrow{f} Y,\ \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X)$$

such that, locally on X and Y,

$$\begin{cases} \bullet & f \text{ identifies with a } C^\infty \text{ [resp. analytic] map from an open subset } U' \\ & \text{of some } \mathbb{R}^n \text{ [resp. } \mathbb{C}^n\text{] to an open subset } V' \text{ of some } \mathbb{R}^m \text{ [resp. } \mathbb{C}^m\text{],} \\ \bullet & \mathcal{O}_Y \to f_* \mathcal{O}_X \text{ is defined by composition with } f. \end{cases}$$

Remarks:

(i) Differential [resp. analytic] manifolds make up a category.
This category has arbitrary sums and finite products.
The contravariant functor

$$(X, \mathcal{O}_X) \longmapsto \mathcal{O}_X(X)$$

is representable in this category by the object

$$\mathbb{R} \qquad \text{[resp. } \mathbb{C}\text{]}.$$

(ii) The category of schemes is defined in the same way:

- A scheme is a ringed space $(X, \mathcal{O}_X)$ such that any point $x \in X$ has an open neighborhood $(U, \mathcal{O}_U)$ which is isomorphic to an “affine scheme”.
- A morphism of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces

$$(X \xrightarrow{f} Y,\ \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X)$$

which locally identifies with a “morphism of affine schemes”.

The dual functorial definition of manifolds

Proposition:

Let $\mathcal{V} =$ category of differential [resp. analytic] manifolds,
$\mathcal{C} =$ full subcategory of open subsets of the $\mathbb{R}^n$'s [resp. $\mathbb{C}^n$'s] and C^∞ [resp. holomorphic] maps,
$J =$ topology on $\mathcal{V}$ or $\mathcal{C}$.

Then the functor

$$\begin{array}{ccl} \mathcal{V} & \longrightarrow & [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] = \widehat{\mathcal{C}} \\ X & \longmapsto & \mathrm{Hom}(\bullet, X) = [U \mapsto \underset{\substack{\| \\ \textit{set of parametrisations} \\ U \to X \textit{ of } X \textit{ by } U}}{\mathrm{Hom}(U, X)}] \end{array}$$

factorises through the full subcategory

$$\widehat{\mathcal{C}}_J$$

of J-sheaves on $\mathcal{C}$ and is fully faithful.

It is an equivalence to the full subcategory of $\widehat{\mathcal{C}}_J$ on J-sheaves $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ such that there exists a globally epimorphic family of morphisms

$$\mathrm{Hom}(\bullet, U_i) \longrightarrow F\,, \qquad i \in I\,,$$

(corresponding to elements of the $F(U_i)$'s)
which are "open" in the sense that,
for any object U of $\mathcal{C}$ and any morphism

$$\mathrm{Hom}(\bullet, U) \longrightarrow F\,,$$

the fiber products in $\widehat{\mathcal{C}}_J$

$$\mathrm{Hom}(\bullet, U_i) \times_F \mathrm{Hom}(\bullet, U)$$

are representable by open subsets of the U_i's.

Remarks:

(i) The same proposition can be written if

$\mathcal{V} =$ category of schemes,
$\mathcal{C} =$ full subcategory of affine schemes.

(ii) The sets

$$\mathrm{Hom}(U, X)$$

can also be denoted $X(U)$.

Their elements can be called

the points of X defined on U,
or the points of X with values in U,
or the U-points of X.

Vector bundles

Definition: Let $(X, \mathcal{O}_X) =$ ringed space.
An $\mathcal{O}_X$-Module $\mathcal{M}$ is called locally free (of some rank r)
if it is locally isomorphic to the $\mathcal{O}_X$-Module $\mathcal{O}_X^r$.

Remarks:

(i) Locally free $\mathcal{O}_X$-Modules make up a full subcategory which has arbitrary finite products.

(ii) If $\mathcal{O}_X$ is a sheaf of commutative rings, this category also has

- tensor products $\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2$ which represent the functors
  $\mathcal{M} \mapsto$ set of bilinear sheaf morphisms $\mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}$
  and are constructed as the sheafifications of the presheaves

$$U \longmapsto \mathcal{M}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{M}_2(U)\,,$$

- alternate powers $\Lambda^k \mathcal{M}$ which represent the functors

  $\mathcal{N} \mapsto$ set of k-linear maps $\mathcal{M} \times \cdots \times \mathcal{M} \to \mathcal{N}$ which are 0 on the diagonals,

  and are constructed as the sheafifications of the presheaves

$$U \longmapsto \Lambda^k \mathcal{M}(U)\,,$$

- “exponentials” or “inner hom” $\mathcal{H}om(\mathcal{M}_1, \mathcal{M}_2)$ which represent the functors

$$\mathcal{M} \longmapsto \mathrm{Hom}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}_1, \mathcal{M}_2)$$

  and are constructed as the sheaves

$$U \longmapsto \mathrm{Hom}(\mathcal{M}_{1|U}, \mathcal{M}_{2|U})\,,$$

- in particular, a contravariant duality functor

$$\mathcal{M} \longmapsto \mathcal{M}^\vee = \mathcal{H}om(\mathcal{M}, \mathcal{O}_X)$$

  which is an involution in the sense that
  $\mathcal{M}^{\vee\vee}$ identifies with $\mathcal{M}$ for any $\mathcal{M}$.

If $\mathcal{M}$ is locally free of rank 1, it is called “invertible” as

$$\mathcal{M} \otimes \mathcal{M}^\vee \quad \text{identifies with} \quad \mathcal{O}_X\,.$$

Lemma:
Let $X =$ differential [resp. analytic] manifold.
Any locally free $\mathcal{O}_X$-Module $\mathcal{M}$ is representable by a (unique up to unique isomorphism) manifold M endowed with a morphism $p : M \to X$
in the sense that, for any morphism $i : U \to X$,

$$\{U \xrightarrow{s} M \mid p \circ s = i\}$$

identifies with

$$(i^* \mathcal{M} \otimes_{i^* \mathcal{O}_X} \mathcal{O}_U)(U) \, .$$

Remark:
The same lemma would apply in any subcategory $\mathcal{C}$ of the category of ringed spaces such that

- $\mathcal{C}$ has finite products,
- the contravariant functor $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$ is representable in $\mathcal{C}$,
- any ringed space which is locally isomorphic to objects of $\mathcal{C}$ is an object of $\mathcal{C}$,
- any morphism of ringed spaces which locally identifies with morphisms of $\mathcal{C}$ is a morphism of $\mathcal{C}$.

Definition:
A vector bundle (of some rank r) on a differential [resp. analytic] manifold X is a manifold M endowed with a morphism $M \to X$ which represents a (rank r) locally free $\mathcal{O}_X$-Module $\mathcal{M}$.

A morphism of vector bundles is a morphism of the associated locally free $\mathcal{O}_X$-Modules.

Remarks:
So, the category of vector bundles on some differential [resp. analytic] manifold X has

- finite products $M_1 \times \cdots \times M_n$,
- tensor products $M_1 \otimes M_2$,
- alternate powers $\Lambda^k M$,
- "inner hom" $Hom(M_1, M_2)$,
- in particular, a duality contravariant functor $M \to M^\vee$,
- a notion of "invertible" vector bundle, which means vector bundle of rank 1.

Cotangent modules and tangent bundles

Definition: Let $A \to B =$ morphism of commutative rings.
A derivation of B, relatively to A, with values in a B-module M, is a map

$$B \xrightarrow{\mathrm{d}} M$$

such that

- it is compatible with addition

$$\mathrm{d}(b_1 + b_2) = \mathrm{d}b_1 + \mathrm{d}b_2\,, \qquad \forall\, b_1, b_2 \in B\,,$$

- it verifies the Leibniz rule

$$\mathrm{d}(b_1 \cdot b_2) = b_1 \cdot \mathrm{d}b_2 + b_2 \cdot \mathrm{d}b_1\,,$$

- the composite

$$A \longrightarrow B \xrightarrow{\mathrm{d}} M \quad \text{is } 0\,.$$

Remark:
Derivations of B, relatively to A, with values in M make up a B-module

$$\mathrm{Der}_{B/A}(M)\,.$$

Proposition: Let $A \to B =$ morphism of commutative rings.
Then the covariant functor

$$\begin{array}{rcl} Mod_B & \longrightarrow & Mod_B \,, \\ M & \longmapsto & \mathrm{Der}_{B/A}(M) \end{array}$$

is representable by a B-module $\Omega_{B/A}$ endowed with a canonical derivation

$$\mathrm{d} : B \longrightarrow \Omega_{B/A}$$

called the "module of differentials" of B.

Sketch of proof: First consider the free B-module

$$\bigoplus_{b \in B} B \cdot \mathrm{d}b$$

generated by basis elements denoted $\mathrm{d}b$, $b \in B$.
Then define $\Omega_{B/A}$ as the quotient of this free module by the submodule generated by the elements

$$\begin{array}{ll} \mathrm{d}(b_1 + b_2) - \mathrm{d}b_1 - \mathrm{d}b_2 \,, & b_1, b_1 \in B \,, \\ \mathrm{d}a \,, & a \in A \,, \\ \mathrm{d}(b_1 b_2) - b_1 \cdot \mathrm{d}b_2 - b_2 \cdot \mathrm{d}b_1 \,, & b_1, b_1 \in B \,. \end{array}$$

Definition:

For any $k \geq 1$,
the B-module of degree k differentials
is defined as the k-th exterior power

$$\Omega^k_{B/A} = \Lambda^k \Omega_{B/A} .$$

Remark:

Any element of $\Omega^k_{B/A}$
is a sum of elements of the form

$$b \cdot \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_k \qquad \text{with} \qquad b, b_1, \ldots, b_k \in B .$$

Lemma: For any $k \geq 1$, there is a well-defined A-linear map

$$\begin{array}{rccc} \mathrm{d}: & \Omega^k_{B/A} & \longrightarrow & \Omega^{k+1}_{B/A}\,, \\ & (b \cdot \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_k) & \longmapsto & \mathrm{d}b \wedge \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_k\,. \end{array}$$

Proof: First, there is a well-defined A-linear map

$$\begin{array}{ccc} \Omega_{B/A} \otimes_B \cdots \otimes_B \Omega_{B/A} & \longrightarrow & \Omega^{k+1}_{B/A}\,, \\ (b \cdot \mathrm{d}b_1 \otimes \cdots \otimes \mathrm{d}b_k) & \longmapsto & \mathrm{d}b \wedge \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_k\,. \end{array}$$

Indeed, elements of the form

$$b \cdot \mathrm{d}b_1 \otimes \cdots \otimes \mathrm{d}b_{i-1} \otimes (\mathrm{d}(b_i b_i') - b_i \mathrm{d}b_i' - b_i' \mathrm{d}b_i) \otimes \mathrm{d}b_{i+1} \otimes \cdots \otimes \mathrm{d}b_k$$

are sent to 0 as

$$\begin{array}{rl} & \mathrm{d}(bb_i) \wedge \mathrm{d}b_i' + \mathrm{d}(bb_i') \wedge \mathrm{d}b_i \\ = & b_i \cdot \mathrm{d}b \wedge \mathrm{d}b_i' + b \cdot \mathrm{d}b_i \wedge \mathrm{d}b_i' + b \cdot \mathrm{d}b_i' \wedge \mathrm{d}b_i + b_i' \cdot \mathrm{d}b \wedge \mathrm{d}b_i \\ = & b_i \cdot \mathrm{d}b \wedge \mathrm{d}b_i' + b_i' \cdot \mathrm{d}b \wedge \mathrm{d}b_i \\ = & \mathrm{d}b \wedge \mathrm{d}(b_i b_i')\,. \end{array}$$

This map is alternate. So it factorises as

$$\Omega^k_{B/A} \longrightarrow \Omega^{k+1}_{B/A}\,.$$

Definition:

The (algebraic) De Rham complex is defined as

$$B \xrightarrow{\mathrm{d}} \Omega_{B/A} \xrightarrow{\mathrm{d}} \Omega^2_{B/A} \longrightarrow \cdots \longrightarrow \Omega^k_{B/A} \xrightarrow{\mathrm{d}} \Omega^{k+1}_{B/A} \longrightarrow \cdots$$

Remark:

The relation $\mathrm{d} \circ \mathrm{d} = 0$

comes from the fact that, by definition,

$$\begin{aligned} & \mathrm{d}(\mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_k) \\ = \; & \mathrm{d}(1 \cdot \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_k) \\ = \; & \mathrm{d}1 \wedge \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_k \\ = \; & 0\,. \end{aligned}$$

Theorem:

(i) If $B = A[X_1, \ldots, X_n]$, then $\Omega_{B/A}$ is the free B-module on the basis elements

$$\mathrm{d}X_1, \ldots, \mathrm{d}X_n$$

endowed with the derivative

$$\begin{array}{rcl} B = A[X_1, \ldots, X_n] & \longrightarrow & \bigoplus\limits_{1 \leq i \leq n} B \cdot \mathrm{d}X_i \,, \\ P(X_1, \ldots, X_n) & \longmapsto & \sum\limits_{i=1}^{n} \frac{\partial P}{\partial X_i}(X_1, \ldots, X_n) \cdot \mathrm{d}X_i \,. \end{array}$$

(ii) If $A = \mathbb{R}$ [resp. $\mathbb{C}$],

$B =$ algebra of C^∞ [resp. holomorphic] functions on an open convex subset U of $\mathbb{R}^n$ [resp. $\mathbb{C}^n$],

$m_x =$ maximal ideal of B consisting of functions which vanish at a point $x \in U$.

Then

$$\mathrm{Im}\left[\Omega_{B/A} \longrightarrow \prod_{x \in U} \varprojlim_{n \in \mathbb{N}} (B/m_x^N) \otimes_B \Omega_{B/A}\right]$$

is the free B-module on the basis elements

$$\mathrm{d}x_1, \ldots, \mathrm{d}x_n$$

endowed with the derivative

$$\begin{array}{rcl} B & \longmapsto & \bigoplus\limits_{1 \leq i \leq n} B \cdot \mathrm{d}x_i \\ f(x_1, \ldots, x_n) & \longmapsto & \sum\limits_{1 \leq i \leq n} \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) \cdot \mathrm{d}x_i \,. \end{array}$$

Proof of the theorem:

(i) Any derivation

$$\mathrm{d} : B \longrightarrow M$$

is entirely determined by the images of $X_1, \ldots, X_n$.

Conversely, the map

$$\begin{array}{rcl} B = A[X_1, \ldots, X_n] & \longrightarrow & B^n\,, \\ P & \longmapsto & \left(\frac{\partial P}{\partial X_i}\right)_{1 \leq i \leq n} \end{array}$$

is a derivation.

(ii) The map

$$\begin{array}{rcl} B & \longrightarrow & B^n\,, \\ f & \longmapsto & \left(\frac{\partial f}{\partial x_i}\right)_{1 \leq i \leq n} \end{array}$$

is a derivation.

The proof follows from the following lemma:

Lemma (Taylor's formula):
For U = convex open subset of $\mathbb{R}^n$ [resp. $\mathbb{C}^n$],
$f = C^\infty$ [resp. holomorphic] function on U,
$a = (a_1, \ldots, a_n)$ = point of U,
N = integer ≥ 1,
we can write

$$f(x) - f(a) = P_N(x) + \sum_{k_1+\cdots+k_n=N+1} (x_1-a_1)^{k_1} \cdots (x_n-a_n)^{k_n} \cdot f_{k_1,\ldots,k_n}(x)$$

where P_N is a polynomial of degree $\leq N$
and the functions $f_{k_1,\ldots,k_n}$ are C^∞ [resp. holomorphic].

Sketch of proof: We take

$$P_N(x) = \sum_{k_1+\cdots+k_n \leq N} \frac{\partial^{k_1+\cdots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(a_1, \ldots, a_n) \cdot \frac{(x_1-a_1)^{k_1}}{k_1!} \cdots \frac{(x_n-a_n)^{k_n}}{k_n!}$$

and, for $k_1 + \cdots + k_n = N+1$,

$$f_{k_1,\ldots,k_n}(x) = \frac{N+1}{k_1! \cdots k_n!} \cdot \int_0^1 (1-t)^N \cdot \frac{\partial^{k_1+\cdots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(a + t(x-a)) \, .$$

Definition:

(i) A local ring is a commutative ring A which has a (unique) maximal ideal m_A such that any element of $A - m$ is invertible.

(ii) If A, B are two local rings, a ring homomorphism
$$A \longrightarrow B$$
is called local if it sends m_A to m_B.

Remark:
Any element of $A - m_A$ is sent to an element of $B - m_B$.

Definition:

(i) A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that, for any point x of X, the fiber
$$\mathcal{O}_{X,x} = \varinjlim_{U \ni x} \mathcal{O}_X(U) \quad \text{is a local ring (with maximal ideal } m_x\text{).}$$

(ii) A morphism of locally ringed spaces is a morphism of ringed spaces
$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$
such that, for any point x of X, the induced morphism
$$\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$
is local.

Definition: Let $(X, \mathcal{O}_X) \xrightarrow{f} (S, \mathcal{O}_S) =$ morphism of locally ringed spaces.

The sheaf of differentials on X relatively to S
is the sheafification $\Omega_{X/S}$ of the presheaf on X

$$\underset{\substack{\| \\ \text{open subset of } X}}{U} \longmapsto \mathrm{Im}\left(\Omega_{\mathcal{O}_X(U)/f^*\mathcal{O}_S(U)} \to \prod_{x \in U} \varprojlim_N (\mathcal{O}_{X,x}/m_x^N) \otimes \Omega_{\mathcal{O}_X(U)/f^*\mathcal{O}_S(U)} \right) .$$

It is endowed with a canonical derivation

$$\mathrm{d} : \mathcal{O}_X \longrightarrow \Omega_{X/S} .$$

Remark:
Let's consider the category of $\mathcal{O}_X$-Modules $\mathcal{M}$ such that, for any U,
the morphism

$$\mathcal{M}(U) \longrightarrow \prod_{x \in U} \varprojlim_N (\mathcal{O}_{X,x}/m_x^N) \otimes \mathcal{M}(U)$$

is injective.
Then $\Omega_{X/S}$ belongs to this category and represents the contravariant functor

$$\mathcal{M} \longmapsto \text{ set of sheaf morphisms } \mathcal{O}_X \longrightarrow \mathcal{M}$$

which

$$\begin{cases} \bullet & \text{are compatible with addition,} \\ \bullet & \text{verify the Leibniz rule,} \\ \bullet & \text{are 0 on } f^*\mathcal{O}_S. \end{cases}$$

Definition:

Let $(X, \mathcal{O}_X) \xrightarrow{f} (S, \mathcal{O}_S)$
$= $ morphism of locally ringed spaces.

For any k, the sheaf of degree k differentials on X relatively to S is the sheafification $\Omega^k_{X/S}$ of the presheaf on X

$$U \longmapsto \Lambda^k \Omega_{X/S}(U)\,.$$

The De Rham complex of X relatively to S is the induced sequence

$$\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_{X/S} \xrightarrow{\mathrm{d}} \Omega^2_{X/S} \longrightarrow \cdots \longrightarrow \Omega^k_{X/S} \xrightarrow{\mathrm{d}} \Omega^{k+1}_{X/S} \longrightarrow \cdots$$

verifying in any degree

$$\mathrm{d} \circ \mathrm{d} = 0\,.$$

The previous theorem implies:

Corollary:

Let X be an n-dimensional differential [resp. analytic] manifold, and S be the point manifold.

Then the sheaf $\Omega_X = \Omega_{X/S}$ is locally free of rank n,

and the sheaves $\Omega_X^k = \Omega_{X/S}^k$ are locally free.

Remark:

More generally, the sheaves $\Omega_{X/S}$ and $\Omega_{X/S}^k$ are locally free if $X \to S$ is a morphism of differential [resp. analytic] manifolds which is locally isomorphic to the projection

$$\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^m$$

[resp. $\quad \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m} \longrightarrow \mathbb{C}^m$].

Definition:

Let $X = n$-dimensional differential [resp. analytic] manifold.

Then its cotangent bundle is

$T_X^\vee =$ vector bundle of rank n associated to the locally free $\mathcal{O}_X$-Module Ω_X,

and its tangent bundle is

$T_X =$ dual vector bundle of $T_X^\vee$.

Remark:

Any morphism of differential [resp. analytic] manifolds

$$f : X \longrightarrow Y$$

induces a morphism of $\mathcal{O}_X$-Modules

$$f^* \Omega_Y \longrightarrow \Omega_X$$

which can be seen as a morphism of vector bundles

$$f^* T_Y^\vee = X \times_Y T_Y^\vee \longrightarrow T_X^\vee$$

or, equivalently,

$$T_X \longrightarrow f^* T_Y = X \times_Y T_Y \, .$$

Remark:

For $U =$ open subset of X,

$$\Gamma(U, T_X) = \{\text{sections } s : U \to T_X \text{ of } p : T_X \to X\}$$

identifies with the set of sheaf morphisms

$$\mathrm{d} : \mathcal{O}_U \longrightarrow \mathcal{O}_U \qquad (\text{where } \mathcal{O}_U = \mathcal{O}_{X|U})$$

such that

- d is compatible with addition,
- d verifies the Leibnitz rule,
- d is 0 on constant functions.

The $\mathcal{O}_X(U)$-module structure of $\Gamma(U, T_X)$ is defined by

- addition of operators $\mathcal{O}_U \to \mathcal{O}_U$,
- multiplication of operators by sections in $\mathcal{O}_X(U)$.

Remark:

- In other words, $\Omega_X^\vee = \mathcal{H}om_{O_X}(\Omega_X, O_X)$ can be seen as a subsheaf of the sheaf

$$\mathcal{H}om_+(\mathcal{O}_X, \mathcal{O}_X) : U \longmapsto \left\{ \begin{array}{l} \text{sheaf morphisms } \mathcal{O}_U \to \mathcal{O}_U \\ \text{which are compatible with} \\ \text{addition and multiplication by constants} \end{array} \right\}.$$

- One denotes
  $\mathcal{D}_X =$ "sheaf of linear partial differential operators"
  $=$ smallest subsheaf of $\mathcal{H}om_+(\mathcal{O}_X, \mathcal{O}_X)$
  which is stable by composition and addition
  and contains $\mathcal{O}_X$ and $\Omega_X^\vee$
  $=$ sheaf of elements of $\mathcal{H}om_+(\mathcal{O}_X, \mathcal{O}_X)$
  which are locally finite sums
  of compositions of elements of $\mathcal{O}_X$ and $\Omega_X^\vee$.
- Any system of linear PDE's can be seen as a $\mathcal{D}_X$-Module $\mathcal{M}$.
  The sheaf of its solutions is

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

Definition:

Let $(X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$
$=$ morphism of locally ringed spaces.

The De Rham cohomology modules of X relatively to S are the cohomology modules

$$H^n_{dR}(X/S)$$

of the cochain complex of $\mathcal{O}_S(S)$-modules $\Omega^\bullet_{X/S}(X)$:

$$0 \to \mathcal{O}_X(X) \xrightarrow{\mathrm{d}} \Omega_{X/S}(X) \xrightarrow{\mathrm{d}} \Omega^2_{X/S}(X) \to \cdots \to \Omega^k_{X/S}(X) \xrightarrow{\mathrm{d}} \Omega^{k+1}_{X/S}(X) \to \cdots$$

Remark:

If $S = \{\bullet\}$ and $\mathcal{O}_S(S) = R$,
the R-modules

$$H^n_{dR}(X) = H^n_{dR}(X/S)$$

are called the De Rham cohomology modules of X.

Remark:

- Any commutative triangle of locally ringed spaces

$$\begin{tikzcd} (X, \mathcal{O}_X) \arrow[rr, "f"] \arrow[dr] & & (Y, \mathcal{O}_Y) \arrow[dl] \\ & (S, \mathcal{O}_S) & \end{tikzcd}$$

  induces a morphism of cochain complexes of $\mathcal{O}_S(S)$-modules

$$\Omega^\bullet_{Y/S}(Y) \longrightarrow \Omega^\bullet_{X/S}(X)$$

  and so a sequence of natural morphisms

$$H^n_{dR}(Y/S) \longrightarrow H^n_{dR}(X/S) \, .$$

- In other words, De Rham cohomology relatively to $(S, \mathcal{O}_S)$ makes up a sequence of contravariant functors from the category of locally ringed spaces over $(S, \mathcal{O}_S)$ to the category of $\mathcal{O}_S(S)$-modules.
- In particular, isomorphic locally ringed spaces have isomorphic De Rham cohomology modules.

Lemma ("Poincaré lemma"):
The De Rham cohomology vector spaces of the differential manifolds

$$\mathbb{R}^d$$

are

$$H^n_{dR}(\mathbb{R}^d) = \begin{cases} \mathbb{R} & \text{if} \quad n = 0\,, \\ 0 & \text{if} \quad n \geq 1\,. \end{cases}$$

Remark:

This lemma also applies
to any differential manifold
which is diffeomorphic to $\mathbb{R}^d$,
in particular any open ball of $\mathbb{R}^d$.

Remark:

If X is an analytic manifold isomorphic to $\mathbb{C}^d$ or an open ball of $\mathbb{C}^d$,
we also have

$$H^n_{dR}(X) = \begin{cases} \mathbb{C} & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \geq 1\,. \end{cases}$$

Proof of the Poincaré lemma:

Any element of $\Omega^k(\mathbb{R}^d)$ has the form

$$\sum_{1\leq i_1<\cdots<i_k\leq d} f_{i_1,\ldots,i_k}(x_1,\ldots,x_n)\cdot \mathrm{d}x_{i_1}\wedge\cdots\wedge \mathrm{d}x_{i_k}\,.$$

For any $k\geq 1$, let

$$\begin{aligned} h^k \quad : \quad & \Omega^k(\mathbb{R}^d) \longrightarrow \Omega^{k-1}(\mathbb{R}^d) \\ & \sum f_{i_1,\ldots,i_k}(x_1,\ldots,x_n)\cdot \mathrm{d}x_{i_1}\wedge\cdots\wedge \mathrm{d}x_{i_k} \\ & \longmapsto \sum x_{i_1}\cdot\left(\int_0^1 \mathrm{d}t\cdot f_{i_1,\ldots,i_k}(0,\ldots,0,tx_{i_1},x_{i_1+1},\ldots,x_n)\right)\cdot \mathrm{d}x_{i_2}\wedge\cdots\wedge \mathrm{d}x_{i_k}\,. \end{aligned}$$

Then we have for $\omega = f(x_1,\ldots,x_n)\cdot \mathrm{d}x_{i_1}\wedge\cdots\wedge \mathrm{d}x_{i_k}$

$$\begin{aligned} \mathrm{d}\circ h^k(\omega) \quad &= \quad \sum_{j>i_1} x_{i_1}\cdot\left(\int_0^1 \mathrm{d}t\cdot\frac{\partial f}{\partial x_j}(0,\ldots,0,tx_i,x_{i_1+1},\ldots,x_n)\right)\cdot \mathrm{d}x_j\wedge \mathrm{d}x_{i_2}\wedge\cdots\wedge \mathrm{d}x_{i_k} \\ &+ \quad f_{i_1,\ldots,i_k}(0,\ldots,0,x_{i_1},x_{i_1+1},\ldots,x_n)\cdot \mathrm{d}x_{i_1}\wedge\cdots\wedge \mathrm{d}x_{i_k}\,, \\ h^{k+1}\circ \mathrm{d}(\omega) \quad &= \quad \sum_{j<i_1} x_j\cdot\int_0^1 \mathrm{d}t\cdot\frac{\partial f}{\partial x_j}(0,\ldots,0,tx_j,x_{j+1},\ldots,x_n)\cdot \mathrm{d}x_{i_1}\wedge\cdots\wedge \mathrm{d}x_{i_n} \\ &- \quad \sum_{j>i_1} x_{i_1}\cdot\int_0^1 \mathrm{d}t\cdot\frac{\partial f}{\partial x_j}(0,\ldots,0,tx_{i_1},x_{i_1+1},\ldots,x_n)\cdot \mathrm{d}x_j\wedge \mathrm{d}x_{i_2}\wedge\cdots\wedge \mathrm{d}x_{i_n}\,. \end{aligned}$$

As a consequence

$$\begin{aligned}
& (\mathrm{d} \circ h^k + h^{k+1} \circ \mathrm{d})(\omega) \\
= \;& f_{i_1,\dots i_k}(0,\dots,0,x_{i_1},x_{i_1+1},\dots,x_n) \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \\
+ \;& \sum_{j<i_1} (f(0,\dots,0,x_j,x_{j+1},\dots,x_n) - f(0,\dots,0,x_{j+1},\dots,x_n)) \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \\
= \;& f(x_1,\dots,x_n) \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} = \omega \,.
\end{aligned}$$

And, in degree 0, for $\omega = f(x_1,\dots,x_n)$,

$$\begin{aligned}
h^1 \circ \mathrm{d}(\omega) &= \sum_j x_j \cdot \int_0^1 \mathrm{d}t \cdot \frac{\partial f}{\partial x_j}(0,\dots,0,tx_j,x_{j+1},\dots,x_n) \\
&= f(x_1,\dots,x_n) - f(0,\dots,0)\,.
\end{aligned}$$

So, the subcomplex of constant functions

$$\mathbb{R} \qquad \text{(in degree 0)}$$

is a homotopy retract of the complex

$$\Omega^\bullet(\mathbb{R}^d)\,.$$

The Poincaré lemma follows.

Partitions of unity

Proposition:
Let $X =$ differential manifold,
$(U_i)_{i \in I} =$ open covering of X.
Then there exists a family of C^∞ functions

$$\varphi_j : X \longrightarrow \mathbb{R}_+$$

such that

- the supports of the φ_j are compact and locally finite,
- the support of any φ_j is contained in some U_i,
- the sum $\sum\limits_j \varphi_j$ is equal to 1 everywhere.

Corollary: There exists a family of C^∞ functions

$$\psi_i : X \longrightarrow \mathbb{R}_+$$

such that

- the support of any ψ_i is contained in U_i,
- the sum $\sum\limits_i \psi_i$ is locally finite and equal to 1 everywhere.

Proof of the proposition (in the case X is countable at infinity):

Suppose X is countable at infinity.

It means X can be written as a union of open subsets

$$X_n \,, \quad n \in \mathbb{N} \,,$$

such that each $\overline{X}_n$ is compact. We can suppose that

$$\overline{X}_n \subset X_{n+1} \,, \qquad \forall\, n \in \mathbb{N} \,.$$

For any $n \in \mathbb{N}$ and any $x \in \overline{X}_n - X_{n-1}$, there is a C^∞ function

$$\varphi_{n,x} : X \longrightarrow \mathbb{R}_+ \qquad \text{with} \qquad \varphi_{n,x}(x) > 0$$

and whose support

$$\begin{cases} \bullet & \text{is compact,} \\ \bullet & \text{is contained in some } U_i, \\ \bullet & \text{has empty intersection with } \overline{X}_{n-2}. \end{cases}$$

Then there is a finite family of points

$$x_{n,1}, \ldots, x_{n,k_n} \in \overline{X}_n - X_{n-1}$$

such that

$$(\varphi_{n,x_{n,1}} + \cdots + \varphi_{n,x_{n,k_n}})(x) > 0 \,, \qquad \forall\, x \in \overline{X}_n - X_{n-1} \,.$$

The sum

$$\varphi = \sum_n \sum_{1 \le i \le k_n} \varphi_{n,x_{n,i}}$$

is locally finite, C^∞ and everywhere > 0.

Corollary:

Let X = differential manifold,
U, V = two open subsets which cover X.
Then there is a short exact sequence of complexes

$$0 \longrightarrow \Omega_X^\bullet(X) \longrightarrow \Omega_U^\bullet(U) \oplus \Omega_V^\bullet(V) \longrightarrow \Omega_{U\cap V}^\bullet(U \cap V) \longrightarrow 0$$

and, as a consequence, a long exact sequence of De Rham cohomology spaces:

$$0 \longrightarrow H_{dR}^\circ(X) \longrightarrow H_{dR}^\circ(U) \oplus H_{dR}^\circ(V) \longrightarrow H_{dR}^\circ(U \cap V) \longrightarrow H_{dR}^1(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{dR}^n(X) \longrightarrow H_{dR}^n(U) \oplus H_{dR}^n(V) \longrightarrow H_{dR}^n(U \cap V) \longrightarrow H_{dR}^{n+1}(X) \longrightarrow \cdots$$

Proof:

Let $\varphi_U, \varphi_V : X \to \mathbb{R}_+$ be C^∞ functions such that

$$\mathrm{supp}(\varphi_U) \subset U\,,\ \mathrm{supp}(\varphi_V) \subset V\,,\ \varphi_U + \varphi_V = 1\,.$$

Then any $\omega \in \Omega_{U\cap V}^k(U \cap V)$ can be written as

$$\omega = \varphi_U \cdot \omega + \varphi_V \cdot \omega$$

where $\varphi_U \cdot \omega$ extended by 0 is in $\Omega_V^k(V)$
$\varphi_V \cdot \omega$ extended by 0 is in $\Omega_U^k(U)$.

Corollary:
Let $X =$ differential manifold
which can be written as a finite union

$$X = U_1 \cup \cdots \cup U_n$$

of open subsets such that all the

$$U_{i_1} \cap \cdots \cap U_{i_k}$$

are diffeomorphic to some $\mathbb{R}^d$ (or open ball of $\mathbb{R}^d$).
Then the De Rham cohomology spaces

$$H^n_{dR}(X)$$

are finite dimensional,
and they are 0 if n is big enough.

Remark:
It can be proven
that any compact differential manifold has such finite open covers.
So it verifies the conclusion of the corollary.

Integration on differential manifolds

If $U =$ open subset of some $\mathbb{R}^n$,
$(f : U \to \mathbb{R}) =$ continuous function
$K =$ compact subset of U
such that $K - K^0$ has measure 0,

then we can consider the well-defined integral

$$\int_K f(x_1, \ldots, x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n = \int_{\overset{\circ}{K}} f(x_1, \ldots, x_n) \cdot \mathrm{d}x_1 \ldots \mathrm{d}x_n \,.$$

Furthermore, if

$$\varphi : V \xrightarrow{\sim} U$$

is a diffeomorphism to U from an open subset

$$V \subseteq \mathbb{R}^n ,$$

we have the formula:

Lemma:
For

$$\varphi = (\varphi_1, \dots, \varphi_n) : V \xrightarrow{\sim} U,$$

there is an equality

$$\begin{aligned} & \int_K f(x_1, \dots, x_n) \mathrm{d}x_1 \dots \mathrm{d}x_n \\ = \; & \int_{\varphi^{-1}(K)} (f \circ \varphi)(y_1, \dots, y_n) \cdot \left| \det \left(\frac{\partial \varphi_i}{\partial y_j} \right) (y_1, \dots, y_n) \right| \cdot \mathrm{d}y_1 \dots \mathrm{d}y_n . \end{aligned}$$

Remark:
If the tangent bundles of $V \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ are identified with

$$V \times \mathbb{R}^n \qquad \text{and} \qquad U \times \mathbb{R}^n,$$

the matrix

$$\left(\frac{\partial \varphi_i}{\partial y_j} \right)_{1 \le i, j \le n}$$

defines the tangent linear map

$$\mathrm{d}\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

and its determinant is the induced scalar morphism

$$\Lambda^n \mathbb{R}^n \longrightarrow \Lambda^n \mathbb{R}^n .$$

Definition:
An orientation on a differential manifold X is a way to associate to any chart
$$\left(U \xrightarrow{x} X\right)$$
where $U =$ connected open subset of $\mathbb{R}^n$
$x =$ diffeomorphism to some open subset of X
a sign
$$\mathrm{sign}(x) \in \{\pm 1\}$$
such that

- if $(U' \xrightarrow{x'} X)$ is deduced from $(U \xrightarrow{x} X)$ by restriction to some open subset $U' \subset U$, then
$$\mathrm{sign}(x') = \mathrm{sign}(x)\,,$$
- if $(U' \xrightarrow{x'} X)$ is deduced from $(U \xrightarrow{x} X)$ by composition with a diffeomorphism $\varphi : U' \xrightarrow{\sim} U$, then
$$\mathrm{sign}(x') = \mathrm{sign}(x) \cdot \mathrm{sign}\left(\det\left(\frac{\partial \varphi_i}{\partial x_j}\right)\right).$$

Remark:
If X has an orientation,
charts $(U \xrightarrow{x} X)$ such that $\mathrm{sign}(x) = +1$
are called well oriented.

Remarks:

(i) For any differential manifold X, there is a sheaf

$$U \longmapsto \mathrm{or}_X(U) = \{\text{orientations of } U\}$$

called the sheaf or_X of orientations of X.
It is locally isomorphic to $\{\pm 1\}$.

(ii) This sheaf may or may not have global sections, i.e. orientations of X.

(iii) A differential manifold X is orientable if and only if there are charts

$$\left(U_i \xrightarrow{x_i} X\right)$$

whose images are an open cover of X
and such that, for any indices i, j,
the maps of change of coordinates

$$\varphi_{i,j} = x_i^{-1} \circ x_j : x_j^{-1}(x_i(U_i) \cap x_j(U_j)) \longrightarrow x_i^{-1}(x_i(U_i) \cap x_j(U_j))$$

verify the condition

$$\mathrm{sign}(\det(\mathrm{d}\varphi_{i,j})) = +1\,.$$

Proposition:

Let $X =$ oriented differential manifold of dimension d.

Then there is a unique way to define integrals

$$\int_K \omega$$

for $K =$ compact subset of X such that $K - \overset{\circ}{K}$ has measure 0,

$\Omega_X^d(U) \ni \omega =$ a differential form of degree d

defined on an open subset U which contains K,

such that

- the integral doesn't change if U is replaced by a smaller $U' \supset K$,
- the integral is linear in ω,
- if $K = K_1 \cup K_2$ and $K_1 \cap K_2$ has measure 0,
$$\int_K \omega = \int_{K_1} \omega + \int_{K_2} \omega\,,$$
- if $\mathbb{R}^d \supset V \xrightarrow[\sim]{\varphi} U \subset X$ is a well oriented chart and $\varphi^*\omega = f(x_1, \dots, x_n)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, then
$$\int_K \omega = \int_{\varphi^{-1}(K)} f(x_1, \dots, x_n)\mathrm{d}x_1 \dots \mathrm{d}x_n\,.$$

Sketch of proof of the proposition:
There is a finite family of well oriented charts

$$U_i \xrightarrow{x_i} X\,,\ 1 \leq i \leq n\,,$$

whose images cover an open neighborhood of K.
Then one can write

$$K = K_1 \cup \cdots \cup K_n$$

where

$$\begin{cases} \bullet \quad \text{each } K_i \text{ is compact and contained in } x_i(U_i), \\ \bullet \quad \text{the boundaries } K_i - \overset{\circ}{K}_i \text{ have measure } 0, \\ \bullet \quad \text{the intersections } K_i \cap K_j \text{ have measure } 0. \end{cases}$$

We must have

$$\int_K \omega = \int_{K_1} \omega + \cdots + \int_{K_n} \omega\,.$$

This reduces the verification of the proposition to the case when X is a (connected) open subset of $\mathbb{R}^d$.
Then the proposition follows from the usual properties of integration and from the lemma.

Stokes' formula

Definition:

Let X = differential manifold of dimension d,
K = closed subset of X.

We say K has a smooth boundary $\partial K = K - \overset{\circ}{K}$ if, for any point $x \in X$, there is a chart

$$\begin{array}{ccccc} \mathbb{R}^d \supset V & \xrightarrow{\sim} & U \subset X \\ \uplus & & \uplus \\ 0 & \longmapsto & x \end{array}$$

such that

- the pull-back of $K \cap U$ is $(]-\infty, 0] \times \mathbb{R}^{d-1}) \cap V$,
- the pull-back of $(\partial K) \cap U$ is $(\{0\} \times \mathbb{R}^{d-1}) \cap V$.

Remarks:

- In this situation, ∂K has an induced structure of differential manifold of dimension $d-1$.
- If X is oriented, ∂K has an induced orientation. We decide that an induced chart

$$(\{0\} \times \mathbb{R}^{d-1}) \cap V \xrightarrow{\sim} (\partial K) \cap U$$

  is well oriented if the starting chart

$$\mathbb{R}^d \supset V \xrightarrow{\sim} U \subset X$$

  is well oriented.

Theorem (Stokes' formula):
Let X = oriented differential manifold of dimension d,
K = compact closed subset with smooth boundary

$$\partial K = K - \overset{\circ}{K} \overset{i}{\hookrightarrow} X\,.$$

Then, for any differential form of degree $d-1$

$$\omega \in \Omega_X^{d-1}(U)$$

defined on an open neighborhood of K, we have

$$\int_K \mathrm{d}\omega = \int_{\partial K} \omega\,.$$

Remark:
For $X = \mathbb{R}$ and $K = [a, b]$, this formula is just

$$\int_a^b \mathrm{d}t \cdot f'(t) = f(b) - f(a)$$

for any C^∞ function defined in an open neighborhood of $[a, b]$.

Sketch of proof:

Using partitions of unity,
we reduce to proving that if

$$\omega = \sum_{1 \leq i \leq n} f_i(x_1, \ldots, x_n) \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n$$

is a differential form of degree $n-1$ on $\mathbb{R}^n$ with compact support,
then

$$\int_{]-\infty,0] \times \mathbb{R}^{n-1}} \mathrm{d}\omega = \int_{\{0\} \times \mathbb{R}^{n-1}} \omega \,.$$

Indeed, for $i \geq 2$,

$$\int_{-\infty}^{+\infty} \mathrm{d}x_i \cdot \frac{\partial f_i}{\partial x_i}(x_1, \ldots, x_n) = 0$$

while for $i = 1$

$$\int_{-\infty}^{0} \mathrm{d}x_1 \cdot \frac{\partial f_1}{\partial x_1}(x_1, \ldots, x_n) = f_1(0, x_2, \ldots, x_n) \,.$$

Corollary:

Let $X =$ differential manifold,
$(\Delta_k \xrightarrow{x} X) =$ smooth k-simplex of X,
$\omega =$ differential form of degree $k-1$
defined on some open neighborhood of $x(\Delta_k)$ in X.

Then we have

$$\int_{\Delta_k} x^*(\mathrm{d}\omega) = \sum_{0 \leq i \leq k} (-1)^{i-1} \cdot \int_{\Delta_{k-1}} (x \circ \partial_i^k)^* \omega \,.$$

Remark:

A smooth k-simplex of X
is a continuous map $\Delta_k \to X$
which is the restriction of a C^∞ map

$$U \to X$$

defined on some open neighborhood of Δ_k in $\mathbb{R}^k$.

Sketch of proof: Recall
$$\Delta_k = \{(t_1, \dots, t_k) \in \mathbb{R}^k \mid 0 \le t_1 \le \cdots \le t_k \le 1\}.$$
For any i, $0 \le i \le k$, the affine map
$$\partial_i^k : \Delta_{k-1} \longrightarrow \Delta_k$$
is
$$(t_1, \dots, t_{k-1}) \longmapsto \begin{cases} (0, t_1, \dots, t_{k-1}) & \text{if} \quad i = 0\,, \\ (t_1, \dots, t_i, t_i, t_{i+1}, \dots, t_{k-1}) & \text{if} \quad 1 \le i \le k-1\,, \\ (t_1, \dots, t_{k-1}, 1) & \text{if} \quad i = k\,. \end{cases}$$
If t is a point of $\mathring{\Delta}_{k-1} = \{(t_1, \dots, t_{k-1}) \in \mathbb{R}^{k-1} \mid 0 < t_1 < \cdots < t_{k-1} < 1\}$, the affine isomorphism
$$\begin{array}{ccl} \mathbb{R} \times \mathbb{R}^{k-1} & \xrightarrow{\sim} & \mathbb{R}^k \\ (t_0, t_1, \dots, t_{k-1}) & \longmapsto & \begin{cases} (-t_0, t_1, \dots, t_{k-1}) & \text{if} \quad i = 0\,, \\ (t_1 + t_0, \dots, t_i + t_0, t_i, t_{i+1}, \dots, t_{k-1}) & \text{if} \quad 1 \le i \le k-1\,, \\ (t_1, \dots, t_{k-1}, 1 + t_0) & \text{if} \quad i = k\,, \end{cases} \end{array}$$
induces an isomorphism of an open neighborhood of
$$(0, t) \quad \text{in} \quad]-\infty, 0] \times \mathbb{R}^{k-1}$$
to an open neighborhood of
$$\partial_i^k(t) \quad \text{in} \quad \Delta_k\,.$$

Furthermore, the associated linear isomorphism

$$\mathbb{R} \times \mathbb{R}^{k-1} \xrightarrow{\sim} \mathbb{R}^k$$

is defined by the matrix

$$\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{if} \quad i = 0\,,$$

$$\begin{array}{r} i \text{ lines} \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \right. \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 & 1 & \vdots & & \vdots \\ 0 & \vdots & & \vdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{if} \quad 1 \leq i \leq k-1\,,$$

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{if} \quad i = k\,.$$

Lastly, the determinant of this matrix is

$$(-1)^{i-1}\,.$$

This corollary implies:

Proposition:

Let X = differential manifold,

$C_\bullet^X$ = chain complex of $\mathbb{R}$-vector spaces
whose basis elements are the continuous maps

$$x : \Delta_k \longrightarrow X\,,$$

$C_X^\bullet = \mathrm{Hom}(C_\bullet^X, \mathbb{R})$ = dual cochain complex,

$C_\bullet^{X,\mathrm{sm}}$ = subcomplex generated by the basis elements

$$x : \Delta_k \longrightarrow X$$

which are smooth,

$C_{X,\mathrm{sm}}^\bullet = \mathrm{Hom}(C_\bullet^{X,\mathrm{sm}}, \mathbb{R})$ = dual cochain complex which is a quotient of $C_X^\bullet$,

$\Omega_X^\bullet$: De Rham complex of X.

Then the bilinear maps

$$\begin{array}{ccc} \Omega_X^k(X) \times C_k^{X,\mathrm{sm}} & \longrightarrow & \mathbb{R}\,, \\ \left(\omega, \Delta_k \xrightarrow{x} X\right) & \longmapsto & (-1)^k \cdot \displaystyle\int_{\Delta_k} x^*(\omega) \end{array}$$

define a morphism of cochain complexes

$$\Omega_X^\bullet(X) \longrightarrow C_{X,\mathrm{sm}}^\bullet\,.$$

Proposition:

Let $X =$ differential manifold.

Then the natural chain morphism

$$C_\bullet^{X,\mathrm{sm}} \longrightarrow C_\bullet^X$$

and its dual

$$C_X^\bullet \longrightarrow C_{X,\mathrm{sm}}^\bullet$$

are quasi-isomorphisms.

In other words, they identify the associated homology (or cohomology) spaces.

Corollary: So there are natural maps

$$H^k_{dR}(X) \longrightarrow H^k(X, \mathbb{R})$$

or, equivalently, bilinear pairings

$$\langle \bullet, \bullet \rangle : H^k_{dR}(X) \times H_k(X, \mathbb{R}) \longrightarrow \mathbb{R}\,.$$

Remark: If an element of $H^k_{dR}(X)$ is represented by

$$\omega \in \Omega^k_X(X) \qquad \text{such that} \qquad \mathrm{d}\omega = 0$$

and an element of $H_k(X, \mathbb{R})$ is represented by

$$c = \sum_x c_x \cdot x \qquad \text{such that} \qquad \mathrm{d}c = 0$$

and furthermore, $c_x \in \mathbb{Z}$, $\forall \left(\Delta_k \xrightarrow{x} X\right)$, then the associated number

$$\langle \omega, c \rangle \in \mathbb{R}$$

is called a period of ω.

The linear map $\langle \omega, \bullet \rangle$ induces a morphism of abelian groups

$$H_k(X, \mathbb{Z}) \longrightarrow H_k(X, \mathbb{R}) \xrightarrow{\langle \omega, \bullet \rangle} \mathbb{R}$$

whose image is the subgroup of periods of ω.

Sketch of proof of the proposition:

As the functor

$$\mathrm{Hom}(\bullet, \mathbb{R}) : \mathrm{Vect}_{\mathbb{R}}^{\mathrm{op}} \longrightarrow \mathrm{Vect}_{\mathbb{R}}$$

preserves exact sequences,
It is enough to prove that the chain complex morphism

$$C_\bullet^{X,\mathrm{sm}} \longrightarrow C_\bullet^X$$

is a quasi-isomorphism.
Denote $H_k^{\mathrm{sm}}(X, \mathbb{R})$ the homology spaces of $C_\bullet^{X,\mathrm{sm}}$.

The proof consists in the following steps:

- If $f : X \to Y$ is a C^∞ map, show that the induced morphisms

  $$H_k(X, \mathbb{R}) \longrightarrow H_k^{\mathrm{sm}}(Y, \mathbb{R})$$

  are invariant by C^∞-deformations of f.
- Deduce that the proposition is true when X is C^∞-contractible.
- Reduce the verification to an open cover.
- Deduce that the proposition is true when X has a finite open cover whose intersections are C^∞-contractible.
- Show the general case.

Step 1: invariance by smooth deformations

Let $X, Y =$ differential manifolds.
Any C^∞ map $X \xrightarrow{f} Y$
induces a commutative square of morphisms of chain complexes:

$$\begin{tikzcd} C_\bullet^{X,\mathrm{sm}} \arrow[r, hook] \arrow[d, "f_\bullet"'] & C_\bullet^X \arrow[d, "f_\bullet"] \\ C_\bullet^{Y,\mathrm{sm}} \arrow[r, hook] & C_\bullet^Y \end{tikzcd}$$

We say that two C^∞ maps

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

are C^∞-homotopic
if there exists an open interval $]a, b[\supset [0, 1]$
and a C^∞ map

$$h :]a, b[\times X \longrightarrow Y$$

such that

$$h(0, \bullet) = f \qquad \text{and} \qquad h(1, \bullet) = g\,.$$

Lemma: If two C^∞ maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are C^∞-homotopic, the induced morphisms of chain complexes

$$C_\bullet^{X,\mathrm{sm}} \underset{g_\bullet}{\overset{f_\bullet}{\rightrightarrows}} C_\bullet^{Y,\mathrm{sm}}$$

are chain homotopic.
In particular, they induce the same morphisms

$$H_k^{\mathrm{sm}}(X,\mathbb{R}) \longrightarrow H_k^{\mathrm{sm}}(Y,\mathbb{R}), \quad \forall\, k \in \mathbb{N}.$$

Proof: We already associated to h a chain homotopy

$$h_\bullet = \left(h_k : C_k^X \longrightarrow C_{k+1}^Y\right)$$

such that, for any k, the morphisms

$$C_k^X \underset{g_k}{\overset{f_k}{\rightrightarrows}} C_k^Y$$

verify

$$f_k - g_k = d \circ h_k + h_{k-1} \circ d\,.$$

Furthermore, it is obvious on the construction that, as h is C^∞,
any h_k sends $\quad C_k^{X,\mathrm{sm}} \hookrightarrow C_k^X$
to $\quad C_{k+1}^{X,\mathrm{sm}} \hookrightarrow C_{k+1}^X$.

Step 2: the case of C^∞-contractible manifolds

A differential manifold X is called C^∞-contractible if there exists a point

$$\{\bullet\} \xrightarrow{x} X$$

such that the composed morphism

$$s \circ p : X \xrightarrow{p} \{\bullet\} \xrightarrow{x} X$$

is C^∞-homotopic to id_X.
It follows from Step 1 that the canonical morphisms

$$H_k^{\mathrm{sm}}(X, \mathbb{R}) \longrightarrow H_k^{\mathrm{sm}}(\{\bullet\}, \mathbb{R})$$

are isomorphisms, just as the morphisms

$$H_k(X, \mathbb{R}) \longrightarrow H_k(\{\bullet\}, \mathbb{R})\,.$$

But we have

$$C_\bullet^{\{\bullet\},\mathrm{sm}} = C_\bullet^{\{\bullet\}}$$

and a fortiori

$$H_k^{\mathrm{sm}}(\{\bullet\}, \mathbb{R}) = H_k(\{\bullet\}, \mathbb{R})\,, \quad \forall\, k\,.$$

We conclude that, if X is C^∞-contractible, the morphisms

$$H_k^{\mathrm{sm}}(X, \mathbb{R}) \longrightarrow H_k(X, \mathbb{R})$$

are isomorphisms.

Step 3: reduction to an open cover

Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of X. Recall that we denoted

$$C_\bullet^{X,\mathcal{U}} \hookrightarrow C_\bullet^X$$

the subcomplex of $C_\bullet^X$ generated by the simplices of X

$$x : \Delta_k \longrightarrow X$$

which factorise through at least one of the U_i's.
In the same way, we can denote

$$C_\bullet^{X,\mathrm{sm},\mathcal{U}} \hookrightarrow C_\bullet^{X,\mathrm{sm}}$$

the subcomplex generated by the smooth simplices of X

$$x : \Delta_k \longrightarrow X$$

which factorise through at least one of the U_i's.
Using barycentric subdivisions, we constructed a morphism

$$r : C_\bullet^X \longrightarrow C_\bullet^{X,\mathcal{U}}$$

such that the composite

$$C_\bullet^{X,\mathcal{U}} \overset{i}{\hookrightarrow} C_\bullet^X \xrightarrow{r} C_\bullet^{X,\mathcal{U}}$$

is id, and a chain homotopy from the composite

$$C_\bullet^X \xrightarrow{r} C_\bullet^{X,\mathcal{U}} \xrightarrow{i} C_\bullet^X$$

to id.

Lemma:

(i) The retraction

$$r : C_\bullet^X \longrightarrow C_\bullet^{X,\mathcal{U}}$$

sends $C_\bullet^{X,\mathrm{sm}}$ to $C_\bullet^{X,\mathrm{sm},\mathcal{U}}$, and the chain homotopy

$$h = \left(C_k^X \longrightarrow C_{k+1}^X \right)$$

sends each $C_k^{X,\mathrm{sm}}$ to $C_{k+1}^{X,\mathrm{sm}}$.

(ii) The morphism of chain complexes

$$C_\bullet^{X,\mathrm{sm},\mathcal{U}} \hookrightarrow C_\bullet^{X,\mathrm{sm}}$$

is a quasi-isomorphism, just as

$$C_\bullet^{X,\mathcal{U}} \hookrightarrow C_\bullet^X \, .$$

Proof of the lemma:
It results from the fact that barycentric subdivisions of a smooth simplex

$$x : \Delta_k \longrightarrow X$$

are smooth simplices.

Corollary of the lemma: Suppose $X = U \cup V$
and we already know that the morphisms

$$\begin{array}{rcl} C_\bullet^{U,\mathrm{sm}} & \hookrightarrow & C_\bullet^{U}\,, \\ C_\bullet^{V,\mathrm{sm}} & \hookrightarrow & C_\bullet^{V}\,, \\ C_\bullet^{U\cap V,\mathrm{sm}} & \hookrightarrow & C_\bullet^{U\cap V} \end{array}$$

are quasi-isomorphisms.
Then we can conclude that the morphism

$$C_\bullet^{X,\mathrm{sm}} \hookrightarrow C_\bullet^{X}$$

is a quasi-isomorphism.

Proof:
Let $\mathcal{U}$ be the open cover of X by U and V.
We have two short exact sequences of chain complexes

$$0 \longrightarrow C_U^{U\cap V,\mathrm{sm}} \longrightarrow C_\bullet^{U,\mathrm{sm}} \oplus C_\bullet^{V,\mathrm{sm}} \longrightarrow C_\bullet^{X,\mathcal{U},\mathrm{sm}} \longrightarrow 0\,,$$
$$0 \longrightarrow C_\bullet^{U\cap V} \longrightarrow C_\bullet^{U} \oplus C_\bullet^{V} \longrightarrow C_\bullet^{X,\mathcal{U}} \longrightarrow 0\,,$$

and associated long exact sequences of homology:

$$\begin{tikzcd}
\cdots \arrow[r] & H_k^{\mathrm{sm}}(U\cap V,\mathbb{R}) \arrow[r] \arrow[d] & H_k^{\mathrm{sm}}(U,\mathbb{R}) \oplus H_k^{\mathrm{sm}}(V,\mathbb{R}) \arrow[r] \arrow[d] & H_k^{\mathrm{sm}}(X,\mathbb{R}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_k(U\cap V,\mathbb{R}) \arrow[r] & H_k(U,\mathbb{R}) \oplus H_k(V,\mathbb{R}) \arrow[r] & H_k(X,\mathbb{R}) \arrow[r] & \cdots
\end{tikzcd}$$

The “five lemma” allows to conclude.

Step 4: the case when X has a finite contractible open cover

This means X has a finite open cover

$$X = U_1 \cup \cdots \cup U_n$$

such that the U_i's are C^∞-contractible as well as all non empty intersections

$$U_{i_1} \cap \cdots \cap U_{i_m} .$$

In that case, we can conclude that

$$C_\bullet^{X,\mathrm{sm}} \hookrightarrow C_\bullet^X$$

is a quasi-isomorphism.

The proof is by induction on n,
using Step 2 and the corollary of Step 3.

Step 5: the general case

Let $X =$ arbitrary differential manifold.
Let $I =$ ordered set of open subsets

$$U \subset X$$

such that

$$\begin{cases} \bullet & U \text{ is relatively compact, meaning } \overline{U} \text{ is compact,} \\ \bullet & U \text{ has a finite } C^\infty\text{-contractible open cover.} \end{cases}$$

It can be proved that any compact subset

$$K \subset X$$

is contained in an element U of I.

It follows that

$$\begin{cases} \bullet & \text{the ordered set } I \text{ is filtering,} \\ \bullet & X \text{ is the union of the } U \in I. \end{cases}$$

So we can write

$$C_\bullet^X = \varinjlim_{U \in I} C_\bullet^U$$

and

$$C_\bullet^{X,\mathrm{sm}} = \varinjlim_{U \in I} C_\bullet^{U,\mathrm{sm}} .$$

As the functor

$$\varinjlim_{I}$$

preserves exact sequences (as I is filtering), we have for any k

$$H_k(X,\mathbb{R}) = \varinjlim_{U\in I} H_k(U,\mathbb{R})\,,$$

$$H_k^{\rm sm}(X,\mathbb{R}) = \varinjlim_{U\in I} H_k^{\rm sm}(U,\mathbb{R})\,.$$

The conclusion follows from Step 4.

De Rham's theorem

Theorem:
Let $X =$ differential manifold which is countable at infinity.

(i) The chain complexes morphism

$$\Omega_X^\bullet \longrightarrow C_{X,\mathrm{sm}}^\bullet$$

is a quasi-isomorphism.

(ii) The De Rham cohomology spaces

$$H_{dR}^k(X)$$

identify with singular cohomology spaces

$$H^k(X, \mathbb{R}) .$$

In other words, the period morphisms induce isomorphisms

$$H_{dR}^k(X) \longrightarrow \mathrm{Hom}(H_k(X, \mathbb{R}), \mathbb{R}) .$$

Proof that (ii) is equivalent to (i):
This follows from the previous proposition.

Step 1: invariance by C^∞-deformations

Lemma:

If two C^∞ maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ between differential manifolds are C^∞-homotopic, the induced morphisms of cochain complexes

$$\Omega_Y^\bullet(Y) \underset{g^*}{\overset{f^*}{\rightrightarrows}} \Omega_X^\bullet(X)$$

are cochain homotopic.
In particular they induce the same morphisms

$$H_{dR}^k(Y) \longrightarrow H_{dR}^k(X)\,, \qquad \forall\, k \in \mathbb{N}\,.$$

Proof of the lemma: It is enough to consider the case when

$$Y = \,]a,b[\, \times X \qquad \text{with} \qquad]a,b[\, \supset [0,1]$$

and f, g are

$$\begin{array}{rccl} & X & \longrightarrow &]a,b[\, \times X = Y\,, \\ f: & x & \longmapsto & (0,x)\,, \\ g: & x & \longmapsto & (1,x)\,. \end{array}$$

We consider the associated restriction maps

$$\Omega_Y^k(Y) \overset{f}{\underset{g}{\rightrightarrows}} \Omega_X^k(X)\,.$$

We want to define homomorphisms

$$h^k : \Omega_Y^k(Y) \longrightarrow \Omega_X^{k-1}(X)$$

such that

$$g - f = \mathrm{d} \circ h^k + h^{k+1} \circ \mathrm{d}$$

in any degree k.
Let's consider a covering of X by open subsets U_i which are diffeomorphic to some open subset of $\mathbb{R}^d$ with coordinates $x_1, \ldots, x_d$.
Let's define

$$h^k : \Omega_Y^k(]a,b[\, \times U_i) \longrightarrow \Omega_X^{k-1}(U_i)$$

by

$$\begin{aligned}
\omega \;\; = & \sum_{\underline{i}=(i_1<\cdots<i_k)} f_{\underline{i}}(t, x_1, \ldots, x_d) \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \\
+ & \sum_{\underline{j}=(j_1<\cdots<j_{k-1})} f_{\underline{j}}(t, x_1, \ldots, x_d) \cdot \mathrm{d}t \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{k-1}} \\
& \longmapsto \sum_{\underline{j}} \left(\int_0^1 f_{\underline{j}}(t, x_1, \ldots, x_d) \cdot \mathrm{d}t \right) \cdot \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{k-1}}\,.
\end{aligned}$$

These definitions match on the intersections of the U_i's and define global morphisms

$$h^k : \Omega^k_Y(Y) \longrightarrow \Omega^{k-1}_X(X)\,.$$

Furthermore, we compute locally

$$\mathrm{d} \circ h^k(\omega) = \sum_{1 \leq j \leq d} \sum_{\underline{j}} \left(\int_0^1 \frac{\partial f_{\underline{j}}}{\partial x_j}(t, x_1, \ldots, x_d) \cdot \mathrm{d}t \right) \cdot \mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{k-1}}$$

and

$$\begin{aligned} h^{k+1} \circ \mathrm{d}(\omega) &= \sum_{\underline{i}} \left(\int_0^1 \frac{\partial f_{\underline{i}}}{\partial t}(t, x_1, \ldots, x_d) \cdot \mathrm{d}t \right) \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \\ &- \sum_{1 \leq j \leq d} \sum_{\underline{j}} \left(\int_0^1 \frac{\partial f_{\underline{j}}}{\partial x_j}(t, x_1, \ldots, x_d) \cdot \mathrm{d}t \right) \cdot \mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{k-1}}\,. \end{aligned}$$

This concludes the proof of the lemma.

Step 2: the case when X is C^∞-contractible

If X is C^∞-contractible, the canonical morphism

$$\Omega^\bullet_{\{\bullet\}}(\{\bullet\}) \longrightarrow \Omega^\bullet_X(X)$$

is a quasi-isomorphism.
As has already been proved,

$$C^\bullet_{\{\bullet\},\mathrm{sm}} \longrightarrow C^\bullet_{X,\mathrm{sm}}$$

is a quasi-isomorphism as well.
So the verification of the theorem in the case when X is contractible is reduced to the case

$$X = \{\bullet\}.$$

In that case, $\Omega^\bullet_{\{\bullet\}}\{\bullet\}$ is equal to $\mathbb{R}$ concentrated in degree 0 and

$$\Omega^\bullet_{\{\bullet\}}(\{\bullet\}) \longrightarrow C^\bullet_{\{\bullet\},\mathrm{sm}}$$

is a quasi-isomorphism.

Step 3: reduction to an open cover

It has already been proved that if

$$X = U \cup V$$

is an open cover of a differential manifold X, the sequence

$$0 \longrightarrow \Omega^\bullet_X(X) \longrightarrow \Omega^\bullet_X(U) \oplus \Omega^\bullet_X(V) \longrightarrow \Omega^\bullet_X(U \cap V) \longrightarrow 0$$

is a short exact sequence.

Using the associated long exact sequence of cohomology

$$\cdots \longrightarrow H^k_{dR}(X) \longrightarrow H^k_{dR}(U) \oplus H^k_{dR}(V) \longrightarrow H^k_{dR}(U \cap V) \longrightarrow H^{k+1}_{dR}(X) \longrightarrow \cdots$$

and its natural morphism to the long exact sequence

$$\cdots \longrightarrow H^k(X,\mathbb{R}) \longrightarrow H^k(U,\mathbb{R}) \oplus H^k(V,\mathbb{R}) \longrightarrow H^k(U \cap V,\mathbb{R}) \longrightarrow H^{k+1}(X,\mathbb{R}) \longrightarrow \cdots$$

we can conclude according to the “five lemma”:

Corollary:

Suppose $X = U \cup V$

and we already know that the morphisms of cochain complexes

$$\begin{array}{rcl} \Omega_X^\bullet(U) & \longrightarrow & C_{U,\mathrm{sm}}^\bullet , \\ \Omega_X^\bullet(V) & \longrightarrow & C_{V,\mathrm{sm}}^\bullet , \\ \Omega_X^\bullet(U \cap V) & \longrightarrow & C_{U \cap V,\mathrm{sm}}^\bullet \end{array}$$

are quasi-isomorphisms.

Then we can conclude that the morphism

$$\Omega_X^\bullet(X) \longrightarrow C_{X,\mathrm{sm}}^\bullet$$

is also a quasi-isomorphism.

Step 4: the case when X has a finite contractible open cover

Recall it means X has a finite open cover

$$X = U_1 \cup \cdots \cup U_n$$

such that the U_i's are C^∞-contractible as well as all non empty intersections

$$U_{i_1} \cap \cdots \cap U_{i_m} .$$

The proof of the theorem in that case is by induction on n, using Step 2 and the corollary of Step 3.

Remark:

One can prove that any compact differential manifold admits such a finite contractible open cover.

De Rham cohomology with compact support

Definition:
Let $(X, \mathcal{O}_X)$ = ringed space,
$\mathcal{M} = \mathcal{O}_X$-Module on X.

(i) The support of a section $m \in \mathcal{M}(X)$ is the smallest closed subset Z of X such that the restriction of m to the open subset $X - Z$ is 0.

(ii) The submodule of

$$\mathcal{M}(X) = \Gamma(X, \mathcal{M})$$

consisting of sections m whose support is compact is denoted

$$\Gamma_c(X, \mathcal{M})$$

and called the $\mathcal{O}_X(X)$-module of sections of $\mathcal{M}$ with compact support.

Remark:
Of course, if X is compact, we always have

$$\Gamma_c(X, \mathcal{M}) = \Gamma(X, \mathcal{M}) .$$

Remarks:

(i) For any morphism of $\mathcal{O}_X$-Modules

$$\mathcal{M}_1 \longrightarrow \mathcal{M}_2 \,,$$

the morphism

$$\Gamma(X, \mathcal{M}_1) \longrightarrow \Gamma(X, \mathcal{M}_2)$$

restricts to a morphism

$$\Gamma_c(X, \mathcal{M}_1) \longrightarrow \Gamma_c(X, \mathcal{M}_2)$$

as any closed subspace of a compact subspace is compact.

(ii) Suppose any compact subspace of X is closed (which is true in particular if X is Hausdorff).
Then for any $\mathcal{O}_X$-Module $\mathcal{M}$ and any open subsets

$$U_1 \subset U_2 \subset X \,,$$

there is a natural morphism of $\mathcal{O}_X(U_2)$-modules

$$\Gamma_c(U_1, \mathcal{M}) \longrightarrow \Gamma_c(U_2, \mathcal{M}) \,.$$

It associates to any section

$$m \in \Gamma_c(U_1, \mathcal{M}) \quad \text{with compact support } Z \subset U_1$$

the unique section of $\Gamma_c(U_2, \mathcal{M})$ whose restriction to U_1 is m and whose restriction to $U_2 - Z$ is 0.

Definition:
Let $(X, \mathcal{O}_X) \xrightarrow{f} (S, \mathcal{O}_S)$
$= $ morphism of ringed spaces.
The De Rham cohomology with compact support of X over S is defined as the family of cohomology spaces

$$H^k_{dR,c}(X/S)\,, \qquad k \in \mathbb{N}\,,$$

of the subcomplex

$$\Gamma_c(X, \Omega^\bullet_{X/S})$$

of the De Rham complex

$$\Gamma(X, \Omega^\bullet_{X/S}) = \Omega^\bullet_{X/S}(X)\,.$$

Remarks:

(i) There are induced morphisms

$$H^k_{dR,c}(X/S) \longrightarrow H^k_{dR}(X/S)\,, \quad k \in \mathbb{N}\,.$$

(ii) If X is Hausdorff, there is a natural morphism of complexes for any open subset U of X

$$\Gamma_c(U, \Omega^\bullet_{X/S}) \longrightarrow \Gamma_c(X, \Omega^\bullet_{X/S})$$

and so induced morphisms

$$H^k_{dR,c}(U/S) \longrightarrow H^k_{dR,c}(X/S)\,.$$

The ring structure of De Rham cohomology

Lemma:
Let $(X, \mathcal{O}_X) \xrightarrow{f} (S, \mathcal{O}_S)$
$=$ morphism of ringed spaces.
Then the operation

$$(\omega, \omega') \longmapsto \omega \wedge \omega'$$

defines morphisms of sheaves

$$\Omega^k_{X/S} \times \Omega^{k'}_{X/S} \longrightarrow \Omega^{k+k'}_{X/S}$$

which verify the following properties:

- they are bilinear with respect to $f^*\mathcal{O}_S$,
- they are associative,
- they verify the commutation rule
$$\omega' \wedge \omega = (-1)^{kk'} \cdot \omega \wedge \omega' ,$$
- they verify the rule of differentiation
$$\mathrm{d}(\omega \wedge \omega') = (\mathrm{d}\omega) \wedge \omega' + (-1)^k \omega \wedge (\mathrm{d}\omega') ,$$
- the support of $\omega \wedge \omega'$ is contained in the intersection of the supports of ω and ω'.

Corollary:
In the same context, there are natural bilinear morphisms

$$\begin{array}{rcl} (\omega,\omega') & \longmapsto & \omega\wedge\omega' \\ H^k_{dR}(X/S)\times H^{k'}_{dR}(X/S) & \longrightarrow & H^{k+k'}_{dR}(X/S)\,, \\ H^k_{dR,c}(X/S)\times H^{k'}_{dR}(X/S) & \longrightarrow & H^{k+k'}_{dR,c}(X/S)\,, \\ H^k_{dR}(X/S)\times H^{k'}_{dR,c}(X/S) & \longrightarrow & H^{k+k'}_{dR,c}(X/S)\,, \\ H^k_{dR,c}(X/S)\times H^{k'}_{dR,c}(X/S) & \longrightarrow & H^{k+k'}_{dR,c}(X/S)\,. \end{array}$$

They are associative and verify the commutation rule

$$\omega'\wedge\omega = (-1)^{kk'}\cdot\omega\wedge\omega'\,.$$

Remark:
This applies in particular to differential manifolds X
(considered over the point manifold $\{\bullet\}$).
We can associate to X its De Rham cohomology spaces with compact support

$$H^k_{dR,c}(X)\,,\qquad k\in\mathbb{N}$$

together with the morphisms $H^k_{dR,c}(X)\to H^k_{dR}(X)$
and the product operations $(\omega,\omega')\mapsto\omega\wedge\omega'$ as above.

Proposition:
The De Rham cohomology with compact support of the differential variety $\mathbb{R}^d$ is

$$H^k_{dR,c}(\mathbb{R}^d) = \begin{cases} \mathbb{R} & \text{if } k = d\,, \\ 0 & \text{otherwise.} \end{cases}$$

Proof:
Let's consider the sphere of dimension d

$$X = S^d = \{(t_0, t_1, \ldots, t_d) \in \mathbb{R}^{d+1} \mid t_0^2 + t_1^2 + \cdots + t_d^2 = 1\}\,.$$

We already know that its De Rham cohomology is

$$H^k_{dR}(X) = H^k(X, \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } k = d \\ 0 & \text{if } k \neq 0, d\,. \end{cases}$$

We observe that, if P is a point of $X = S^d$

$$U = X - \{P\} \quad \text{is diffeomorphic to } \mathbb{R}^d\,.$$

Let's choose a sequence of open neighborhoods

$$U_n \text{ of } P \text{ in } X\,,\ n \in \mathbb{N}\,,$$

such that

- for any n, $\overline{U_{n+1}} \subset U_n$,
- the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is $\{P\}$,
- each U_n is diffeomorphic to a ball of $\mathbb{R}^d$.

Then we have a short exact sequence of complexes

$$0 \longrightarrow \Gamma_c(U, \Omega_X^\bullet) \longrightarrow \Gamma(X, \Omega_X^\bullet) \longrightarrow \varinjlim_n \Gamma(U_n, \Omega_X^\bullet) \longrightarrow 0\,.$$

Indeed, for any element ω of some $\Gamma(U_n, \Omega_X^k)$,
there exists an element ω' of $\Gamma(X, \Omega_X^k)$
which coincides with ω on some $U_{n'}$, $n' > n$.
As the functor $\varinjlim_n$ respects exact sequences,
the cohomology spaces of the complex $\varinjlim_n \Gamma(U_n, \Omega_X^\bullet)$ are the colimits

$$\varinjlim_n H_{dR}^k(U_n) = \begin{cases} \mathbb{R} & \text{if } k = 0\,, \\ 0 & \text{if } k \neq 0\,. \end{cases}$$

Furthermore, the morphism

$$H^{\circ}_{dR}(X) \longrightarrow \varinjlim_{n} H^{\circ}_{dR}(U_n)$$

identifies with the identity morphism

$$\mathbb{R} \longrightarrow \mathbb{R}\,.$$

So, the long exact sequence of cohomology associated to our short exact sequence of complexes yields

$$H^k_{dR,c}(U) = \begin{cases} \mathbb{R} & \text{if } k = d\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For the computation of De Rham cohomology with compact support, we can use:

Lemma:
Let X = differential manifold with an open cover $X = U \cup V$.
Then the complex

$$0 \longrightarrow \Gamma_c(U \cap V, \Omega_X^\bullet) \longrightarrow \Gamma_c(U, \Omega_X^\bullet) \oplus \Gamma_c(V, \Omega_X^\bullet) \longrightarrow \Gamma_c(X, \Omega_X^\bullet) \longrightarrow 0$$

is a short exact sequence of complexes,
and there is an associated long exact sequence of cohomology:

$$\cdots \to H^k_{dR,c}(U \cap V) \to H^k_{dR,c}(U) \oplus H^k_{dR,c}(V) \to H^k_{dR,c}(X) \to H^{k+1}_{dR,c}(U \cap V) \to \cdots$$

Proof:
Consider a partition of unity $1 = \varphi_U + \varphi_V$
where φ_U, φ_V are C^∞ functions whose supports are contained in U and V.
Then, any element $\omega \in \Gamma_c(X, \Omega_X^k)$ can be written

$$\omega = \varphi_U \cdot \omega + \varphi_V \cdot \omega$$

with

$$\varphi_U \cdot \omega \in \Gamma_c(U, \Omega_X^k) \quad \text{and} \quad \varphi_V \cdot \omega \in \Gamma_c(V, \Omega_X^k)\,.$$

Corollary:

Let $X =$ differential manifold which can be written as a finite union

$$X = U_1 \cup \cdots \cup U_n$$

of open subsets $U_1, \ldots, U_n$ which are diffeomorphic to $\mathbb{R}^d$ (or, equivalently, to balls of $\mathbb{R}^d$) as well as their intersections

$$U_{i_1} \cap \cdots \cap U_{i_m} .$$

Then the De Rham cohomology spaces with compact support

$$H^k_{dR,c}(X)$$

are finite dimensional.

Remark:

This corollary applies in particular
to any compact differential manifold.

The Poincaré pairing

Lemma:
Let $X =$ oriented differential manifold of dimension d.
Then the integration form

$$\begin{array}{rcl} \Gamma_c(X, \Omega_X^d) & \longrightarrow & \mathbb{R}\,, \\ \omega & \longmapsto & \displaystyle\int_X \omega \end{array}$$

defines a linear map

$$H^d_{dR,c}(X) \longrightarrow \mathbb{R}\,.$$

Proof:
If $\omega \in \Gamma_c(X, \Omega_X^d)$ can be written

$$\omega = \mathrm{d}\omega' \quad \text{with} \quad \omega' \in \Gamma_c(X, \Omega_X^{d-1})\,,$$

then Stokes' formula implies

$$\int_X \omega = 0$$

as X has no boundary.

Corollary:

Let $X =$ oriented differential manifold of dimension d.
Then the composition of the product

$$\begin{array}{rcl} H^k_{dR,c}(X) \times H^{d-k}_{dR}(X) & \longrightarrow & H^d_{dR,c}(X)\,, \\ (\omega, \omega') & \longmapsto & \omega \wedge \omega' \end{array}$$

for any $k \in \{0, 1, \ldots, d\}$
and of the integration form

$$H^d_{dR,c}(X) \longrightarrow \mathbb{R}$$

yields a bilinear pairing

$$H^k_{dR,c}(X) \times H^{d-k}_{dR}(X) \longrightarrow \mathbb{R}\,.$$

The Poincaré duality

Theorem:
Let X = oriented differential manifold of dimension d.
Then, for any k, the pairing

$$\begin{array}{rcl} H^k_{dR,c}(X) \times H^{d-k}_{dR}(X) & \longrightarrow & \mathbb{R}\,, \\ (\omega, \omega') & \longmapsto & \displaystyle\int_{\mathbb{R}} \omega \wedge \omega' \end{array}$$

induces an isomorphism

$$H^{d-k}_{dR}(X) \xrightarrow{\sim} H^k_{dR,c}(X)^\vee = \mathrm{Hom}(H^k_{dR,c}(X), \mathbb{R})\,.$$

Remark:
If the De Rham cohomology spaces of X are finite-dimensional, in particular if X has a finite C^∞-contractible open cover, the morphisms

$$H^k_{dR,c}(X) \longrightarrow H^{d-k}_{dR}(X)^\vee = \mathrm{Hom}(H^{d-k}_{dR}(X), \mathbb{R})$$

are also isomorphisms.

Remarks:

(i) The theorem applies in particular
to any oriented differential manifold X of dimension d
which is compact.
In that case, we have perfect pairings

$$H_{dR}^{k}(X) \times H_{dR}^{d-k}(X) \longrightarrow \mathbb{R}$$

which means in particular that the spaces

$$H_{dR}^{k}(X) = H^{k}(X, \mathbb{R}) \qquad \text{and} \qquad H_{dR}^{d-k}(X) = H^{d-k}(X, \mathbb{R})$$

always have the same dimension.

(ii) Combining this theorem with the de Rham theorem,
we get that, for any differential manifold X,
De Rham cohomology with compact support

$$H_{dR,c}^{k}(X)$$

identifies with singular homology

$$H_{d-k}(X, \mathbb{R})\,.$$

Partial proof of the theorem:
Suppose X can be written as a finite union

$$X = U_1 \cup \cdots \cup U_n$$

of open subsets $U_1, \dots, U_n$ which are diffeomorphic to $\mathbb{R}^d$ as well as their intersections $U_1 \cap \cdots \cap U_{i_n}$.
Then one can prove by induction on n that X verifies Poincaré duality.
If $n = 1$, the result is already known as

$$H^k_{dR}(\mathbb{R}^d) = \begin{cases} \mathbb{R} & \text{if} \quad k = 0\,, \\ 0 & \text{if} \quad k \neq 0\,, \end{cases}$$

and

$$H^k_{dR,c}(\mathbb{R}^d) = \begin{cases} \mathbb{R} & \text{if} \quad k = d\,, \\ 0 & \text{if} \quad k \neq d\,. \end{cases}$$

If $n \geq 2$, write

$$U = U_1 \cup \cdots \cup U_{n-1}$$

and

$$V = U_n\,.$$

We can suppose the result is already known for U, V and $U \cap V$.

The Poincaré pairing induces a morphism of long exact sequences:

$$\begin{array}{ccccccc} \cdots \to H_{dR}^{d-k}(X) & \to & H_{dR}^{d-k}(U) \oplus H_{dR}^{d-k}(V) & \to & H_{dR}^{d-k}(U \cap V) & \to & H_{dR}^{d-k+1}(X) \to \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to H_{dR,c}^{k}(X)^{\vee} & \to & H_{dR,c}^{k}(U)^{\vee} \oplus H_{dR,c}^{k}(V)^{\vee} & \to & H_{dR,c}^{k}(U \cap V) & \to & H_{dR,c}^{k-1}(X) \to \cdots \end{array}$$

The conclusion follows from the “five lemma”.

The cohomology class of a submanifold

Definition: Let $X =$ differential manifold of dimension d.
A closed submanifold of X of codimension k is a closed subspace

$$Y \hookrightarrow X$$

such that, for any point $y \in Y$,
there exists an open neighborhood U of y in X
and a diffeomorphism to a ball of $\mathbb{R}^d$

$$U \xrightarrow{\sim} B = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1^2 + \cdots + x_d^2 < 1\}$$

sending $Y \cap U$ to $B \cap \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 = 0, \dots, x_k = 0\}$.

Remark:
Equivalently, a closed subset $Y \subset X$ is a submanifold of codimension d
if, in an open neighborhood $U \subset X$ of any point $y \in Y$,
it can be defined by k equations

$$f_1 = 0, \dots, f_k = 0$$

where $f_1, \dots, f_k : U \to \mathbb{R}$ are C^∞ functions
whose differentials are linearly independent at y.

Definition:

Let $X =$ oriented differential manifold of dimension d,

$(Y \overset{i}{\hookrightarrow} X) =$ closed submanifold of X endowed with an orientation.

The cohomology class of Y is the unique element

$$\mathrm{cl}_Y \in H^k_{dR}(X)$$

such that, for any differential form with compact support

$$\omega \in \Gamma_c(X, \Omega_X^{d-k}) \quad \text{verifying} \quad \mathrm{d}\omega = 0\,,$$

we have

$$\int_X \mathrm{cl}_Y \wedge \omega = \int_Y i^*\omega\,.$$

Remark:

If $\omega \in \Gamma(X, \Omega_X^{d-k})$ has compact support,

then $i^*\omega \in \Gamma(Y, \Omega_Y^{d-k})$ also has compact support and $\int_Y i^*\omega$ is well defined.

This defines a linear form on $H^{d-k}_{dR,c}(X)$ which, by Poincaré duality,

is represented by a unique element $\mathrm{cl}_Y \in H^k_{dR}(X)$.

Compatibility of cohomology classes with intersections

Definition:
Let X = differential manifold of dimension d.
Two closed submanifolds

$Y \hookrightarrow X$ of codimension k,

$Y' \hookrightarrow X$ of codimension k',

are said to intersect transversely if, for any point $y \in Y$, there exists an open neighborhood of y in X and a diffeomorphism to a ball of $\mathbb{R}^d$

$$U \xrightarrow{\sim} B = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1^2 + \cdots + x_d^2 < 1\}$$

sending $Y \cap U$ to $B \cap \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 = 0, \ldots, x_k = 0\}$
and $Y' \cap U$ to $B' \cap \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_{k+1} = 0, \ldots, x_{k+k'} = 0\}$.

Remark:
Equivalently, Y and Y' intersect transversely if, in an open neigborhood $U \subset X$ of any point $y \in Y \cap Y'$, Y and Y' can be defined by equations

$f_1 = 0, \ldots, f_k = 0$

and $f_{k+1} = 0, \ldots, f_{k+k'} = 0$

where $f_1, \ldots, f_{k+k'} : U \to \mathbb{R}$ are C^∞ functions whose differentials are linearly independent at y.

Lemma: Let $X =$ differential manifold of dimension d,
$Y, Y' =$ two closed submanifolds of codimensions k, k' which intersect transversely.

Suppose X and the closed submanifolds Y, Y', $Y \cap Y'$ are endowed with orientations.

At any element $y \in Y \cap Y'$, choose local coordinates

$$x_1, \ldots, x_d \quad \text{of} \quad X$$

such that Y and Y' are respectively defined by

$x_1 = 0, \ldots, x_k = 0$

and $x_{k+1} = 0, \ldots, x_{k+k'} = 0$.

Define an "intersection sign"

$$\mathrm{sign}(y)$$

as the product of the signs of the coordinate systems

$$\begin{array}{ll} x_1, \ldots, x_d & \text{of} \quad X\,, \\ x_{k+1}, \ldots, x_d & \text{of} \quad Y\,, \\ x_1, \ldots, x_k, x_{k+k'+1}, \ldots x_d & \text{of} \quad Y' \\ x_{k+k'+1}, \ldots, x_d & \text{of} \quad Y \cap Y' \end{array}$$

with respect to the chosen orientations of X, Y, Y', $Y \cap Y'$. Then

(i) $\mathrm{sign}(y)$ doesn't depend on the choice of $x_1, \ldots, x_d$,

(ii) it is locally constant on $Y \cap Y'$.

Proof:

(i) Let's consider another system of coordinates $y_1, \dots, y_d$ verifying the same conditions in a neighborhood of y.
At y, the diffeomorphism of change of coordinates has a differential matrix of the form

$$\begin{pmatrix} A & 0 & 0 \\ * & B & 0 \\ * & * & C \end{pmatrix}$$

where A, B, C are square matrices of ranks k, k' and $d - k - k'$.
The corresponding determinants are

$$\begin{array}{ll} \det(A) \cdot \det(B) \cdot \det(C) & \text{for the change of coordinates of } X, \\ \det(B) \cdot \det(C) & \text{for the change of coordinates of } Y, \\ \det(A) \cdot \det(C) & \text{for the change of coordinates of } Y', \\ \det(C) & \text{for the change of coordinates of } Y \cap Y'. \end{array}$$

Their product is

$$\det(A)^2 \cdot \det(B)^2 \cdot \det(C)^4$$

whose sign is always $+1$.

(ii) is an obvious consequence of (i) and the definition of $\mathrm{sign}(y)$.

Theorem:

Let $X =$ oriented differential manifold of dimension d,

$Y, Y' =$ closed submanifolds of X which intersect transversely.

Suppose Y, Y' and $Y \cap Y'$ are endowed with orientations.

Decompose $Y \cap Y'$ in connected components

$$Y \cap Y' = \coprod_i Y_i$$

and associate to any connected component Y_i the intersection sign

$$\mathrm{sign}(Y_i)$$

defined by the previous lemma.

Then we have the formula

$$\mathrm{cl}_Y \wedge \mathrm{cl}_{Y'} = \sum_i \mathrm{sign}(Y_i) \cdot \mathrm{cl}_{Y_i}$$

in $H_{dR}^{k+k'}(X)$.

Sketch of proof of the theorem:

Step 1: Reduction to a pull-back formula

Denoting $i_Y : Y \hookrightarrow X$,
we have for any $\omega \in H_{dR,c}^{k+k'}(X)$

$$\int \mathrm{cl}_Y \wedge \mathrm{cl}_{Y'} \wedge \omega = \int_Y i_Y^*(\mathrm{cl}_{Y'}) \wedge i_Y^*(\omega)\,.$$

So we just have to prove that the pull-back

$$i_Y^*(\mathrm{cl}_{Y'}) \in H_{dR}^{k'}(Y)$$

is equal to the sum

$$\sum_i \mathrm{sign}(Y_i) \cdot \mathrm{cl}_{Y_i}^Y$$

where, for any i, $\mathrm{cl}_{Y_i}^Y$ is the cohomology class of

$$Y_i \hookrightarrow Y$$

in $H_{dR}^{k'}(Y)$.

Step 2: Lifting to relative cohomology

We are going to lift the class

$$\mathrm{cl}_{Y'} \in H^{k'}_{dR}(X)$$

to a refined class

$$\mathrm{cl}_{Y'} \in H^{k'}_{dR}(X, Y')$$

in a "relative cohomology space" $H^{k}_{dR}(X, Y')$ where it can be computed locally. For this we need the following general definition:

Definition:

For any morphism of cochain [resp. chain] complexes,

$$A^\bullet \xrightarrow{u} B^\bullet \qquad [\text{resp.} \quad A_\bullet \longrightarrow B_\bullet \;],$$

the cone of u is the cochain [resp. chain] complex

$$C_u^\bullet \qquad [\text{resp.} \quad C_\bullet^u \;]$$

defined by

$$C_u^k = A^k \oplus B^{k-1} \qquad [\text{resp.} \quad C_k^u = A_k \oplus B_{k+1} \;]$$

and the differentials

$$\begin{pmatrix} d & 0 \\ u_k & -d \end{pmatrix}.$$

Remarks:

(i) If $C^\bullet$ is the cone of $A^\bullet \xrightarrow{u} B^\bullet$ and $B[-1]^\bullet$ is defined by $B[-1]^k = B^{k-1}$ with differentials $-d$, the canonical short exact sequence of complexes

$$0 \longrightarrow B[-1]^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet \longrightarrow 0$$

yields a long exact sequence of cohomology:

$$\cdots \longrightarrow H^{k-1}(B^\bullet) \longrightarrow H^k(C^\bullet) \longrightarrow H^k(A^\bullet) \xrightarrow{u} H^k(B^\bullet) \longrightarrow \cdots$$

(ii) Any commutative square of complexes

$$\begin{tikzcd} A^\bullet \arrow[r, "u"] \arrow[d] & B^\bullet \arrow[d] \\ A'^\bullet \arrow[r, "u'"] & B'^\bullet \end{tikzcd}$$

yields a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B[-1]^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & A^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B'[-1]^\bullet \arrow[r] & C'^\bullet \arrow[r] & A'^\bullet \arrow[r] & 0 \end{tikzcd}$$

if $C^\bullet, C'^\bullet$ are the cones of u and u'.

According to the "five lemma",

$$C^\bullet \longrightarrow C'^\bullet$$

is a quasi-isomorphism if $A^\bullet \to A'^\bullet$ and $B^\bullet \to B'^\bullet$ are quasi-isomorphisms.

Definition: Let Z = closed subset of X = differential manifold of dimension d.

(i) Let $\Gamma(X, Z, \Omega_X^\bullet)$ be the cone of the morphism

$$\Omega_X^\bullet(X) \longrightarrow \Omega_X^\bullet(X - Z)$$

and $H_{dR}^k(X, Z)$ its cohomology spaces.

(ii) Let $\Gamma_c(X, Z, \Omega_X^\bullet)$ be the cone of the morphism

$$\Gamma_c(X - Z, \Omega_X^\bullet) \longrightarrow \Gamma_c(X, \Omega_X^\bullet)$$

and $H_{dR,c}^k(X, Z)$ its cohomology spaces.

Remark: According to Stokes' formula, we have for any differential form ω of degree k on X or $X - Z$ and any differential form ω' of degree $d - k - 1$ with compact support

$$\int_X \mathrm{d}(\omega \wedge \omega') = 0$$

and so

$$\int_X \mathrm{d}\omega \wedge \omega' = (-1)^{k-1} \cdot \int_X \omega \wedge \mathrm{d}\omega' .$$

This implies:

Lemma:
For any k, integration of forms of degree d on X and $Z - X$ defines isomorphisms

$$H^k_{dR}(X,Z) \xrightarrow{\sim} H^{d-k}_{dR,c}(X,Z)^\vee = \mathrm{Hom}(H^{d-k}_{dR,c}(X,Z),\mathbb{R})$$

which lift to a quasi-isomorphism of cochain diagrams

$$\begin{array}{ccc} \Gamma(X,Z,\Omega^\bullet_X) & \longrightarrow & \Gamma_c(X,Z,\Omega^\bullet_X)^\vee[d] \\ & & \| \\ & & \mathrm{Hom}(\Gamma_c(X,Z,\Omega^\bullet_X),\mathbb{R})[d] \end{array}$$

induced by the commutative square

$$\begin{tikzcd} \Gamma(X,\Omega^\bullet_X) \arrow[r] \arrow[d] & \Gamma(X-Z,\Omega^\bullet_X) \arrow[d] \\ \Gamma_c(X,\Omega^\bullet_X)^\vee[d] \arrow[r] & \Gamma_c(X-Z,\Omega^\bullet_X)^\vee[d] \end{tikzcd}$$

where

- the differentials of the bottom row have been modified by factors $(-1)^{d-k-1}$,
- the two vertical arrows are quasi-isomorphisms.

Remark:
For any complex A, $A[d]$ denotes the complex whose indices have been shifted by $k \mapsto k+d$ and whose differentials have been modified by the factor $(-1)^d$.

Corollary: Suppose Z is an oriented submanifold of X of codimension k. Then the linear form

$$\begin{array}{rcl} \Gamma_c(X,Z,\Omega_X^\bullet)^{d-k} = \Gamma_c(X,\Omega_X^{d-k}) \oplus \Gamma_c(X-Z,\Omega_X^{d-k+1}) & \longrightarrow & \mathbb{R} \\ (\omega_1,\omega_2) & \longmapsto & \displaystyle\int_Z i_Z^*(\omega_1) \end{array}$$

defines an element

$$\mathrm{cl}_Y \in H^k_{dR}(X,Z) = H^{d-k}_{dR,c}(X,Z)^\vee$$

which lifts the already defined cohomology class

$$\mathrm{cl}_Y \in H^k_{dR}(X) = H^{d-k}_{dR,c}(X)^\vee\,.$$

Proof: Indeed, the linear form

$$(\omega_1,\omega_2) \longmapsto \int_Z i_Z^*(\omega_1)$$

vanishes on all pairs (ω_1,ω_2) which are in the image of

$$\begin{array}{rcl} \Gamma_c(X,\Omega_X^{d-k-1}) \oplus \Gamma_c(X-Z,\Omega_X^{d-k}) & \longrightarrow & \Gamma_c(X,\Omega_X^{d-k}) \oplus \Gamma_c(X-Z,\Omega_X^{d-k+1})\,, \\ (\omega_1',\omega_2') & \longmapsto & (\mathrm{d}\omega_1+\omega_2',\mathrm{d}\omega_2')\,. \end{array}$$

Step 3: excision

Definition:
Let X = topological space,
Z = closed subspace,
R = coefficient ring for singular (co)homology.
We denote

$$C_\bullet^{X,Z} \qquad \text{[resp.} \quad C^\bullet_{X,Z}\ \text{]}$$

the cone of the morphism of complexes

$$C_\bullet^{X-Z} \longrightarrow C_\bullet^X \qquad \text{[resp.} \quad C^\bullet_X \longrightarrow C^\bullet_{X-Z}\ \text{]}$$

and

$$H_k(X, Z, R) \qquad \text{[resp.} \quad H^k(X, Z, R)\ \text{]}$$

their associated (co)-homology invariants.

Remark: By definition,

$$C^\bullet_{X,Z} \quad \text{identifies with} \quad \mathrm{Hom}(C_\bullet^{X,Z}, R)$$

and there are induced isomorphisms

$$H^k(X, Z, R) \xrightarrow{\sim} \mathrm{Hom}(H_k(X, Z, R), R)$$

if R is a field.

Remark:

If $X =$ differential manifold,
  $R = \mathbb{R}$,
the commutative square

$$\begin{tikzcd} \Gamma(X, \Omega_X^\bullet) \arrow[r] \arrow[d] & \Gamma(X - Z, \Omega_R^\bullet) \arrow[d] \\ C_{X,\mathrm{sm}}^\bullet \arrow[r] & C_{X-Z,\mathrm{sm}}^\bullet \end{tikzcd}$$

whose vertical arrows are quasi-isomorphisms
(according to De Rham's theorem)
induces a quasi-isomorphism

$$\Gamma(X, Z, \Omega_X^\bullet) \longrightarrow C_{X,Z,\mathrm{sm}}^\bullet = \text{ cone of } C_{X,\mathrm{sm}}^\bullet \longrightarrow C_{X-Z,\mathrm{sm}}^\bullet$$

and so isomorphisms:

$$\begin{array}{ccc} H_{dR}^k(X, Z) & \xrightarrow{\sim} & H^k(X, Z, \mathbb{R}) \\ & & \| \\ & & \mathrm{Hom}(H_k(X, Z, \mathbb{R}), \mathbb{R}) \end{array}$$

Lemma:

Let $X =$ topological space,
$Z =$ closed subspace,
$R =$ coefficient ring,
and $U =$ open subset of X which contains Z.

Then the morphism of complexes

$$C_\bullet^{U,Z} \longrightarrow C_\bullet^{X,Z}$$

induced by the commutative square

$$\begin{tikzcd} C_\bullet^{U-Z} \arrow[r] \arrow[d] & C_\bullet^{U} \arrow[d] \\ C_\bullet^{X-Z} \arrow[r] & C_\bullet^{X} \end{tikzcd}$$

is a quasi-isomorphism, inducing identifications

$$H_k(U,Z,R) \xrightarrow{\sim} H_k(X,Z,R)\,.$$

Proof of the lemma: Denote $\mathcal{U}$ the open cover of X by $X - Z$ and U.
Recall $C_\bullet^{X,\mathcal{U}}$ is the subcomplex of $C_\bullet^X$ generated by simplices $\Delta_k \xrightarrow{x} X$ which factorise through $X - Z$ or U, and the morphism

$$C_\bullet^{X,\mathcal{U}} \longrightarrow C_\bullet^X$$

is a quasi-isomorphism.
Let

$$C_\bullet^{X,Z,\mathcal{U}} = \text{ cone of } C_\bullet^{X-Z} \longrightarrow C_\bullet^{X,\mathcal{U}} .$$

The commutative square

$$\begin{tikzcd} C_\bullet^{X-Z} \arrow[r] \arrow[d, equal] & C_\bullet^{X,\mathcal{U}} \arrow[d] \\ C_\bullet^{X-Z} \arrow[r] & C_\bullet^X \end{tikzcd}$$

yields a quasi-isomorphism

$$C_\bullet^{X,Z,\mathcal{U}} \longrightarrow C_\bullet^{X,Z}$$

and the morphism $C_\bullet^{U,Z} \to C_\bullet^{X,Z}$ factorises through the morphism $C_\bullet^{U,Z} \to C_\bullet^{X,Z,\mathcal{U}}$ induced by the commutative square:

$$\begin{tikzcd} C_\bullet^{U-Z} \arrow[r] \arrow[d] & C_\bullet^{U} \arrow[d] \\ C_\bullet^{X-Z} \arrow[r] & C_\bullet^{X,\mathcal{U}} \end{tikzcd}$$

We observe that the quotient complexes associated to the embeddings

$$\begin{array}{rcl} C_\bullet^{U-Z} & \hookrightarrow & C_\bullet^{X-Z}, \\ C_\bullet^{U} & \hookrightarrow & C_\bullet^{X,\mathcal{U}} \end{array}$$

identify.

This implies that the quotient complex associated to the induced embedding

$$C_\bullet^{U,Z} \hookrightarrow C_\bullet^{X,Z,\mathcal{U}}$$

is quasi-isomorphic to 0.

This means that the morphism

$$C_\bullet^{U,Z} \longrightarrow C_\bullet^{X,Z,\mathcal{U}}$$

is a quasi-isomorphism.

Corollary of the lemma:

Let $X =$ differential manifold,
$Z =$ closed subset,
$U =$ open subset which contains Z.

Then the natural morphism of complexes

$$\Gamma(X, Z, \Omega_X^\bullet) \longrightarrow \Gamma(U, Z, \Omega_X^\bullet)$$

is a quasi-isomorphism, yielding isomorphisms

$$H_{dR}^k(X, Z) \longrightarrow H_{dR}^k(U, Z)\,.$$

Step 4: reduction to the case of a vector bundle

Definition:
Let $X =$ differential manifold,
$(Z \overset{i}{\hookrightarrow} X) =$ closed submanifold of codimension k.
The normal tangent bundle of Z in X is the vector bundle

$$N_{Z/X} \qquad \text{over} \qquad Z$$

which is associated to the dual of the locally free $\mathcal{O}_Z$-Module of rank k

$$\mathrm{Ker}(i^*\Omega_X \longrightarrow \Omega_Y)$$

or, equivalently, to the locally free $\mathcal{O}_Z$-Module

$$\mathcal{N}_{Z/X} = \mathrm{Coker}(\Omega_Y^\vee \longrightarrow i^*\Omega_X^\vee)\,.$$

Remark: Any orientation of X induces an orientation of $N_{Z/X}$.
If Z is also oriented, the fibers of the projection

$$N_{Z/X} \longrightarrow Z$$

are oriented. Locally over Z, $N_{Z/X}$ is isomorphic to $Z \times \mathbb{R}^k$ and the orientation of its fibers is induced by an orientation of $\mathbb{R}^k$.

Proposition:
Let $X =$ differential manifold,
$Z =$ closed submanifold of codimension k,
$N = N_{Z/X} =$ normal tangent bundle of Z in X
endowed with its 0 section $Z \hookrightarrow N$.
Then:

(i) For any coefficient ring R, the relative (co)homology modules

$$H_k(X, Z, R) \quad \text{and} \quad H_k(N, Z, R)$$

$$\text{[resp.} \quad H^k(X, Z, R) \quad \text{and} \quad H^k(N, Z, R)\text{]}$$

identify.

(ii) In particular, the relative De Rham cohomology spaces

$$H^k_{dR}(X, Z) \quad \text{and} \quad H^k_{dR}(N, Z)$$

identify.

Sketch of proof: One can prove that there exist open neighborhoods
U of Z in X (called a tubular neighborhood),
V of Z in $N = N_{X/Z}$
and a diffeomorphism

$$U \xrightarrow{\sim} V$$

which tranforms $Z \hookrightarrow X$ into $Z \hookrightarrow V$.
Then the proposition is a consequence of excision.

Step 5: localisation in the case of a vector bundle

Proposition:

Let $Z =$ differential manifold,

$N =$ vector bundle of rank k over Z

endowed with its canonical projection $p : N \to Z$

and its 0 section $Z \hookrightarrow N$,

$R =$ coefficient ring.

Then the presheaf on Z

$$\underset{\substack{\parallel \\ \text{open subset of } Z}}{U} \longmapsto H^k(p^{-1}(U), p^{-1}(U) \cap Z, R)$$

is a sheaf of R-modules which is locally free of rank 1.

Sketch of proof:

If U is an open subset of Z which is C^∞-contractible and such that $p^{-1}(U)$ is isomorphic to $U \times \mathbb{R}^k$, the relative cohomology modules

$$H^i(p^{-1}(U), p^{-1}(U) \cap Z, R)$$

identify with the modules

$$H^i(\mathbb{R}^k, \{0\}, R) .$$

We have a long exact equence

$$\longrightarrow H^i(\mathbb{R}^k, \{0\}, R) \longrightarrow H^i(\mathbb{R}^k, R) \longrightarrow H^i(\mathbb{R}^k - \{0\}, R) \longrightarrow H^{i+1}(\mathbb{R}^k, \{0\}, R) \longrightarrow \cdots$$

where we know

$$H^i(\mathbb{R}^k, R) = \begin{cases} R & \text{if} \quad i = 0\,, \\ 0 & \text{if} \quad i \neq 0 \end{cases}$$

and

$$H^i(\mathbb{R}^k - \{0\}, R) = \begin{cases} R & \text{if} \quad i = 0 \text{ or } i = k-1\,, \\ 0 & \text{if} \quad i \neq 0, k-1 \end{cases}$$

as the sphere S^{k-1} is a homotopy retract of $\mathbb{R}^k - \{0\}$.
So we have

$$H^i(\mathbb{R}^k, \{0\}, R) = \begin{cases} R & \text{if } i = k\,, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the sheaf of R-modules associated to the presheaf

$$U \longmapsto H^i(p^{-1}(U), p^{-1}(U) \cap Z, R)$$

is 0 if $i \neq k$ and is locally free of rank 1 if $i = k$.

The conclusion of the proposition follows for general sheaf-theoretic reasons.

Step 6: conclusion of the proof of the proposition:

Let's come back to X, Y, $Y'$and $Y \cap Y' = \coprod_i Y_i$.

According to the previous proposition, the formula

$$i_Y^* \mathrm{cl}_{Y'} = \sum_i \mathrm{sign}(Y_i) \cdot \mathrm{cl}_{Y_i}^Y$$

in $H_{dR}^{k'}(Y, Y \cap Y')$ can be checked locally. So we are reduced to the case where

$$\begin{array}{rcl} X = N & = & \mathbb{R}^k \times \mathbb{R}^{k'} \times \mathbb{R}^{d-k-k'}\,, \\ Y & = & \{0\} \times \mathbb{R}^{k'} \times \mathbb{R}^{d-k-k'}\,, \\ Y' = Z & = & \mathbb{R}^k \times \{0\} \times \mathbb{R}^{d-k-k'}\,, \\ Y \cap Y' & = & \{0\} \times \{0\} \times \mathbb{R}^{d-k-k'}\,. \end{array}$$

We can suppose that X, Y, Y' and $Y \cap Y'$ are endowed with the orientations deduced from the usual orientations of $\mathbb{R}^k, \mathbb{R}^{k'}, \mathbb{R}^{d-k}$.

In that case, the class

$$\mathrm{cl}_{Y'} \in H_{dR}^{k'}(X, Y') \quad \text{with} \quad X = \mathbb{R}^{k'} \times Y'\,,$$

and the class

$$\mathrm{cl}_{Y \cap Y'}^Y \in H_{dR}^{k'}(Y, Y \cap Y') \quad \text{with} \quad Y = \mathbb{R}^{k'} \times (Y \cap Y')\,,$$

are deduced by pull-back from the class associated to

$$\{0\} \hookrightarrow \mathbb{R}^{k'}$$

in $H_{dR}^{k'}(\mathbb{R}^{k'}, \{0\})$.

It is interesting to express this class concretely:

Proposition:
Let's consider the spherical coordinates on $\mathbb{R}^n - \{0\}$ defined by the diffeomorphism

$$\begin{array}{ccc}]0,+\infty[\times S^{n-1} & \xrightarrow{\sim} & \mathbb{R}^n - \{0\} \\ (\rho, u) & \longmapsto & \rho \cdot u\,, \end{array}$$

the invariant volume form ω_S on S^{n-1} and the total volume V of S^{n-1}.
The cohomology class of

$$\{0\} \hookrightarrow \mathbb{R}^n \quad \text{in} \quad H^n_{dR}(\mathbb{R}^n, \mathbb{R}^n - \{0\})$$

is represented by the closed form

$$(-\mathrm{d}\omega, -\omega) \quad \text{in} \quad \Omega^n_{\mathbb{R}^n}(\mathbb{R}^n) \oplus \Omega^{n-1}_{\mathbb{R}^n}(\mathbb{R}^n - \{0\})$$

with $\omega = V^{-1} \cdot \omega_S$ and $\mathrm{d}\omega = 0$.

Proof: For any closed form

$$(f, \mathrm{d}f) \in \Omega^0_{\mathbb{R}^n,c}(\mathbb{R}^n) \oplus \Omega^1_{\mathbb{R}^n,c}(\mathbb{R}^n - \{0\})\,,$$

we have

$$-\int_{\mathbb{R}^n} f \cdot \mathrm{d}\omega - \int_{\mathbb{R}^n-\{0\}} \mathrm{d}f \wedge \omega = -\int_{\mathbb{R}^n-\{0\}} \mathrm{d}(f \cdot \omega) = \lim_{\rho \mapsto 0} \int_{\rho \cdot S} f \cdot \omega = f(0)\,.$$

The Lefschetz fixed points formula

Theorem:

Let X = oriented compact differential manifold of dimension d,
$f = C^\infty$-map $X \to X$
whose graph $\Gamma_f \hookrightarrow X \times X$ intersects
the diagonal $\Gamma_{\mathrm{id}} \hookrightarrow X \times X$ transversely.

For any point $x \in X$ such that $f(x) = x$, denote
$\mathrm{sign}_f(x)$ = sign of the intersection of Γ_f and Γ_{id} at x.

Then we have

$$\begin{aligned}\sum_{\substack{x \in X \\ f(x)=x}} \mathrm{sign}_f(x) &= \sum_{0 \le k \le d} (-1)^k \cdot \mathrm{Tr}(f^*, H^k_{dR}(X)) \\ &= \mathrm{Tr}(f^*, H^\bullet_{dR}(X))\,.\end{aligned}$$

Remarks:

(i) Even if Γ_f does not intersect Γ_{i_d} transversely, one can prove there exists $g : X \to X$ such that

$\begin{cases} \bullet \quad f \text{ and } g \text{ are } C^\infty\text{-homotopic,} \\ \bullet \quad \text{the intersection of } \Gamma_{i_d} \text{ and } \Gamma_g \text{ is transverse,} \end{cases}$

and, therefore,

$$\mathrm{Tr}(f, H^\bullet_{dR}(X)) = \mathrm{Tr}(g, H^\bullet_{dR}(X)) = \sum_{\substack{x \in X \\ g(x) = x}} \mathrm{sign}_g(x) \, .$$

(ii) The Euler-Poincaré characteristic of X is

$$\sum_k (-1)^k \cdot \dim H^k_{dR}(X) = \mathrm{Tr}(\mathrm{id}, H^\bullet_{dR}(X)) \, .$$

Proof of the theorem: The Lefschetz formula follows from the previous theorem combined with two other results:

- the Künneth formula which expresses the cohomology spaces of a product $X \times Y$ in terms of the cohomology spaces of its factors X, Y,
- the computation of the cohomology class of the diagonal

$$\Delta : X \hookrightarrow X \times X \, .$$

Step 1: the Künneth formula

Let $X, Y =$ differential manifolds,
$X \times Y =$ their product endowed with the projections

$$p_1 : X \times Y \longrightarrow X\,, \quad p_2 : X \times Y \longrightarrow Y\,.$$

The formula

$$(\omega_1, \omega_2) \longmapsto p_1^* \omega_1 \wedge p_2^* \omega_2$$

defines morphisms

$$\Gamma(X, \Omega_X^{k_1}) \otimes_{\mathbb{R}} \Gamma(Y, \Omega_Y^{k_2}) \longrightarrow \Gamma(X \times Y, \Omega_{X \times Y}^{k_1+k_2})$$

and

$$\Gamma_c(X, \Omega_X^{k_1}) \otimes_{\mathbb{R}} \Gamma_c(Y, \Omega_Y^{k_2}) \longrightarrow \Gamma_c(X \times Y, \Omega_{X \times Y}^{k_1+k_2})$$

such that

$$(\mathrm{d}(p_1^* \omega_1 \wedge p_2^* \omega_2) = p_1^*(\mathrm{d}\omega_1) \wedge p_2^* \omega_2 + (-1)^{k_1} \cdot p_1^* \omega_1 \wedge p_2^*(\mathrm{d}\omega_1)\,.$$

So it induces morphisms

$$H_{dR}^{k_1}(X) \otimes_{\mathbb{R}} H_{dR}^{k_2}(Y) \longrightarrow H_{dR}^{k_1+k_2}(X \times Y)$$

and

$$H_{dR,c}^{k_1}(X) \otimes_{\mathbb{R}} H_{dR,c}^{k_2}(Y) \longrightarrow H_{dR,c}^{k_1+k_2}(X \times Y)\,.$$

Proposition:

Let $X, Y =$ differential manifolds.

(i) If X or Y is a finite union

$$U_1 \cup \cdots \cup U_n$$

of open subsets which are C^∞-contractible as well as their intersections $U_{i_1} \cap \cdots \cap U_{i_m}$, then the morphisms

$$\bigoplus_{k_1+k_2=k} H^{k_1}_{dR}(X) \otimes H^{k_2}_{dR}(Y) \longrightarrow H^k_{dR}(X \times Y)$$

are isomorphisms.

(ii) The morphisms

$$\bigoplus_{k_1+k_2=k} H^{k_1}_{dR,c}(X) \otimes H^{k_2}_{dR,c}(Y) \longrightarrow H^k_{dR,c}(X \times Y)$$

are always isomorphisms.

Proof:

We need the following algebraic lemma:

Lemma:
Let $R =$ commutative field,
$A, B =$ two cochain complexes of R-vector spaces

$$(\cdots \longrightarrow A^{k-1} \longrightarrow A^k \longrightarrow A^{k+1} \longrightarrow \cdots)$$

and

$$(\cdots \longrightarrow B^{k-1} \longrightarrow B^k \longrightarrow B^{k+1} \longrightarrow \cdots)$$

which are 0 in degrees $k \ll 0$.

Let $A \otimes B =$ the complex whose degree k component is

$$\bigoplus_{k_1+k_2=k} A^{k_1} \otimes_R B^{k_2}$$

and whose differential is defined by

$$\begin{array}{rcl} A^{k_1} \otimes_R B^{k_2} & \longrightarrow & (A^{k_1+1} \otimes_R B^{k_2}) \oplus (A^{k_1} \otimes_R B^{k_2+1}) \\ (a \otimes b) & \longmapsto & (\mathrm{d}a \otimes_R b, (-1)^{k_1} \cdot a \otimes_R \mathrm{d}b) \, . \end{array}$$

Then the natural morphisms

$$\bigoplus_{k_1+k_2=k} H^{k_1}(A) \otimes_R H^{k_2}(B) \longrightarrow H^k(A \otimes B)$$

are isomorphisms.

Proof of the lemma:

- As the functor $\otimes_R$ is exact, the statement is true if A or B is concentrated in one degree.
- Let's denote m_A and m_B the biggest integers such that

$$A^{k_1} = 0\,,\ \forall\, k_1 < m_A \qquad \text{and} \qquad B^{k_2} = 0\,,\ \forall\, k_2 < m_B\,.$$

  Then the statement is obvious in all degrees $k < m_A + m_B$.
- For any m, let's denote $A^{=m}$ [resp. $A^{>m}$] the complex which coincides with A in degree $k = m$ [resp. in degrees $k > m$] and is 0 elsewhere.

  Then there are short exact sequences of complexes

$$0 \longrightarrow A^{>m_A} \longrightarrow A \longrightarrow A^{=m_A} \longrightarrow 0\,,$$

$$0 \longrightarrow A^{>m_A} \otimes B \longrightarrow A \otimes B \longrightarrow A^{=m_A} \otimes B \longrightarrow 0\,,$$

  and an associated morphism of long exact sequences:

$$
\begin{tikzcd}
\cdots \arrow[r] & \bigoplus\limits_{k_1+k_2=k} H^{k_1}(A^{>m_A})\otimes_R H^{k_2}(B) \arrow[r] \arrow[d] & \bigoplus\limits_{k_1+k_2=k} H^{k_1}(A)\otimes_R H^{k_2}(B) \arrow[r] \arrow[d] & A^{m_A}\otimes_R H^{k-m_A}(B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^k(A^{>m_A}\otimes B) \arrow[r] & H^k(A\otimes B) \arrow[r] & H^k(A^{=m_A}\otimes B) \arrow[r] & \cdots
\end{tikzcd}
$$

- Using the “five lemma”, the statement is proved by decreasing induction on m_A if A only has finitely many non zero components.
- The statement for A and B in degree k reduces to the statement for $A/A^{>m}$ and B if $m+m_B>k$.

Proof of the proposition:

(i) According to the lemma, we have to prove that if X has the form of the statement $X = U_1 \cup \cdots \cup U_n$, then the morphism of cochain complexes

$$\Gamma(X, \Omega_X^\bullet) \otimes_{\mathbb{R}} \Gamma(Y, \Omega_Y^\bullet) \longrightarrow \Gamma(X \times Y, \Omega_{X\times Y}^\bullet)$$

is a quasi-isomorphism.

The proof is by induction on n.

If $n = 1$, $X = U_1$ is C^∞-contractible, Y is a C^∞-retract of $X \times Y$ and the natural morphism

$$\begin{array}{rcl} \mathbb{R} = \Gamma(\{\bullet\}, \Omega_{\{\bullet\}}^\bullet) & \longrightarrow & \Gamma(X, \Omega_X^\bullet)\,, \\ \Gamma(Y, \Omega_Y^\bullet) & \longrightarrow & \Gamma(X \times Y, \Omega_{X\times Y}^\bullet) \end{array}$$

are quasi-isomorphisms.

If $n \geq 2$, let's denote $U = U_1 \cup \cdots \cup U_{n-1}$, $V = U_n$
and suppose the result is already know for U, V and $U \cap V$.

Then the result for $X = U \cup V$ follows from the "five lemma" applied to the morphism of long exact sequences deduced from the morphism of short exact sequences of complexes:

$$\begin{tikzcd}
0 \arrow[r] & \begin{array}{c}\Gamma(X,\Omega_X^\bullet)\\ \otimes\Gamma(Y,\Omega_Y^\bullet)\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\Gamma(U,\Omega_X^\bullet)\otimes\Gamma(Y,\Omega_Y^\bullet)\\ \oplus\Gamma(V,\Omega_X^\bullet)\otimes\Gamma(Y,\Omega_Y^\bullet)\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\Gamma(U\cap V,\Omega_X^\bullet)\\ \otimes\Gamma(Y,\Omega_Y^\bullet)\end{array} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X \times Y, \Omega_{X\times Y}^\bullet) \arrow[r] & \begin{array}{c}\Gamma(U\times Y,\Omega_{X\times Y}^\bullet)\\ \oplus\Gamma(V\times Y,\Omega_{X\times Y}^\bullet)\end{array} \arrow[r] & \Gamma((U\cap V)\times Y, \Omega_{X\times Y}^\bullet) \arrow[r] & 0
\end{tikzcd}$$

(ii) If X is diffeomorphic to $\mathbb{R}^d$ and Y is diffeomorphic to $\mathbb{R}^{d'}$,

$$\Gamma_c(X, \Omega_X^\bullet), \Gamma_c(Y, \Omega_Y^\bullet) \quad \text{and} \quad \Gamma_c(X \times Y, \Omega_{X\times Y}^\bullet)$$

are quasi-isomorphic to $\mathbb{R}$ concentrated in degrees $d, d', d + d'$.
The statement of the proposition follows.
If

$$\begin{array}{rcl} X & = & U_1 \cup \cdots \cup U_n \,, \\ Y & = & V_1 \cup \cdots \cup V_{n'} \,, \end{array}$$

where $U_1, \ldots, U_n$ [resp. $V_1, \ldots, V_{n'}$] are diffeomorphic to some $\mathbb{R}^d$ [resp. $\mathbb{R}^{d'}$] as well as the intersections

$$U_{i_1} \cap \cdots \cap U_{i_m} \qquad \text{[resp.} \quad V_{j_1} \cap \cdots \cap V_{j_{m'}} \ \text{]},$$

the statement of the proposition is proved by induction on n and n', using the "five lemma" in the same way as in (i).

In general, let $\mathcal{U}$ and $\mathcal{V}$ be the ordered sets of open subsets

$$U \subset X \qquad V \subset Y$$

which can be written in the above form

$$U = U_1 \cup \cdots \cup U_n \qquad V = V_1 \cup \cdots \cup V_{n'} \,.$$

One can prove that $\mathcal{U}$ and $\mathcal{V}$ are filtered ordered sets and that

$$X = \varinjlim_{U \in \mathcal{U}} U \,, \qquad Y = \varinjlim_{V \in \mathcal{V}} V \,.$$

Then the result follows from the formulas

$$H^{k_1}_{dR,c}(X) = \varinjlim_{U \in \mathcal{U}} H^{k_1}_{dR,c}(U) \,, \; H^{k_2}_{dR,c}(Y) = \varinjlim_{V \in \mathcal{V}} H^{k_2}_{dR,c}(V) \,,$$

$$H^{k}_{dR,c}(X \times Y) = \varinjlim_{\substack{U \in \mathcal{U} \\ V \in \mathcal{V}}} H^{k}_{dR,c}(U \times V) \,.$$

Step 2: the cohomology class of the diagonal

Proposition:
Let $X =$ oriented compact differential manifold of dimension d.
Then the cohomology class

$$\mathrm{cl}_\Delta \in H^d_{dR}(X \times X) = \bigoplus_{0\leq k\leq d} H^{d-k}_{dR}(X) \otimes H^k_{dR}(X)$$

of the diagonal submanifold

$$\Delta : X \hookrightarrow X \times X$$

is the sum

$$\sum_{0\leq k\leq d} (-1)^k \cdot \sum_i \omega_i^* \otimes \omega_i$$

where, for any degree k,

- the family (ω_i) is a basis of the space $H^k_{dR}(X)$,
- the family (ω_i^*) is the dual basis of the space $H^{n-k}_{dR}(X)$ identified to the dual space $H^k_{dR}(X)^\vee$ by the pairing

$$\begin{array}{rcl} H^{d-k}_{dR}(X) \times H^k_{dR}(X) & \longrightarrow & \mathbb{R}\,, \\ (\ell, \omega) & \longmapsto & \int_X \ell \wedge \omega\,. \end{array}$$

Proof of the proposition:
For any basis element

$$\omega_{i_1} \otimes \omega_{i_2}^* \in H_{dR}^k(X) \otimes H_{dR}^{d-k}(X) \hookrightarrow H_{dR}^d(X \times X)\,,$$

we have by definition of the cohomology class of $X \overset{\Delta}{\hookrightarrow} X \times X$

$$\int_{X\times X} \mathrm{cl}_\Delta \wedge (\omega_{i_1} \otimes \omega_{i_2}^*) = \int_X \omega_{i_1} \wedge \omega_{i_2}^*$$

while

$$\begin{aligned} & \int_{X\times X} \left(\sum_{k'} (-1)^{k'} \cdot \sum_i \omega_i^* \otimes \omega_i \right) \wedge (\omega_{i_1} \otimes \omega_{i_2}^*) \\ = \;& \sum_i \left(\int_X \omega_i^* \wedge \omega_{i_1} \right) \cdot \left(\int_X \omega_i \wedge \omega_{i_2}^* \right) \\ = \;& \sum_i \left(\int_X \omega_{i_1} \wedge \omega_i^* \right) \cdot \left(\int_X \omega_{i_2}^* \wedge \omega_i \right) \\ = \;& \int_X \omega_{i_1} \wedge \omega_{i_2}^*\,. \end{aligned}$$

Step 3: conclusion of the proof of the Lefschetz formula

We have

$$\sum_{\substack{x\in X\\ f(x)=x}} \mathrm{sign}_f(x) = \int_{X\times X} \mathrm{cl}_f \cdot \mathrm{cl}_\Delta = \int_X (\mathrm{id}_X, f)^* \mathrm{cl}_\Delta$$

by definition of the cohomology class cl_f of $X \xhookrightarrow{(\mathrm{id}_X, f)} X \times X$.

Pulling back the formula for cl_Δ, we get

$$\int_X (\mathrm{id}_X, f)^* \mathrm{cl}_\Delta = \sum_{0\le k\le d} (-1)^k \cdot \sum_i \int_X \omega_i^* \wedge f^* \omega_i = \sum_{0\le k\le d} (-1)^k \cdot \mathrm{Tr}(f^*, H^k_{dR}(X)) \, .$$