Cohomology of toposes

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Chapter I:

Basic features of homology and cohomology

First examples: singular homology and cohomology

From topology or geometry to linear structures

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- ciao2

First vague definition:

Homology and cohomology
consist in associating
“linear invariants” to “geometric objects”.

More concretely:

- “Geometric objects” may be
  topological spaces (or more generally Grothendieck sites),
  differential manifolds,
  analytic manifolds,
  algebraic varieties over a field,
  schemes over a ring,
  Grothendieck sites endowed with an extra “geometric” structure.
- “Linear invariants” may be
  vector spaces over a coefficient field,
  modules over a coefficient ring (in particular, abelian groups),
  diagrams of vector spaces or modules and linear maps.

Why linear invariants?

Linear structures are extremely amenable to computation.
In particular, one can define:

Definition:

(i) A diagram of linear structures

$$A \xrightarrow{d} B \xrightarrow{d'} C$$

is called a sequence if

$$d' \circ d = 0$$

or, equivalently,

$$\mathrm{Im}(d) \subseteq \mathrm{Ker}(d') .$$

(ii) Such a sequence is called exact if, furthermore, there is an equivalence

$$\mathrm{Ker}(d') = \mathrm{Im}(d) .$$

(iii) In general, the difference between the implication

$$\mathrm{Im}(d) \subseteq \mathrm{Ker}(d')$$

and the corresponding equivalence is represented by the quotient linear structure

$$H = \mathrm{Ker}(d')/\mathrm{Im}(d) .$$

Remarks:

- A longer diagram of linear structures

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$$

is called a complex [resp. an exact complex] if each subdiagram of the form

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

is a sequence [resp. an exact sequence].

- Exact complexes of the form

$$0 \longrightarrow A \xrightarrow{d} B \xrightarrow{d'} C \longrightarrow 0$$

are also called "short exact sequences"
(characterized by $\mathrm{Ker}(d) = 0$, $\mathrm{Im}(d) = \mathrm{Ker}(d')$, $\mathrm{Im}(d') = C$).

- A complex of the form

$$\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots$$

[resp.

$$\cdots \longrightarrow X_{n-1} \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \cdots \ ]$$

is called a chain complex [resp. cochain complex].

What is the meaning of the word “invariant”?

First meaning of “invariant”:

Isomorphisms between geometric objects
should yield linear isomorphisms between their invariants.

Remark:

Here, the word “isomorphism” may mean
- homeomorphism between topological spaces,
- diffeomorphism between differential manifolds,
- holomorphic diffeomorphism between analytic manifolds,
- algebraic isomorphism between algebraic varieties or schemes,
- …

Corollary:

If two geometric objects have invariants which are not linearly isomorphic
(for instance vector spaces with different dimensions),
then they are not isomorphic.

Stronger form of the first meaning of "invariant":

All natural transforms between geometric objects should yield linear maps (or linear morphisms) between their invariants, in such a way that:

(1) The identity transform of any geometric object should yield the identity isomorphism of the associated linear invariant (or invariants).

(2) The composite of 2 natural transforms between geometric objects $X \to Y \to Z$ should yield the composite of the associated linear maps.

Remark:

Conditions (1) and (2) exactly say that invariants should make up a functor from the geometric category under consideration to a linear category.

Functors may respect the direction of arrows or reverse them:

Definition:

A cohomology [resp. homology] functor
is a contravariant [resp. equivariant] functor

$$
\begin{array}{rcrcll}
H & : & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{L} & [\text{resp. } \mathcal{C} \longrightarrow \mathcal{L}\ ] \\
 & & X & \longmapsto & H(X) & \\
 & & (X \xrightarrow{f} Y) & \longmapsto & (H(Y) \xrightarrow{f^*} H(X)) & [\text{resp. } H(X) \xrightarrow{f_*} H(Y)\ ]
\end{array}
$$

from a "geometric" category $\mathcal{C}$ to a linear category $\mathcal{L}$.

Remark:

The "geometric" category $\mathcal{C}$ can be

topological spaces $+$ continuous maps,
differential manifolds $+$ C^∞-maps,
analytic manifolds $+$ holomorphic maps,
algebraic varieties or schemes $+$ algebraic morphisms,
sites $+$ morphisms between the associated toposes,
ringed sites $+$ morphisms of ringed sites,
...

Corollary of this definition:

For any homology [resp. cohomology] functor

$$H : \mathcal{C} \longrightarrow \mathcal{L} \qquad [\text{resp. } \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{L}\ ],$$

and for any geometric object (or “space”)

$$X \in \mathrm{Ob}(\mathcal{C})\,,$$

the linear structure

$$H(X)$$

is endowed with an action of the group

$$\mathrm{Aut}(X) = \{\text{isomorphisms } X \xrightarrow{\sim} X\}$$

and even of the monoid

$$\mathrm{End}(X) = \{\text{morphisms } X \longrightarrow X\} \qquad [\text{resp. } \mathrm{End}(X)^{\mathrm{op}}\ ].$$

Second meaning of “invariant”:

The (co)homology invariants $H(X)$
of geometric objects (or spaces) X
can be defined or computed
in many different ways.

Remark:

Usually, (co)homology invariants are much easier to define than to compute.

The general definitions are well adapted to prove the (numerous and most important) formal properties of these invariants, but not to compute them.

First, let’s remark that in linear algebra one can associate to complexes of linear structures invariants which can be computed in many different ways:

Definition:

(i) The (co)homology invariants of a complex A of (co)chains

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

$$\text{[resp.} \qquad \cdots \longrightarrow A_{n-1} \xrightarrow{d_n} A_n \xrightarrow{d_{n+1}} A_{n+1} \longrightarrow \cdots \text{]}$$

are the quotients

$$H_n(A) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1})$$

$$\text{[resp.} \qquad H^n(A) = \mathrm{Ker}(d_{n+1})/\mathrm{Im}(d_n) \text{].}$$

(ii) A morphism of complexes of (co)chains A and B is a commutative diagram of linear morphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots & (A) \\
\cdots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots & (B)
\end{tikzcd}$$

It trivially induces a sequence of linear morphisms

$$H_n(A) \longrightarrow H_n(B) \qquad \text{[resp. } H^n(A) \longrightarrow H^n(B) \text{]}$$

indexed by the integers n.

(iii) Such a morphism of complexes

$$A \longrightarrow B$$

is called a "quasi-isomorphism" if all the induced linear morphisms between the (co)homology invariants of A and B

$$H_n(A) \longrightarrow H_n(B) \qquad [\text{resp. } H^n(A) \longrightarrow H^n(B)\ ]$$

are isomorphisms.

This general fact allows to formulate a much more precise definition of the notions of homology or cohomology:

Definition:

A (co)homology theory on a geometric category $\mathcal{C}$ is a sequence of functors

$$H_n : \mathcal{C} \longrightarrow \mathcal{L} \qquad [\text{resp. } H^n : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{L}\ ]$$

to a category $\mathcal{L}$ of linear structures, such that:

(i) For any space X of $\mathcal{C}$, the invariants

$$H_n(X) \qquad [\text{resp. } H^n(X)\ ]$$

are defined as the (co)homology invariants of some (co)chain complexes A which can be constructed from X.

These complexes A are not unique, but they are all related by natural quasi-isomorphisms which induce canonical isomorphisms of their (co)homology invariants.

(ii) For any morphism $X \to Y$ of $\mathcal{C}$, the induced linear morphisms

$$H_n(X) \longrightarrow H_n(Y) \qquad [\text{resp. } H^n(Y) \longrightarrow H^n(X)\,]$$

are induced by a natural morphism of co(chain) complexes

$$A \longrightarrow B \qquad [\text{resp. } B \longrightarrow A\,]$$

for some choice of complexes A and B which define the invariants $H_n(X)$ and $H_n(Y)$ [resp. $H^n(X)$ and $H^n(Y)$].

Third possible meaning of “invariant”:

In most concrete cohomology theories, the cohomology invariants $H^n(X)$ of geometric objects X remain unchanged under deformations of X.

Deformations mean homotopy equivalences:

Definition:
Let $\mathcal{C}$ be a geometric category such that

- products $X \times Y$ are well-defined in $\mathcal{C}$,
- $\mathcal{C}$ has a “point” (or “terminal”) object $\{\bullet\}$ such that any space X has a unique morphism
$$X \longrightarrow \{\bullet\}$$
and, as a consequence, $X \times \{\bullet\} = X$ for any X,
- $\mathcal{C}$ has a “line” object $\mathbb{L}$ endowed with 2 points
$$\{\bullet\} \xrightarrow{0} \mathbb{L}\,, \qquad \{\bullet\} \xrightarrow{1} \mathbb{L}\,,$$
and a morphism
$$\mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L}$$
whose composites with $0 \times \mathrm{id}_{\mathbb{L}}$ and $1 \times \mathrm{id}_{\mathbb{L}}$ are
$$\mathbb{L} \longrightarrow \{\bullet\} \xrightarrow{0} \mathbb{L} \quad \text{and} \quad \mathrm{id}_{\mathbb{L}} : \mathbb{L} \longrightarrow \mathbb{L}\,.$$

Then:

(i) Two morphisms of $\mathcal{C}$

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

are called "homotopic" (denoted $f \sim g$) if there exists a morphism

$$\mathbb{L} \times X \longrightarrow Y \qquad \text{(called a geometric homotopy)}$$

whose composites with $0 \times \mathrm{id}_X$ and $1 \times \mathrm{id}_X$ are f and g.

(ii) Two morphisms of $\mathcal{C}$

$$f : X \longrightarrow Y \quad \text{and} \quad g : Y \to X$$

are called an "homotopy equivalence" if

$$g \circ f \sim \mathrm{id}_X \quad \text{and} \quad f \circ g \sim \mathrm{id}_Y \,.$$

Remark:
In the situation of (ii), $Y \to X$ is called a retract if $f \circ g = \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.
In particular, $\{\bullet\} \xrightarrow{0} \mathbb{L}$ is a retract.

Examples:

The line object $\mathbb{L}$ usually is

$[0, 1]$ if $\mathcal{C} =$ topological spaces,

$\mathbb{R}$ (or any open interval) if $\mathcal{C} =$ diff. manifolds,

an open disc of $\mathbb{C}$ if $\mathcal{C} =$ analytic manifolds,

$\mathbb{A}^1 =$ affine line if $\mathcal{C} =$ algebraic varieties.

Definition: Let $\mathcal{C}$ be a geometric category with finite products, point object and line object. A cohomology theory consisting in a sequence of functors

$$H^n : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{L}$$

is called "homotopy invariant" if, for any pair of morphisms

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

which are homotopic, one has

$$H^n(f) = H^n(g) : H^n(Y) \longrightarrow H^n(X)\,, \quad \forall\, n\,.$$

Remark:
This implies that for any homotopy equivalence

$$\left(X \xrightarrow{f} Y, Y \xrightarrow{g} X\right)$$

the induced linear morphisms

$$H^n(X) \underset{H^n(g)}{\overset{H^n(f)}{\rightleftarrows}} H^n(Y)$$

are inverse to each other.
In particular, the $H^n(\mathbb{L})$ and $H^n(\{\bullet\})$ identify.

Let's recall that the invariants

$$H^n(X) \quad \text{and} \quad H^n(Y)$$

should be associated to two cochain complexes

$$A = (\cdots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow A_{n+1} \longrightarrow \cdots)$$

$$B = (\cdots \longrightarrow B_{n-1} \longrightarrow B_n \longrightarrow B_{n+1} \longrightarrow \cdots)$$

and the linear morphisms

$$H^n(f), H^n(g) : H^n(Y) \rightrightarrows H^n(X)$$

should be induced by two morphisms of complexes

$$B \rightrightarrows A.$$

The reason why we would have

$$H^n(f) = H^n(g), \qquad \forall\, n,$$

is that the geometric homotopy

$$f \sim g \quad \text{(consisting in a morphism } \mathbb{L} \times X \to Y)$$

should induce a cochain homotopy in the following sense:

Definition:
A cochain homotopy between two morphisms

$$f = (f_n : B_n \to A_n) \quad \text{and} \quad g = (g_n : B_n \to A_n)$$

from a cochain complex

$$B = (\cdots \longrightarrow B_{n-1} \xrightarrow{d_n} B_n \xrightarrow{d_{n+1}} B_{n+1} \longrightarrow \cdots)$$

to a cochain complex

$$A = (\cdots \longrightarrow A_{n-1} \xrightarrow{d_n} A_n \xrightarrow{d_{n+1}} A_{n+1} \longrightarrow \cdots)$$

is a sequence of linear morphisms

$$(h_n : B_n \longrightarrow A_{n-1})$$

such that, for any n,

$$g_n - f_n = d_n \circ h_n + h_{n+1} \circ d_{n+1} \,.$$

Remark:
Replacing the indices n by $-n$ defines the notion of chain homotopy.

Lemma:

For any pair of morphisms of (co)chain complexes

$$f, g : B \rightrightarrows A$$

which are related by a (co)chain homotopy, the induced linear morphisms

$$H^n(f), H^n(g) : H^n(B) \longrightarrow H^n(A)$$

are all equal.

Variations of (co)homology invariants

Spaces in a geometric category $\mathcal{C}$ are not only related by space morphisms but also by variations in continuous families:

Definition:

Let $\mathcal{C}$ = geometric category with a "point" (or "terminal") object $\{\bullet\}$,
$\begin{pmatrix} X \\ \downarrow \\ S \end{pmatrix}$ = morphism of $\mathcal{C}$ over a base space S which is "continuous" (in the sense that it cannot be written as a disjoint union of subspaces).
If, for any point $\{\bullet\} \xrightarrow{s} S$ of S, the fiber

$$X_s = \{\bullet\} \times_S X$$

is well-defined as an object of $\mathcal{C}$, the family of these fibers X_s can be called a "continuous" family of spaces in $\mathcal{C}$.

Remark:

The conditions of this definition are fulfilled if

- $\mathcal{C}$ is the category of topological spaces,
- $\mathcal{C}$ is the category of algebraic varieties over some base field,
- $\mathcal{C}$ is the category of differential [resp. analytic] manifolds and the morphism $X \to S$ is submersive (or smooth), meaning that for some choices of coordinates it locally looks like the projection of some product

$$\mathbb{R}^m \times \mathbb{R}^n \qquad \text{[resp. } \mathbb{C}^m \times \mathbb{C}^n \text{]}$$

on $\mathbb{R}^m$ [resp. $\mathbb{C}^m$].

This definition raises the following question:

Question:

In the situation of the definition, suppose $\mathcal{C}$ is endowed with (co)homology functors

$$H^n : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{L} \quad \text{or} \quad H_n : \mathcal{C} \longrightarrow \mathcal{L} \, .$$

Then, how do the invariants of the fibers

$$H^n(X_s) \quad \text{or} \quad H_n(X_s)$$

vary with the parameters s?

Remark:

It is generally true that the $H^n(X_s)$ or $H_n(X_s)$ vary nicely (for instance, if they are vector spaces, they have the same dimensions) if the morphism $X \to S$ verifies

- a global condition: it is compact (or "proper") in some sense,
- a local condition: it is smooth (excluding singularities) in some sense.

Warning:

Even if the invariant (co)homology spaces

$$H^n(X_s) \qquad \text{or} \qquad H_n(X_s)$$

have the same dimensions,
they are usually different when considered as representations of the groups

$$\mathrm{Aut}(X_s)\,.$$

Singular homology and cohomology:

We are going to define a first homology and cohomology theory for topological spaces.

The construction consists in several steps:

- defining the simplicial category Δ and the category $\widehat{\Delta}$ of simplicial sets,
- defining a functor
$$\mathrm{Top} \longrightarrow \widehat{\Delta}$$
from topological spaces to simplicial sets,
- defining a functor
$$\widehat{\Delta} \longrightarrow C_R$$
to the category C_R of chain complexes of R-modules (for some coefficient ring R),
- taking the (co)homology invariants of these chain complexes and their dual cochain complexes.

Simplicial sets:

Definition:

Let Δ be the "simplicial category" defined in the following way:

- the objects of Δ are the sets

$$[n] = \{0, 1, \ldots, n\}, \qquad n \in \mathbb{N},$$

- the morphisms of Δ are the strictly increasing maps

$$[m] = \{0, 1, \ldots, m\} \longrightarrow \{0, 1, \ldots, n\} = [n] .$$

Definition:

A simplicial set is a contravariant functor

$$\Delta^{\mathrm{op}} \longrightarrow \mathrm{Set} .$$

The category of simplicial sets is denoted

$$\widehat{\Delta} .$$

Remark: This definition can be made explicit:

- A simplicial set $X_\bullet$ consists in a sequence of sets
$$X_n\,,\quad n\in\mathbb{N}\,,$$
related by maps
$$X_\sigma : X_n \longrightarrow X_m$$
indexed by strictly increasing maps
$$\sigma : \{0,1,\ldots,m\} \longrightarrow \{0,1,\ldots,n\}$$
and such that
$$X_\tau \circ X_\sigma = X_{\sigma\circ\tau} \text{ for any } \{0,1,\ldots,\ell\} \xrightarrow{\tau} \{0,1,\ldots,m\} \xrightarrow{\sigma} \{0,1,\ldots,n\}\,,$$
$$X_{\mathrm{id}} = \mathrm{id}_{X_n} \text{ for any } n\in\mathbb{N}\,.$$
- A morphism of simplicial sets
$$X_\bullet \longrightarrow Y_\bullet$$
is a sequence of maps
$$\alpha_n : X_n \longrightarrow Y_n\,,\quad n\in\mathbb{N}\,,$$
making commutative all diagrams:
$$\begin{tikzcd} X_n \arrow[r, "\alpha_n"] \arrow[d, "X_\sigma"'] & Y_n \arrow[d, "Y_\sigma"] \\ X_m \arrow[r, "\alpha_m"] & Y_m \end{tikzcd}$$

From topological spaces to simplicial sets:

Definition:
Let $\Delta_\bullet$ be the "geometric realization functor"

$$\Delta \longrightarrow \mathrm{Top}$$

defined in the following way:

- For any $n \in \mathbb{N}$, $\Delta_\bullet([n]) = \Delta_n$ is the simplex

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leq x_1 \leq \ldots \leq x_n \leq 1\}$$

  also defined as the convex hull of the points

$$P_i^n = (\underbrace{0, \ldots, 0}_{i \text{ times}}, \underbrace{1, \ldots, 1}_{n-i \text{ times}}), \quad 0 \leq i \leq n.$$

- For any increasing map

$$\sigma : \{0, 1, \ldots, m\} \longrightarrow \{0, 1, \ldots, n\},$$

$$\Delta_\sigma : \Delta_m \longrightarrow \Delta_n$$

  is the unique affine map such that

$$\Delta_\sigma(P_i^m) = P_{\sigma(i)}^n, \quad 0 \leq i \leq m.$$

Definition: Let

$$S : \mathrm{Top} \longrightarrow \widehat{\Delta}$$

be the simplicialization functor defined as

$$\underset{\text{topological space}}{X} \longmapsto \underset{\text{simplicial set}}{X_\bullet = \mathrm{Hom}(\Delta_\bullet, X)}$$

where:

- for any $n \in \mathbb{N}$,

  $$X_n = \mathrm{Hom}(\Delta_n, X)$$

  is the set of continuous maps

  $$\Delta_n \xrightarrow{x} X\,,$$

- for any increasing map $\sigma : [m] \to [n]$,

  $$X_\sigma : X_n \longrightarrow X_m$$

  is the map defined by composition with $\Delta_\sigma : \Delta_n \to \Delta_m$

  $$(\Delta_n \xrightarrow{x} X) \longmapsto (\Delta_m \xrightarrow{x \circ \Delta_\sigma} X)\,.$$

From simplicial sets to chain complexes

Definition:

Let $R =$ (commutative) ring,
$\mathrm{Mod}_R =$ category of R-modules and R-linear maps.

The "free R-modules" functor is

$$\begin{array}{ccc} \mathrm{Set} & \longrightarrow & \mathrm{Mod}_R\,, \\ I & \longmapsto & \bigoplus\limits_{i \in I} R = R^{(I)}\,. \end{array}$$

Remark:

The "free R-modules" functor is left-adjoint to the "forgetful" functor

$$\begin{array}{ccc} \mathrm{Mod}_R & \longrightarrow & \mathrm{Set}\,, \\ M & \longmapsto & \text{underlying set } M\,. \end{array}$$

Corollary:

The "free R-modules" functor defines a functor

$$\widehat{\Delta} = [\Delta^{\mathrm{op}}, \mathrm{Set}] \longrightarrow [\Delta^{\mathrm{op}}, \mathrm{Mod}_R]$$

to the category

$$[\Delta^{\mathrm{op}}, \mathrm{Mod}_R]$$

of contravariant functors

$$\begin{array}{rrcl} M_\bullet \quad : & \Delta^{\mathrm{op}} & \longrightarrow & \mathrm{Mod}_R\,, \\ & [n] & \longmapsto & M_n \\ & ([m] \xrightarrow{\sigma} [n]) & \longmapsto & (M_n \xrightarrow{M_\sigma} M_m)\,. \end{array}$$

Remark:

For any $n \geq 1$, there are exactly $n+1$ strictly increasing maps

$$[n-1] = \{0, 1, \ldots, n-1\} \longrightarrow \{0, 1, \ldots, n\} = [n] \, .$$

They are denoted

$$\partial_n^i \, , \qquad 0 \leq i \leq n \, ,$$

with

$$\mathrm{Im}(\partial_n^i) = \{0, 1, \ldots, n\} - \{i\} \, .$$

The associated affine maps

$$\Delta_{n-1} \longrightarrow \Delta_n$$

are called the faces of Δ_n.

Proposition:
Let $C_R =$ category of chain complexes of R-modules indexed by integers $n \in \mathbb{N}$ (and completed by 0 for $n < 0$).
Then there is a functor

$$[\Delta^{\mathrm{op}}, \mathrm{Mod}_R] \longrightarrow C_R$$

which associates to any contravariant functor

$$M_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Mod}_R$$

the chain complex

$$\cdots \longrightarrow M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{d_1} M_0 \longrightarrow 0$$

where , for any $n \geq 1$,

$$d_n = \sum_{0 \leq i \leq n} (-1)^i \cdot M_{\partial_n^i} .$$

Proof:
Any increasing map $\{0, \ldots, n-1\} \to \{0, \ldots, n\}$ has an image of the form $\{0, \ldots, n\} - \{i_1, i_2\}$ with $i_1 < i_2$. It admits exactly 2 decompositions

$$\partial_n^{i_2} \circ \partial_{n-1}^{i_1} \qquad \text{and} \qquad \partial_n^{i_1} \circ \partial_{n-1}^{i_2 - 1}$$

with opposite associated signs.

Definition:

The composed functor

$$\mathrm{Top} \longrightarrow \widehat{\Delta} = [\Delta^{\mathrm{op}}, \mathrm{Set}] \longrightarrow [\Delta^{\mathrm{op}}, \mathrm{Mod}_R] \longrightarrow C_R$$

associates to any topological space X a chain complex

$$C_\bullet^X$$

and to any continuous map $X \to Y$ a morphism of complexes

$$C_\bullet^X \longrightarrow C_\bullet^Y .$$

The associated homology invariants are called the singular homology functors (with coefficients in R)

$$\begin{array}{rcl} \mathrm{Top} & \longrightarrow & \mathrm{Mod}_R \,, \\ X & \longmapsto & H_n(X, R) \,. \end{array}$$

As $R =$ commutative ring, there is a contravariant functor

$$\begin{array}{rcl} \mathrm{Mod}_R^{\mathrm{op}} & \longrightarrow & \mathrm{Mod}_R \,, \\ M & \longmapsto & \mathrm{Hom}(M, R) \,. \end{array}$$

It induces a contravariant functor

$$C_R^{\mathrm{op}} \longrightarrow C_R^V$$

from the category C_R of chain complexes to the category C_R^V of cochain complexes (indexed by integers $n \in \mathbb{N}$).

Definition: The composed functor

$$\mathrm{Top}^{\mathrm{op}} \longrightarrow C_R^V$$

of $\mathrm{Top} \to C_R$ and $C_R^{\mathrm{op}} \to C_R^V$ associates to any topological space X a cochain complex

$$C_X^\bullet$$

and to any continuous map $X \to Y$ a morphism of complexes

$$C_Y^\bullet \longrightarrow C_X^\bullet \,.$$

The associated cohomology invariants are called the singular cohomology functors (with coefficients in R)

$$\begin{array}{rcl} \mathrm{Top}^{\mathrm{op}} & \longrightarrow & \mathrm{Mod}_R \,, \\ X & \longmapsto & H^n(X, R) \,. \end{array}$$

Remarks:

(i) For any free module

$$M = R^{(I)} = \bigoplus_{i \in I} R\,,$$

its dual module $\mathrm{Hom}(M, R)$ identifies with the product

$$\prod_{i \in I} R = R^I\,.$$

This applies in particular to all components

$$C_n^X = \bigoplus_{x \in \mathrm{Hom}(\Delta_n, X)} R$$

of the chain complex $C_\bullet^X$ associated to a topological space X.

(ii) If R is a field, the dualizing functor

$$\begin{array}{rcl} \mathrm{Mod}_R^{\mathrm{op}} & \longrightarrow & \mathrm{Mod}_R\,, \\ M & \longmapsto & \mathrm{Hom}(M, R) \end{array}$$

is "exact" in the sense that it respects exact complexes.
This implies that for any topological space X

$$H^n(X, R) = \mathrm{Hom}(H_n(X, R), R)\,.$$

Homotopy invariance for singular homology and cohomology

Theorem:

Let $X, Y =$ topological spaces
$f, g : X \rightrightarrows Y =$ continuous maps

which are homotopic in the sense that there exists a continuous map

$$\begin{array}{rcl} h : [0,1] \times X & \longrightarrow & Y\,, \\ (t,x) & \longmapsto & h(t,x) \end{array}$$

such that $h(0, \bullet) = f$ and $h(1, \bullet) = g$.

Then the associated chain complex morphisms

$$f_*, g_* : C_\bullet^X \rightrightarrows C_\bullet^Y$$

are chain homotopic.

Remark:

As a consequence, the cochain complex morphisms

$$f^*, g^* : C_Y^\bullet \longrightarrow C_X^\bullet$$

are cochain homotopic.

Corollary:

For any pair of continuous maps

$$f, g : X \rightrightarrows Y$$

which are homotopic,

the associated linear maps

$$f_*, g_* : H_n(X, R) \rightrightarrows H_n(Y, R)$$

or

$$f^*, g^* : H^n(Y, R) \rightrightarrows H^n(X, R)$$

are equal.

Proof of the theorem: We have to define a sequence of linear maps

$$h_n : C_n^X \longrightarrow C_{n+1}^Y , \qquad n \in \mathbb{N} ,$$

such that for any n

$$f_n - g_n : C_n^X \longrightarrow C_n^Y$$

is equal to

$$d_{n+1} \circ h_n + h_{n-1} \circ d_n .$$

Let's recall that C_n^X is the free R-module on the set of continuous maps

$$x \in \Delta_n \longrightarrow X$$

and C_{n+1}^Y is the free R-module on the set of continuous maps

$$y : \Delta_{n+1} \longrightarrow Y .$$

Then one can define h_n by

$$x \longmapsto \sum_{1 \leq i \leq n+1} (-1)^{i-1} y_{x,i}$$

where each $y_{x,i} : \Delta_{n+1} \to Y$ is

$$\Delta_{n+1} = \{(t_1, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} \mid 0 \leq t_1 \leq \ldots \leq t_{n+1} \leq 1\} \longrightarrow Y ,$$

$$(t_1, \ldots, t_{n+1}) \longmapsto h(t_i, x(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n+1})) .$$

Indeed, $d_{n+1} \circ h_n + h_{n-1} \circ d_n$ sends each basis vector

$$x : \Delta_n \longrightarrow X$$

to the sum of the basis vectors

$$f \circ x = \begin{cases} \Delta_n & \longrightarrow & Y\,, \\ (t_1, \ldots, t_n) & \longmapsto & h(0, x(t_1, \ldots, t_n)) \end{cases}$$

with coefficient 1,

$$g \circ x = \begin{cases} \Delta_n & \longrightarrow & Y\,, \\ (t_1, \ldots, t_n) & \longmapsto & h(1, x(t_1, \ldots, t_n)) \end{cases}$$

with coefficient $(-1)^n \cdot (-1)^{n+1} = -1$,

$$\begin{cases} \Delta_n & \longrightarrow & Y\,, \\ (t_1, \ldots, t_n) & \longmapsto & h(t_i, x(t_1, \ldots, t_n)) \end{cases}$$

with coefficient $(-1)^{i-1} \cdot (-1)^{i-1} + (-1)^{i-1} \cdot (-1)^i = 0$,

$$\begin{cases} \Delta_n & \longrightarrow & Y\,, \\ (t_1, \ldots, t_n) & \longmapsto & h(t_i, x(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_j, t_j, \ldots, t_n)) \end{cases}$$

with coefficient $(-1)^{i-1} \cdot (-1)^j + (-1)^{i-1} \cdot (-1)^{j-1} = 0$
and

$$\begin{cases} \Delta_n & \longrightarrow & Y\,, \\ (t_1, \ldots, t_n) & \longmapsto & h(t_i, x(t_1, \ldots, t_j, t_j, \ldots t_{i-1}, t_{i+1}, \ldots, t_n) \end{cases}$$

with coefficient $(-1)^{i-2} \cdot (-1)^j + (-1)^{i-1} \cdot (-1)^j = 0$.

Corollary of the theorem:

For any topological space X which is contractible, i.e. homotopic to $\{\cdot\}$, we have

$$H_n(X) = \begin{cases} R & \text{if} \quad n = 0\,, \\ 0 & \text{if} \quad n \geq 1\,, \end{cases}$$

and

$$H^n(X) = \begin{cases} R & \text{if} \quad n = 0\,, \\ 0 & \text{if} \quad n \geq 1\,. \end{cases}$$

Remark:

This applies in particular to

$\mathbb{R}^m$ and any ball of $\mathbb{R}^m$.

Compatibility of singular (co)homology with open covers

Theorem:

Let $X =$ topological space,
$\quad\mathcal{U} =$ open cover of X
$\quad\quad= $ family of open subsets $(U_i)_{i\in I}$ such that

$$X = \bigcup_{i\in I} U_i\,.$$

Let $C_\bullet^{X,\mathcal{U}} =$ subcomplex of $C_\bullet^X$ consisting in the free R-modules $C_n^{X,\mathcal{U}}$ on the families of continuous maps $\Delta_n \to X$ whose image is contained in one of the U_i's.

Then the embedding chain morphism

$$C_\bullet^{X,\mathcal{U}} \xrightarrow{i} C_\bullet^X$$

is a chain retract, which means that there exists a chain morphism

$$C_\bullet^X \xrightarrow{r} C_\bullet^{X,\mathcal{U}}$$

such that $r \circ i = \mathrm{id}$ and $i \circ r$ is chain homotopic to i.

Corollary:

(i) The chain morphism

$$C_\bullet^{X,\mathcal{U}} \longrightarrow C_\bullet^X$$

is a quasi-isomorphism.

(ii) If $C^\bullet_{X,\mathcal{U}}$ is the cochain complex defined as the transform of $C_\bullet^{X,\mathcal{U}}$ by $\mathrm{Hom}(\bullet, R)$, then the projection cochain morphism

$$p : C^\bullet_X \longrightarrow C^\bullet_{X,\mathcal{U}}$$

has a section

$$s : C^\bullet_{X,\mathcal{U}} \longrightarrow C^\bullet_X$$

such that

$$p \circ s = \mathrm{id}$$

and $s \circ p$ is cochain homotopic to id.

(iii) The cochain morphism

$$p : C^\bullet_X \longrightarrow C^\bullet_{X,\mathcal{U}}$$

is a quasi-isomorphism.

Consider the case where the open cover $\mathcal{U}$ of X consists in two elements U_1, U_2. Then the embeddings

$$\begin{tikzcd} & U_1 \arrow[dr, hook] & \\ U_1 \cap U_2 \arrow[ur, hook] \arrow[dr, hook] & & X = U_1 \cup U_2 \\ & U_2 \arrow[ur, hook] & \end{tikzcd}$$

induce two chain morphisms

$$C_\bullet^{U_1 \cap U_2} \longrightarrow C_\bullet^{U_1} \oplus C_\bullet^{U_2} \longrightarrow C_\bullet^{X,\mathcal{U}}$$

such that, for any n,

$$0 \longrightarrow C_n^{U_1 \cap U_2} \longrightarrow C_n^{U_1} \oplus C_n^{U_2} \longrightarrow C_n^{X,\mathcal{U}} \longrightarrow 0$$

is a short exact sequence.
We also have dual cochain morphisms

$$C^\bullet_{X,\mathcal{U}} \longrightarrow C^\bullet_{U_1} \oplus C^\bullet_{U_2} \longrightarrow C^\bullet_{U_1 \cap U_2}$$

and, as the $C_n^{X,\mathcal{U}}$ are free R-modules, each

$$0 \longrightarrow C^n_{X,\mathcal{U}} \longrightarrow C^n_{U_1} \oplus C^n_{U_2} \longrightarrow C^n_{U_1 \cap U_2} \longrightarrow 0$$

is a short exact sequence.

By now, we can apply the most important

Lemma ("snake lemma"):
Let $A, B, C =$ three chain [resp. cochain] complexes of R-modules related by two morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that, for any n,

$$0 \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow 0$$

$$[\text{resp.} \qquad 0 \longrightarrow A^n \xrightarrow{f_n} B^n \xrightarrow{g_n} C^n \longrightarrow 0\ ]$$

is a short exact sequence.
Then there are natural morphisms

$$H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \qquad [\text{resp. } H^n(C) \xrightarrow{\partial_{n+1}} H^{n+1}(A)\ ]$$

such that the sequences

$$\longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow \cdots$$

$$[\text{resp.} \quad \longrightarrow H^n(A) \longrightarrow H^n(B) \longrightarrow H^n(C) \longrightarrow H^{n+1}(A) \longrightarrow H^{n+1}(B) \longrightarrow \cdots\ ]$$

are long exact sequences.

We deduce from the lemma and the corollary:

Proposition:

Let $X =$ topological space,
$U_1, U_2 =$ two open subsets such that $X = U_1 \cup U_2$.

Then there are natural long exact sequences

$$\longrightarrow H_n(U_1 \cap U_2, R) \longrightarrow H_n(U_1, R) \oplus H_n(U_2, R) \longrightarrow H_n(X, R)$$
$$\longrightarrow H_{n-1}(U_1 \cap U_2, R) \longrightarrow H_{n-1}(U_1, R) \oplus H_{n-1}(U_2, R) \longrightarrow \cdots$$

and

$$\longrightarrow H^n(X, R) \longrightarrow H^n(U_1, R) \oplus H^n(U_2, R) \longrightarrow H^n(U_1 \cap U_2, R)$$
$$\longrightarrow H^{n+1}(X, R) \longrightarrow H^{n+1}(U_1, R) \oplus H^{n+1}(U_2, R) \longrightarrow \cdots$$

This proposition implies:

Corollary:

Let $X =$ topological space
which has a finite open cover
$X = U_1 \cup \ldots \cup U_m$
by open subsets U_i which are contractible
as well as their intersections $U_{i_1} \cap \cdots \cap U_{i_k}$.

Then, the (co)homology R-modules

$$H_n(X, R) \qquad \text{and} \qquad H^n(X, R)$$

are 0 for $n > m$ and they are finitely presented if R is a nœtherian ring.

In particular, they are finite-dimensional if R is a field.

Sketch of proof of the theorem:

Step 1: Definition of barycentric subdivisions

For any points $P_0, P_1, \ldots, P_n \in \mathbb{R}^m$, there is a unique affine map

$$\Delta_n \longrightarrow \mathbb{R}^m$$

sending $P_i^n = (\underbrace{0,\ldots,0}_{i \text{ times}}, \underbrace{1,\ldots,1}_{n-i \text{ times}})$ to P_i, $0 \leq i \leq n$.

It can be denoted $\ell_{P_0,P_1,\ldots,P_n}$.

The notion of barycentric subdivision of $\ell_{P_0,P_1,\ldots,P_n}$ is defined by induction on n.

If $n = 0$, it is $\ell = \ell_{P_0}$.

If $n \geq 1$, it is a linear map of the form

$$\ell_{Q_0,\ldots,Q_n} : \Delta_n \longrightarrow \mathbb{R}^m$$

where

$$Q_0 = \frac{P_0 + P_1 + \ldots + P_n}{n+1},$$

and

$$\ell_{Q_1,\ldots,Q_n} : \Delta_{n-1} \longrightarrow \mathbb{R}^m$$

is a barycentric subdivision of a face of $\ell_{P_0,P_1,\ldots,P_n}$.

Step 2: Estimation of the size of barycentric subdivisions

Denoting

$$\|\ell_{P_0,\ldots,P_n}\| = \max_{t,t'\in\Delta_n} \|\ell_{P_0,\ldots,P_n}(t) - \ell_{P_0,\ldots,P_n}(t')\|$$

for some choice of norm $\|\cdot\|$ on $\mathbb{R}^m$,

check by induction on n that

$$\|\ell_{Q_0,\ldots,Q_n}\| \leq \frac{n}{n+1} \cdot \|\ell_{P_0,\ldots,P_n}\|$$

for any barycentric subdivision

$$\ell_{Q_0,\ldots,Q_n} \qquad \text{of} \qquad \ell_{P_0,\ldots,P_n}\,.$$

Step 3: Subdivision and homotopy

For $X =$ topological space, construct a chain complex morphism

$$s = (s_n : C_n^X \longrightarrow C_n^X)_{n\in\mathbb{N}}$$

and a chain homotopy

$$(h_n : C_n^X \longrightarrow C_{n+1}^X)_{n\geq -1}$$

such that

$$\mathrm{id} - s_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n \,, \qquad \forall\, n\,,$$

and, for any basis vector of C_n^X

$$x : \Delta_n \longrightarrow X\,,$$

$s_n(x)$ is a linear combination of basis vectors of the form

$$x \circ \ell : \Delta_n \longrightarrow X$$

where each ℓ is a barycentric subdivision of

$$\mathrm{id}_{\Delta_n} : \Delta_n \longrightarrow \Delta_n \subset \mathbb{R}^n\,.$$

Step 4: Iteration of subdivisions

Let X = topological space,
$\mathcal{U} = (U_i)_{i \in I}$ = open cover of X.

Deduce from Step 2 that for any basis vector

$$x : \Delta_n \longrightarrow X\,,$$

there exists an integer $N \in \mathbb{N}$ such that

$$(\underbrace{s_n \circ \ldots \circ s_n}_{N \text{ times}})(x)$$

is a linear combination of basis vectors

$$x \circ \ell : \Delta_n \longrightarrow X$$

whose images $x \circ \ell(\Delta_n)$ are all contained in at least one of the U_i's.

Step 5: End of the construction

Using Step 3 and Step 4, construct a chain complex morphism

$$r = (r_n : C_n^X \longrightarrow C_n^X)_{n\in\mathbb{N}}$$

and a chain homotopy

$$(h'_n : C_n^X \longrightarrow C_{n+1}^X)_{n\geq -1}$$

such that

$$\mathrm{id} - r_n = d_{n+1} \circ h'_n + h'_{n-1} \circ d_n\,, \quad \forall\, n\,,$$

and, for any basis vector

$$x : \Delta_n \longrightarrow X\,,$$

$r_n(x)$ is a linear combination of basis vectors

$$x \circ \ell : \Delta_n \longrightarrow X$$

whose images are contained in at least one of the U_i's and, furthermore,

$$r_n(x) = x$$

if (and only if) the image of x is contained in one of the U_i's.

Geometric realization of simplicial sets

Definition:

For any simplicial set

$$X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}\,,$$

its geometric realization $|X_\bullet|$ is defined as the colimit in Top

$$\varinjlim_{\int X_\bullet} \Delta_\bullet\,.$$

where: $\int X_\bullet$ is the category defined as

$$\begin{cases} \text{objects} = \text{pairs } ([n], x) \text{ with } x \in X_n, \\ \text{morphisms } ([m], y) \to ([n], x) \text{ are morphisms} \\ \qquad\qquad \sigma : [m] \to [n] \text{ of } \Delta \text{ such that } X_\sigma(x) = y, \end{cases}$$

and $\Delta_\bullet$ is the functor

$$\begin{array}{rcl} \int X_\bullet & \longrightarrow & \mathrm{Top}\,, \\ ([n], s) & \longmapsto & \Delta_n\,, \\ (([m], y) \to ([n], x)) & \longmapsto & \Delta_m \xrightarrow{X_\sigma} \Delta_n\,. \end{array}$$

Remarks:

(i) The functor

$$\begin{array}{ccc} \widehat{\Delta} & \longrightarrow & \mathrm{Top}\,, \\ X_\bullet & \longmapsto & |X_\bullet| \end{array}$$

is left adjoint to the functor

$$\begin{array}{ccc} \mathrm{Top} & \longrightarrow & \widehat{\Delta}\,, \\ X & \longmapsto & ([n] \longmapsto \mathrm{Hom}(\Delta_n, X))\,. \end{array}$$

In particular, it respects arbitrary colimits.
It can be proved that it also respects finite limits.

(ii) A topologic space which is homeomorphic to some

$$|X_\bullet|$$

is called triangulable,
and such a homeomorphism is called a triangulation of X.
Of course, if a triangulation exists, it is not unique.
It can be proved that differential manifolds always admit triangulations.

For any simplicial set $X_\bullet$ defines an associated chain complex

$$R^{(X_\bullet)} = (C_n^{X_\bullet} = R^{(X_n)})_{n\in\mathbb{N}}$$

and also the singular chain complex of $|X_\bullet|$

$$(C_n^{|X_\bullet|} = R^{(\mathrm{Hom}(\Delta_n, |X_\bullet|))})_{n\in\mathbb{N}}\,.$$

By definition of $|X_\bullet|$, each element $x \in X_n$ defines a continuous map

$$\Delta_n \longrightarrow |X_\bullet|$$

associated to the object $([n], x)$ of $\int X_\bullet$.
This defines a family of linear maps

$$C_n^{X_\bullet} \longrightarrow C_n^{|X_\bullet|}\,, \quad n \in \mathbb{N}\,,$$

which form a chain complex morphism.
Applying the functor $\mathrm{Hom}(\bullet, R)$, we also get a cochain complex morphism

$$(C^n_{|X_\bullet|} \longrightarrow C^n_{X_\bullet})_{n\in\mathbb{Z}}\,.$$

Theorem:

For any simplicial set $X_\bullet$, the natural chain and cochain morphisms

$$(C_n^{X_\bullet} \longrightarrow C_n^{|X_\bullet|})_{n\in\mathbb{N}}\,,$$

$$(C^n_{|X_\bullet|} \longrightarrow C^n_{X_\bullet})_{n\in\mathbb{N}}$$

are quasi-isomorphisms.

As a consequence, any triangulation of a topological space X

$$|X_\bullet| \xrightarrow{\sim} X$$

allows to compute the singular (co)homology of X
as the (co)homology of the indexing simplicial set $X_\bullet$.

Corollary:

Let $X =$ topological space
which admits a triangulation
$$|X_\bullet| \xrightarrow{\sim} X$$
by some simplicial set $X_\bullet$ verifying
$$X_n = \emptyset \text{ if } n > m$$
and $X_n =$ finite set, $\forall\, n$.

Then, the (co)homology R-modules

$$H_n(X,R) \qquad \text{and} \qquad H^n(X,R)$$

are 0 for $n > m$ and they are finitely presented if R is nœtherian.

In particular, they are finite-dimensional if R is a field.

Remark:

The alternate sum of the dimensions

$$\sum_{0 \le n \le m} (-1)^n \cdot \dim H^n(X) = \sum_{0 \le n \le m} (-1)^n \cdot (\# X_n)$$

is called the Euler-Poincaré characteristic.

Corollary:

(i) The singular (co)homology modules

$$H_n(S^d, R) \qquad \text{or} \qquad H^n(S^d, R)$$

of a sphere of dimension $d \geq 1$

$$S^d = \{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1} \mid x_0^2 + \cdots + x_d^2 = 1\}$$

are

$$\begin{cases} R & \text{for } n = 0 \text{ and } n = d, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Any continuous map

$$B^d \longrightarrow B^d$$

from a closed ball of dimension $d \geq 1$

$$B^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1^2 + \cdots + x_d^2 \leq 1\}$$

to itself has at least one fixed point.

Proof of (ii) from (i):
Suppose that there exists a continuous map

$$f : B^d \longrightarrow B^d$$

without any fixed point.
Then one can associate to any element $x \in B^d$ the intersection point $r(x)$ of $S^{d-1} \subset B^d$ and the half-line from $f(x)$ to x.
This defines a continuous map

$$r : B^d \longrightarrow S^{d-1}$$

such that $r \circ i = \mathrm{id}$ for $i : S^{d-1} \hookrightarrow B^d$.
Furthermore, the map

$$\begin{array}{ccc} [0,1] \times B^d & \longrightarrow & B^d \\ (t,x) & \longmapsto & t \cdot r(x) + (1-t) \cdot x \end{array}$$

is a homotopy from id to $i \circ r$.
So S^{d-1} would be a retract of B^d.
This is impossible as their singular homology invariants are different.

Proof of (i) from the theorem:

The boundary $\partial\Delta_{d+1}$ of Δ_{d+1} is homeomorphic to S^d.

Let $\Delta_\bullet^{d+1}$ = canonical (finite) triangulation of Δ_{d+1},
$\partial\Delta_\bullet^{d+1}$ = canonical (finite) triangulation of $\partial\Delta_{d+1}$.

Then $\partial\Delta_\bullet^{d+1}$ is a simplicial subset of $\Delta_\bullet^{d+1}$ and the maps

$$\partial\Delta_n^{d+1} \longrightarrow \Delta_n^{d+1}$$

are one-to-one except for $n = d+1$ where

$$\partial\Delta_{d+1}^{d+1} = \emptyset \qquad \text{and} \qquad \Delta_{d+1}^{d+1} = \{\bullet\}.$$

The associated morphism of chain complexes

$$R^{(\partial\Delta_\bullet^{d+1})} \longrightarrow R^{\Delta_\bullet^{d+1}}$$

is an isomorphism at all degrees n except $n = d+1$ where it is

$$0 \longrightarrow R\,.$$

As Δ_{d+1} is contractible, the associated long exact sequence of homology decomposes into short exact sequences

$$
\begin{array}{ll}
0 \longrightarrow H_n(S^d, R) \longrightarrow 0 & \text{for } n > d, \\
0 \longrightarrow R \longrightarrow H_d(S^d, R) \longrightarrow 0\,, & \\
0 \longrightarrow H_n(S^d, R) \longrightarrow 0 & \text{for } 0 < n < d, \\
0 \longrightarrow H_0(S^d, R) \longrightarrow H_0(\Delta^d, R) = R \longrightarrow 0\,. &
\end{array}
$$

Sketch of proof of the theorem:

Step 1: The case of Δ_d and its triangulation $\Delta_\bullet^d$

The component Δ_0^d of $\Delta_\bullet^d$ is the set $\{P_0, P_1, \ldots, P_d\}$ of the vertices of Δ_d and, for $n > 0$, Δ_n^d is the set of its n-dimensional faces denoted

$$(P_{i_0}, P_{i_1}, \ldots, P_{i_n}) \qquad \text{with} \qquad 0 \leq i_0 < i_1 < \cdots < i_n \leq d\,.$$

The chain complex $R^{(\Delta_\bullet^0)}$ is R in degree 0 and 0 in degrees $n > 0$.
Consider the two chain complex morphisms

$$R^{(\Delta_\bullet^0)} \xrightarrow{i_\bullet} R^{(\Delta_\bullet^d)} \qquad \text{and} \qquad R^{(\Delta_\bullet^d)} \xrightarrow{r_\bullet} R^{(\Delta_\bullet^0)}$$

defined in degree 0 by

$$i_0(1) = P_0 \qquad \text{and} \qquad r_0(P_k) = 1\,, \quad 0 \leq k \leq d\,.$$

Then $r_\bullet$ is a chain retraction of $i_\bullet$ in the sense that

$$r_\bullet \circ i_\bullet = \mathrm{id}$$

and $i_\bullet \circ r_\bullet$ is chain homotopic to id.

Indeed, we have for any n

$$\mathrm{id} - i_n \circ r_n = d_{n+1} \circ h_n + d_n \circ h_{n-1}$$

if the $h_n : R^{(\Delta_n^d)} \to R^{(\Delta_{n+1}^d)}$ are defined by

$$(P_{i_0}, P_{i_1}, \ldots, P_{i_n}) \longmapsto \begin{cases} (P_0, P_{i_0}, \ldots, P_{i_n}) & \text{if } i_0 > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Step 2: The case of a finite simplicial set $X_\bullet$

Let d be the biggest integer such that $X_d \neq \emptyset$.
The proof is by induction on d and $\# X_d$.
Choose an element x of X_d.
It defines a morphism of simplicial sets

$$\Delta^d_\bullet \longrightarrow X_\bullet\,.$$

If $X'_\bullet$ is the simplicial set deduced from $X_\bullet$ by removing the element x, the pull-back of the subobject

$$X'_\bullet \hookrightarrow X_\bullet$$

along the morphism $\Delta^d_\bullet \to X_\bullet$ is

$$\partial\Delta^d_\bullet \hookrightarrow \Delta^d_\bullet\,.$$

So there is a short exact sequence of chain complexes

$$0 \longrightarrow R^{(\partial\Delta^d_\bullet)} \longrightarrow R^{(\Delta^d_\bullet)} \oplus R^{(X'_\bullet)} \longrightarrow R^{(X_\bullet)} \longrightarrow 0$$

and an induced long exact sequence of homology.

On the other hand, we have a closed embedding

$$|X'_\bullet| \hookrightarrow |X_\bullet|$$

and a continuous map

$$|\Delta^d_\bullet| \longrightarrow |X_\bullet|$$

which induces a homeomorphism

$$|\Delta^d_\bullet| - |\partial\Delta^d_\bullet| \xrightarrow{\sim} |X_\bullet| - |X'_\bullet| = U\,.$$

There exists an open neighborhood V of $|X'_\bullet|$ in $|X_\bullet|$ such that

- $|X'_\bullet|$ is a retract of V,
- $|\partial\Delta^d_\bullet|$ is a retract of $U \cap V$ seen as an open subset of $|\Delta^d_\bullet| - |\partial\Delta^d_\bullet|$.

Furthermore, the embedding

$$U \subset |\Delta^d_\bullet|$$

is an homotopy equivalence.

So there is a natural morphism from the long exact sequence of homology associated with

$$
\begin{tikzcd}
 & U \arrow[dr, hook] & \\
U \cap V \arrow[ur, hook] \arrow[dr, hook] & & {|X_\bullet| = U \cup V} \\
 & V \arrow[ur, hook] &
\end{tikzcd}
$$

to the long exact sequence of homology defined by

$$0 \longrightarrow R^{(\partial \Delta^d_\bullet)} \longrightarrow R^{(\Delta^d_\bullet)} \oplus R^{(X'_\bullet)} \longrightarrow R^{(X_\bullet)} \longrightarrow 0\,.$$

It consists of isomorphisms according to the most important lemma:

Lemma:
Consider a commutative diagram of R-modules

$$\begin{tikzcd} A_5 \arrow[r] \arrow[d, "u_5"] & A_4 \arrow[r] \arrow[d, "u_4"] & A_3 \arrow[r] \arrow[d, "u_3"] & A_2 \arrow[r] \arrow[d, "u_2"] & A_1 \arrow[d, "u_1"] \\ B_5 \arrow[r] & B_4 \arrow[r] & B_3 \arrow[r] & B_2 \arrow[r] & B_1 \end{tikzcd}$$

where

$\begin{cases} \bullet & \text{the two horizontal sequences are exact,} \\ \bullet & u_5, u_4, u_2, u_1 \text{ are isomorphisms.} \end{cases}$

Then u_3 is also an isomorphism.

Proof:
Diagram chasing ...

End of the sketch of proof of the theorem:

Step 3: From finite simplicial sets of arbitrary simplicial sets

Let $X_\bullet$ = arbitrary simplicial set,

I = ordered set of finite simplicial subsets $X_\bullet^i \hookrightarrow X_\bullet$.

Then I is filtering and we have

$$X_\bullet = \varinjlim_{i \in I} X_\bullet^i \qquad \text{in} \qquad \widehat{\Delta}$$

therefore

$$R^{(X_\bullet)} = \varinjlim_{i \in I} R^{(X_\bullet^i)}$$

and

$$|X_\bullet| = \varinjlim_{i \in I} |X_\bullet^i| \qquad \text{in} \qquad \mathrm{Top}\,.$$

Furthermore, any continuous map

$$\Delta_n \longrightarrow |X_\bullet|$$

factorizes through the subspaces

$$|X_\bullet^i| \hookrightarrow |X_\bullet|$$

associated to some finite $X_\bullet^i$.

This implies that

$$C_\bullet^{|X_\bullet|} = \varinjlim_{i \in I} C_\bullet^{|X_\bullet^i|} .$$

As I is filtering, the functor $\varinjlim\limits_{i \in I}$ respects finite limits as well as colimits.

So it commutes with homology functors and we can conclude.