Topos Theory

Lectures 9-10: Geometric morphisms

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Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism. The natural notion of morphism of geometric morphisms if that of geometric transformation.

Definition

(i) Let $\mathscr{E}$ and $\mathscr{F}$ be toposes. A geometric morphism $f : \mathscr{E} \to \mathscr{F}$ consists of a pair of functors $f_* : \mathscr{E} \to \mathscr{F}$ (the direct image of f) and $f^* : \mathscr{F} \to \mathscr{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathscr{E} \to \mathscr{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

- Grothendieck toposes and geometric morphisms between them form a category, denoted by $\mathfrak{BTop}$.
- Given two toposes $\mathscr{E}$ and $\mathscr{F}$, geometric morphisms from $\mathscr{E}$ to $\mathscr{F}$ and geometric transformations between them form a category, denoted by **Geom**$(\mathscr{E}, \mathscr{F})$.

Examples of geometric morphisms

- A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The direct image $\mathbf{Sh}(f)_*$ sends a sheaf $F \in Ob(\mathbf{Sh}(X))$ to the sheaf $\mathbf{Sh}(f)_*(F)$ defined by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for any open subset V of Y. The inverse image $\mathbf{Sh}(f)^*$ acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.
- Every Grothendieck topos $\mathscr{E}$ has a unique geometric morphism $\mathscr{E} \to \mathbf{Set}$. The direct image is the global sections functor $\Gamma : \mathscr{E} \to \mathbf{Set}$, sending an object $e \in \mathscr{E}$ to the set $Hom_{\mathscr{E}}(1_{\mathscr{E}}, e)$, while the inverse image functor $\Delta : \mathbf{Set} \to \mathscr{E}$ sends a set S to the coproduct $\coprod_{s \in S} 1_{\mathscr{E}}$.
- For any site $(\mathscr{C}, J)$, the pair of functors formed by the inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ and the associated sheaf functor $a : [\mathscr{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ yields a geometric morphism $i : \mathbf{Sh}(\mathscr{C}, J) \to [\mathscr{C}^{\text{op}}, \mathbf{Set}]$.

Inclusions and surjections

Definition

- A geometric morphism $f : \mathscr{E} \to \mathscr{F}$ is said to be a geometric inclusion if the direct image functor $f_* : \mathscr{E} \to \mathscr{F}$ is full and faithful.
- A geometric morphism $f : \mathscr{E} \to \mathscr{F}$ is said to be a surjection if the inverse image functor $f^* : \mathscr{F} \to \mathscr{E}$ reflects isomorphisms i.e. for any arrow u in $\mathscr{F}$, if $f^*(u)$ is an isomorphism then u is an isomorphism.

Theorem

Every geometric morphism can be factored, uniquely up to canonical equivalence, as a surjection followed by an inclusion.

The notion of locale

Definition

- A frame is a complete lattice A satisfying the infinite distributive law
$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$
- A frame homomorphism $h : A \to B$ is a mapping preserving finite meets and arbitrary joins.
- We write **Frm** for the category of frames and frame homomorphisms.

Fact

A poset is a frame if and only if it is a complete Heyting algebra.

Note that we have a functor $\mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$ which sends a topological space X to its lattice $\mathscr{O}(X)$ of open sets and a continuous function $f : X \to Y$ to the function $\mathscr{O}(f) : \mathscr{O}(Y) \to \mathscr{O}(X)$ sending an open subset V of Y to the open subset $f^{-}(V)$ of X. This motivates the following

Definition

The category **Loc** of locales is the dual $\mathbf{Frm}^{\mathrm{op}}$ of the category of frames (a locale is an object of the category **Loc**).

Pointless topology

Pointless topology is an attempt to do Topology without making reference to the points of topological spaces but rather entirely in terms of their open subsets and of the inclusion relation between them. For example, notions such as connectedness or compactness of a topological space can be entirely reformulated as properties of its lattice of open subsets:

- A space X is connected if and only if for any $a, b \in \mathscr{O}(X)$ such that $a \wedge b = 0$, $a \vee b = 1$ implies either $a = 1$ or $b = 1$;
- A space X is compact if and only if whenever $1 = \bigvee_{i \in I} a_i$ in $\mathscr{O}(X)$, there exist a finite subset $I' \subseteq I$ such that $1 = \bigvee_{i \in I'} a_i$.

Pointless topology thus provides tools for working with locales *as they were* lattices of open subsets of a topological space (even though not all of them are of this form). On the other hand, a locale, being a complete Heyting algebra, can also be studied by using an algebraic or logical intuition.

The dual nature of the concept of locale

This interplay of topological and logical aspects in the theory of locales is very interesting and fruitful; in fact, important 'topological' properties of locales translate into natural logical properties, via the identification of locales with complete Heyting algebras:

Example

Locales	Complete Heyting algebras
Extremally disconnected locales	Complete De Morgan algebras
Almost discrete locales	Complete Boolean algebras

Sheaves on a locale

Definition

Given a locale L, the topos $\mathbf{Sh}(L)$ of sheaves on L is defined as $\mathbf{Sh}(L, J_L)$, where J_L is the Grothendieck topology on L (regarded as a poset category) given by:

$$\{a_i \mid i \in I\} \in J_L(a) \text{ if and only if } \bigvee_{i \in I} a_i = a\,.$$

Theorem

- *The assignment* $L \to \mathbf{Sh}(L)$ *is the object-map of a full and faithful (pseudo-)functor from the category* $\mathbf{Loc}$ *of locales to the category* $\mathfrak{BTop}$ *of Grothendieck toposes.*
- *For any locale L, there is a Heyting algebra isomorphism* $L \cong \mathrm{Sub}_{\mathbf{Sh}(L)}(1_{\mathbf{Sh}(L)})$.

The assignment $L \to \mathbf{Sh}(L)$ indeed brings (pointless) Topology into the world of Grothendieck toposes; in fact, important topological properties of locales can be expressed as topos-theoretic invariants (i.e. properties of toposes which are stable under categorical equivalence) of the corresponding toposes of sheaves. These invariants can in turn be used to give definitions of topological properties for Grothendieck toposes.

For further reading

S. Mac Lane and I. Moerdijk.
Sheaves in geometry and logic: a first introduction to topos theory
Springer-Verlag, 1992.