Topos Theory

Lectures 5-6: Sheaves on a site

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Presheaves on a topological space

Definition
Let $X$ be a topological space. A presheaf $\mathcal{F}$ on $X$ consists of the data:

(i) for every open subset $U$ of $X$, a set $\mathcal{F}(U)$ and

(ii) for every inclusion $V \subseteq U$ of open subsets of $X$, a function $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ subject to the conditions

- $\rho_{U,U}$ is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$ and
- if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

The maps $\rho_{U,V}$ are called restriction maps, and we sometimes write $s|_V$ instead of $\rho_{U,V}(s)$, if $s \in \mathcal{F}(U)$.

A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ on a topological space $X$ is a collection of maps $\mathcal{F}(U) \to \mathcal{G}(U)$ which is compatible with respect to restriction maps.

Remark
Categorically, a presheaf $\mathcal{F}$ on $X$ is a functor $\mathcal{F} : \mathcal{O}(X)^{op} \to \text{Set}$, where $\mathcal{O}(X)$ is the poset category corresponding to the lattice of open sets of the topological space $X$ (with respect to the inclusion relation).

A morphism of presheaves is then just a natural transformation between the corresponding functors.

So we have a category $[\mathcal{O}(X)^{op}, \text{Set}]$ of presheaves on $X$. 
Sheaves on a topological space

Definition
A sheaf \( \mathcal{F} \) on a topological space \( X \) is a presheaf on \( X \) satisfying the additional conditions

(i) if \( U \) is an open set, if \( \{ V_i \mid i \in I \} \) is an open covering of \( U \), and if \( s, t \in \mathcal{F}(U) \) are elements such that \( s|_{V_i} = t|_{V_i} \) for all \( i \), then \( s = t \);
(ii) if \( U \) is an open set, if \( \{ V_i \mid i \in I \} \) is an open covering of \( U \), and if we have elements \( s_i \in \mathcal{F}(V_i) \) for each \( i \), with the property that for each \( i, j \in I \), \( s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j} \), then there is an element \( s \in \mathcal{F}(U) \) (necessarily unique by (i)) such that \( s|_{V_i} = s_i \) for each \( i \).

A morphism of sheaves is defined as a morphism of the underlying presheaves.

Remark
Categorically, a sheaf is a functor \( \mathcal{O}(X)^{\text{op}} \rightarrow \text{Set} \) which satisfies certain conditions expressible in categorical language entirely in terms of the poset category \( \mathcal{O}(X) \) and of the usual notion of covering on it. The category \( \text{Sh}(X) \) of sheaves on a topological space \( X \) is thus a full subcategory of the category \([\mathcal{O}(X)^{\text{op}}, \text{Set}]\) of presheaves on \( X \).

This paves the way for a significant categorical generalization of the notion of sheaf, leading to the notion of Grothendieck topos.
The associated sheaf functor

**Theorem**

*Given a presheaf \( \mathcal{F} \), there is a sheaf \( a(\mathcal{F}) \) and a morphism \( \theta : \mathcal{F} \rightarrow a(\mathcal{F}) \), with the property that for any sheaf \( \mathcal{G} \), and any morphism \( \phi : \mathcal{F} \rightarrow \mathcal{G} \), there is a unique morphism \( \psi : a(\mathcal{F}) \rightarrow \mathcal{G} \) such that \( \psi \circ \theta = \phi \).*

The sheaf \( a(\mathcal{F}) \) is called the **sheaf associated** to the presheaf \( \mathcal{F} \).

**Remark**

*Categorically, this means that the inclusion functor \( i : \text{Sh}(X) \rightarrow [\mathcal{O}(X)^{\text{op}},\text{Set}] \) has a left adjoint \( a : [\mathcal{O}(X)^{\text{op}},\text{Set}] \rightarrow \text{Sh}(X) \).*

The left adjoint \( a : [\mathcal{O}(X)^{\text{op}},\text{Set}] \rightarrow \text{Sh}(X) \) is called the **associated sheaf functor**.
Examples of sheaves

Examples

• the sheaf of continuous real-valued functions on any topological space
• the sheaf of regular functions on a variety
• the sheaf of differentiable functions on a differentiable manifold
• the sheaf of holomorphic functions on a complex manifold

In each of the above examples, the restriction maps of the sheaf are the usual set-theoretic restrictions of functions to a subset.

Remark

Sheaves arising in Mathematics are often equipped with more structure than the mere set-theoretic one; for example, one may wish to consider sheaves of modules (resp. rings, abelian groups, ...) on a topological space $X$.

The natural categorical way of looking at these notions is to consider them as models of certain (geometric) theories in a category $\mathbf{Sh}(X)$ of sheaves of sets.
The sheaf of cross-sections of a bundle

Definition

- For any topological space $X$, a continuous map $p : Y \to X$ is called a bundle over $X$. In fact, the category of bundles is the slice category $\text{Top}/X$.
- Given an open subset $U$ of $X$, a cross-section over $U$ of a bundle $p : Y \to X$ is a continuous map $s : U \to Y$ such that the composite $p \circ s$ is the inclusion $i : U \hookrightarrow X$. Let

$$\Gamma_p U = \{ s \mid s : U \to Y \text{ and } p \circ s = i : U \to X \}$$

denote the set of all such cross-sections over $U$.
- If $V \subseteq U$, one has a restriction operation $\Gamma_p U \to \Gamma_p V$. The functor $\Gamma_p : \mathcal{O}(X)^{\text{op}} \to \text{Set}$ obtained in this way is a sheaf and is called the sheaf of cross-sections of the bundle $p$. 
The bundle of germs of a presheaf

Definition

• Given any presheaf \( \mathcal{F} : \mathcal{O}(X)^{\text{op}} \to \text{Set} \) on a space \( X \), a point \( x \in X \), two open neighbourhoods \( U \) and \( V \) of \( x \), and two elements \( s \in \mathcal{F}(U), t \in \mathcal{F}(V) \). We say that \( s \) and \( t \) have the same germ at \( x \) when there is some open set \( W \subseteq U \cap V \) with \( x \in W \) and \( s|_W = t|_W \). This relation ‘to have the same germ at \( x \)' is an equivalence relation, and the equivalence class of any one such \( s \) is called the germ of \( s \) at \( x \), in symbols \( \text{germ}(s) \).

• Let

\[
\mathcal{F}_x = \{ \text{germ}(s) \mid s \in \mathcal{F}(U), \ x \in U \text{ open in } X \}
\]

be the set of all germs at \( x \).

• Let \( \Gamma_p \) be the disjoint union of the \( \mathcal{F}_x \)

\[
\Lambda_p = \{ (x, r) \mid x \in X, r \in \mathcal{F}_x \}
\]

topologized by taking as a base of open sets all the image sets \( \tilde{s}(U) \), where \( \tilde{s} : U \to \Lambda_p \) is the map induced by an element \( s \in \mathcal{F}(U) \) by taking its germs at points in \( U \).

• With respect to this topology, the natural projection map \( \Lambda_p \to X \) becomes a continuous map, called the bundle of germs of the presheaf \( \mathcal{F} \).
Sheaves as étale bundles

Definition

- A bundle \( p : E \to X \) is said to be étale (over \( X \)) when \( p \) is a local homeomorphism in the following sense: for each \( e \in E \) there is an open set \( V \), with \( e \in V \), such that \( p(V) \) is open in \( X \) and \( p|_V \) is a homeomorphism \( V \to p(V) \).

- The full subcategory of \( \text{Top}/X \) on the étale bundles is denoted by \( \text{Etale}(X) \).

Theorem

- For any topological space \( X \), there is a pair of adjoint functors

\[
\Gamma : \text{Top}/X \to [\mathcal{O}(X)^{\text{op}}, \text{Set}], \quad \Lambda : [\mathcal{O}(X)^{\text{op}}, \text{Set}] \to \text{Top}/X,
\]

where \( \Gamma \) assigns to each bundle \( p : Y \to X \) the sheaf of cross-sections of \( p \), while its left adjoint \( \Lambda \) assigns to each presheaf \( \mathcal{F} \) the bundle of germs of \( \mathcal{F} \).

- The adjunction restricts to an equivalence of categories

\[
\text{Sh}(X) \simeq \text{Etale}(X).
\]
In order to ‘categorify’ the notion of sheaf of a topological space, the first step is to introduce an abstract notion of covering on a category.

**Definition**

- Given a category \( \mathcal{C} \) and an object \( c \in \text{Ob}(\mathcal{C}) \), a **sieve** \( S \) in \( \mathcal{C} \) on \( c \) is a collection of arrows in \( \mathcal{C} \) with codomain \( c \) such that

\[
\text{if } f \in S \Rightarrow f \circ g \in S
\]

whenever this composition makes sense.

- We say that a sieve \( S \) is **generated** by a given family of arrows (with common codomain) if it is the smallest sieve which contains all the arrows of the family.

If \( S \) is a sieve on \( c \) and \( h : d \to c \) is any arrow to \( c \), then

\[
h^*(S) := \{ g \mid \text{cod}(g) = d, \ h \circ g \in S \}
\]

is a sieve on \( d \).
Grothendieck topologies II

Definition
A Grothendieck topology on a small category $\mathcal{C}$ is a function $J$ which assigns to each object $c$ of $\mathcal{C}$ a collection $J(c)$ of sieves on $c$ in such a way that

(i) (maximality axiom) the maximal sieve $M_c = \{ f | \text{cod}(f) = c \}$ is in $J(c)$;

(ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;

(iii) (transitivity axiom) if $S \in J(c)$ and $R$ is any sieve on $c$ such that $f^*(R) \in J(d)$ for all $f : d \to c$ in $S$, then $R \in J(c)$.

The sieves $S$ which belong to $J(c)$ for some object $c$ of $\mathcal{C}$ are said to be $J$-covering.
Examples of Grothendieck topologies I

- For any (small) category \( \mathcal{C} \), the **trivial topology** on \( \mathcal{C} \) is the Grothendieck topology in which the only sieve covering an object \( c \) is the maximal sieve \( M_c \).

- The **dense topology** \( D \) on a category \( \mathcal{C} \) is defined by: for a sieve \( S \),

\[
S \in D(c) \quad \text{if and only if} \quad \text{for any } f : d \to c \text{ there exists } g : e \to d \text{ such that } f \circ g \in S .
\]

If \( \mathcal{C} \) satisfies the **right Ore condition** i.e. the property that any two arrows \( f : d \to c \) and \( g : e \to c \) with a common codomain \( c \) can be completed to a commutative square

\[
\begin{array}{ccc}
\bullet & \to & d \\
| & | & | \\
| & | & | \\
\downarrow & \downarrow & \downarrow \\
e & \overset{g}{\to} & c \\
\end{array}
\]

then the dense topology on \( \mathcal{C} \) specializes to the **atomic topology** on \( \mathcal{C} \) i.e. the topology \( J_{at} \) defined by: for a sieve \( S \),

\[
S \in J_{at}(c) \quad \text{if and only if} \quad S \not= \emptyset .
\]
Examples of Grothendieck topologies II

• If $X$ is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathcal{O}(X)}$ on the poset category $\mathcal{O}(X)$: for a sieve $S = \{ U_i \hookrightarrow U \mid i \in I \}$ on $U \in Ob(\mathcal{O}(X))$,

$$S \in J_{\mathcal{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U.$$

• More generally, given a complete Heyting algebra $H$, i.e. a Heyting algebra with arbitrary joins $\bigvee$ (and meets), we can define a Grothendieck topology $J_H$ by:

$$\{ a_i \mid i \in I \} \in J_H(a) \text{ if and only if } \bigvee_{i \in I} a_i = a.$$
The notion of Grothendieck topos I

Definition

- A site is a pair \((\mathcal{C}, J)\) where \(\mathcal{C}\) is a small category and \(J\) is a Grothendieck topology on \(\mathcal{C}\).
- A presheaf on a (small) category \(\mathcal{C}\) is a functor \(P : \mathcal{C}^{\text{op}} \to \text{Set}\).
- Let \(P : \mathcal{C}^{\text{op}} \to \text{Set}\) be a presheaf on \(\mathcal{C}\) and \(S\) be a sieve on an object \(c\) of \(\mathcal{C}\). A matching family for \(S\) of elements of \(P\) is a function which assigns to each arrow \(f : d \to c\) in \(S\) an element \(x_f \in P(d)\) in such a way that

\[
P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d.
\]

An amalgamation for such a family is a single element \(x \in P(c)\) such that

\[
P(f)(x) = x_f \quad \text{for all } f \text{ in } S.
\]
The notion of Grothendieck topos II

- Given a site \((\mathcal{C}, J)\), a presheaf on \(\mathcal{C}\) is a \(J\)-sheaf if every matching family for any \(J\)-covering sieve on any object of \(\mathcal{C}\) has a unique amalgamation.
- The category \(\mathbf{Sh}(\mathcal{C}, J)\) of sheaves on the site \((\mathcal{C}, J)\) is the full subcategory of \([\mathcal{C}^{\text{op}}, \mathbf{Set}]\) on the presheaves which are \(J\)-sheaves.
- A Grothendieck topos is any category of sheaves on a site.

Examples

- For any (small) category \(\mathcal{C}\), \([\mathcal{C}^{\text{op}}, \mathbf{Set}]\) is the category of sheaves \(\mathbf{Sh}(\mathcal{C}, T)\) where \(T\) is the trivial topology on \(\mathcal{C}\).
- For any topological space \(X\), \(\mathbf{Sh}(\mathcal{O}(X), J_{\mathcal{O}(X)})\) is equivalent to the usual category \(\mathbf{Sh}(X)\) of sheaves on \(X\).
For further reading

S. Mac Lane and I. Moerdijk.  
*Sheaves in geometry and logic: a first introduction to topos theory*  