Olivia Caramello

Local operators

For further reading

Topos Theory Lecture 11: Local operators

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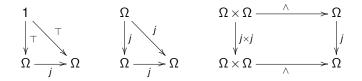
Local operators

For further reading

The concept of local operator

Definition

Let *&* be a topos, with subobject classifier ⊤ : 1 → Ω. A local operator (or Lawvere-Tierney topology) on *&* is an arrow *j* : Ω → Ω in *&* such that the diagrams



commute (where $\wedge: \Omega \times \Omega \to \Omega$ is the meet operation of the internal Heyting algebra Ω).

• Let & be an elementary topos. A universal closure operation on & is a closure operation c on subobjects which commutes with pullback (= intersection) of subobjects.

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Universal closure operations I

Definition

Let c be a universal closure operation on an elementary topos \mathscr{E} .

- A subobject $m : a' \to a$ in \mathscr{E} is said to be *c*-dense if $c(m) = id_a$, and *c*-closed if c(m) = m.
- An object a of & is said to be a c-sheaf if whenever we have a diagram

$$b' \xrightarrow{f'} a$$

 $m \downarrow b$

where *m* is a *c*-dense subobject, there exists exactly one arrow $f: b \rightarrow a$ such that $f \circ m = f'$.

 The full subcategory of *ℰ* on the objects which are *c*-sheaves will be denoted by **sh**_c(*ℰ*).

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Universal closure operations II

Theorem

For any elementary topos \mathscr{E} , there is a bijection between universal closure operations on \mathscr{E} and local operators on \mathscr{E} .

Sketch of proof.

The bijection sends a universal closure operation *c* on \mathscr{E} to the local operator $j_c : \Omega \to \Omega$ given by classifying map of the subobject $c(1 \xrightarrow{\top} \top)$, and a local operator *j* to the closure operation c_j induced by composing classifying arrows with *j*.

Fact

For any local operator j on an elementary (resp. Grothendieck) topos \mathscr{E} , $\mathbf{sh}_{c_j}(\mathscr{E})$ is an elementary (resp. Grothendieck) topos, and the inclusion $\mathbf{sh}_{c_j}(\mathscr{E}) \hookrightarrow \mathscr{E}$ has a left adjoint $\mathbf{a}_j : \mathscr{E} \to \mathbf{sh}_{c_j}(\mathscr{E})$ which preserves finite limits. In fact, local operators on \mathscr{E} also correspond bijectively to (equivalence classes of) geometric inclusions to \mathscr{E} .

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Local operators and Grothendieck topologies

Theorem

If \mathscr{C} is a small category, the Grothendieck topologies J on \mathscr{C} correspond exactly to the local operators on the presheaf topos $[\mathscr{C}^{op}, \mathbf{Set}]$.

Sketch of proof.

The correspondence sends a local operator $j : \Omega \to \Omega$ to the subobject $J \to \Omega$ which it classifies, that is to the Grothendieck topology J on \mathscr{C} defined by:

 $S \in J(c)$ if and only if $j(c)(S) = M_c$

Conversely, it sends a Grothendieck topology *J*, regarded as a subobject $J \rightarrow \Omega$, to the arrow $j : \Omega \rightarrow \Omega$ that classifies it.

In fact, if *J* is the Grothendieck topology corresponding to a local operator *j*, an object of [\mathscr{C}^{op} , **Set**] is a *J*-sheaf (in the sense of Grothendieck toposes) if and only if it is a *c_j*-sheaf (in the sense of universal closure operations).

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🛸 S. Mac Lane and I. Moerdijk.

Sheaves in geometry and logic: a first introduction to topos theory Springer-Verlag, 1992.