Unification and morphogenesis: a topos-theoretic perspective

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Plan of the talk

- The theme of unification in mathematics
- Toposes as unifying 'bridges'
- The duality between topos-theoretic invariants and their manifestations: a mathematical morphogenesis
- The idea of 'bridge'
- Symmetries and completions
- Future directions

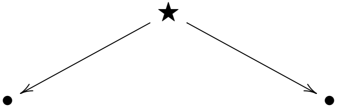
Unification and 'bridges' in mathematics

- Mathematics consists of several distinct areas (e.g., Algebra, Geometry, Analysis, Topology, Number Theory), each characterized by its own language and techniques.
- With time, various connections between the areas have been discovered, leading in some cases to the creation of actual 'bridges' between different mathematical branches (think for example of analytic geometry).
- The importance of 'bridges' between different areas lies in the fact that they make it possible to transfer knowledge and methods between the areas, so that problems formulated in the language of one field can be tacked (and possibly solved) using techniques from a different field.
- Mathematical logic and topos theory turn out to be fundamental tools for investigating the relations between different mathematical theories in a systematic and rigorous way.

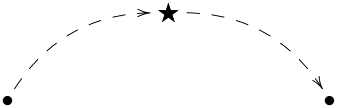
The concept of unification

We can distinguish between two different kinds of unification.

- ‘Static’ unification (through a generalization): two concepts are seen to be special instances of a more general one:



- ‘Dynamic’ unification (through a construction): two objects are related to each other through a third one (usually constructed from each of them), which acts like a ‘bridge’ enabling transfers of information between them.



Transfers of information arise from the process of ‘translating’ properties of (resp. constructions on) the ‘bridge object’ into properties of (resp. constructions on) the two objects.

Unifying theories for mathematics

In relation to the types of unification introduced above, we can say that:

- Set theory and category theory realize a static unification of mathematics, essentially of linguistic nature. Indeed, each of these theories provide an abstract global framework in whose language most of mathematics can be formulated.

  Note that, even though each of them provides a way of expressing and organizing mathematics in one single language, these theories do not offer by themselves effective methods for an actual transfer of knowledge between distinct fields.

- Instead, toposes, as spaces on which the fundamental mathematical invariants are naturally defined, allow one to effectively connect different mathematical theories with each other, and also to study a given theory from a multiplicity of different points of view, thus defining a more substantial, dynamical approach to the problem of 'unifying mathematics'.

The "unifying notion" of topos

In this talk the term 'topos' will always mean 'Grothendieck topos'.

> *"C'est le thème du topos qui est ce "lit", ou cette "rivière profonde" où viennent s'épouser la géométrie et l'algèbre, la topologie et l'arithmétique, la logique mathématique et la théorie des catégories, le monde du continu et celui des structures "discontinues" ou "discrètes". Il est ce que j'ai conçu de plus vaste, pour saisir avec finesse, par un même langage riche en résonances géométriques, une "essence" commune à des situations des plus éloignées les unes des autres provenant de telle région ou de telle autre du vaste univers des choses mathématiques".*
>
> *A. Grothendieck*

Since my Ph.D. studies, I have developed a theory and a number of techniques allowing one to exploit the unifying potential of the notion of topos for establishing 'bridges' across different mathematical theories.

Toposes as unifying 'bridges'

This theory, introduced in the programmatic paper "*The unification of Mathematics via Topos Theory*" of 2010, allows one to exploit the technical flexibility inherent to the concept of topos - most notably, the possibility of presenting a topos in a multitude of different ways - for building unifying 'bridges' useful for transferring notions, ideas and results across different mathematical contexts.

In the last years, besides leading to the solution of a number of long-standing problems in categorical logic, these techniques have generated several substantial applications in different mathematical fields. Still, much remains to be done so that toposes become a key tool universally used for investigating mathematical theories and their relations.

In fact, these 'bridges' have proved useful not only for connecting different mathematical theories with each other, but also for investigating a given theory from multiple points of view.

A few selected applications

Since the theory of topos-theoretic 'bridges' was introduced, several applications of it have been obtained in different fields of Mathematics, such as:

- Model theory (topos-theoretic Fraïssé theorem)
- Proof theory (various results for first-order theories)
- Algebra (topos-theoretic generalization of topological Galois theory)
- Topology (topos-theoretic interpretation/generation of Stone-type and Priestley-type dualities)
- Functional analysis (various results on Gelfand spectra and Wallman compactifications)
- Many-valued logics and lattice-ordered groups (two joint papers with A. C. Russo)
- Cyclic homology, as reinterpreted by A. Connes (work on *"cyclic theories"*, jointly with N. Wentzlaff)
- Algebraic geometry (logical analysis of (co)homological motives, cf. the paper *"Syntactic categories for Nori motives"* joint with L. Barbieri-Viale and L. Lafforgue)

Classifying toposes

In relation to the aim of using toposes as unifying 'bridges', regarding toposes from the perspective of the structures that they classify is particularly relevant.

In the seventies, thanks to the work of a number of categorical logicians, notably including M. Makkai and G. Reyes, it was discovered that:

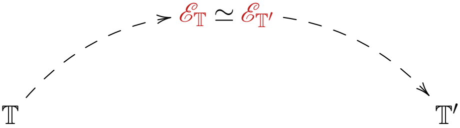
- With any mathematical theory $\mathbb{T}$ (of a very general form) one can canonically associate a topos $\mathscr{E}_{\mathbb{T}}$, called its classifying topos, which represents its 'semantical core'.
- Two given mathematical theories have the same classifying topos (up to equivalence) if and only if they have the same 'semantical core', that is, if and only if they are indistinguishable from a semantic viewpoint. Two such theories are said to be Morita-equivalent.
- Conversely, any topos is the classifying topos of some theory (in fact, of infinitely many theories).
- A topos can thus be seen as a canonical representative for equivalence classes of theories modulo Morita-equivalence.

Toposes as 'bridges'

- The notion of Morita-equivalence formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods', which explains its ubiquity in Mathematics.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted as arising from Morita-equivalences.
- Any two theories which are bi-interpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.
- Moreover, the notion of Morita-equivalence captures the dynamics inherent to the very concept of mathematical theory; indeed, a mathematical theory alone gives rise to an infinite number of Morita-equivalences.
- Topos theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.

Toposes as 'bridges'

- The existence of different theories with the same classifying topos translates, at the technical level, into the existence of different representations for the same topos.
- Topos-theoretic invariants, that is properties of (or construction on) toposes which are invariant with respect to their different representations, can thus be used to transfer information from one theory to another:



- Transfers of information take place by expressing a given invariant in terms of the different representations of the topos.

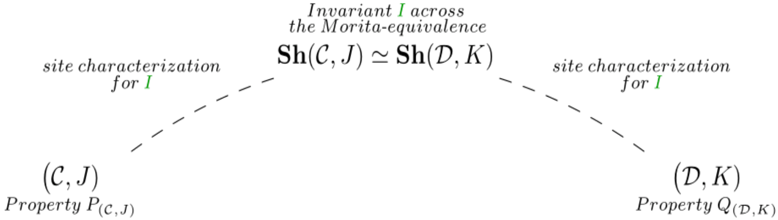
Toposes as 'bridges'

- Different properties (resp. constructions) arising in the context of theories classified by the same topos come thus to be seen as different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.
- Every invariant behaves in this context as a 'pair of glasses' enabling to discern some information hidden in the given Morita equivalence; different invariants allow to englighten and transfer different information.
- This methodology is technically feasible because the relationship between a topos and its representations is very natural, enabling us to transfer invariants across different representations (and hence, between different theories) in an effective (though generally non-trivial, and even, in some cases, very complex) way.

The 'bridge-building' technique

- Decks of 'bridges': Morita-equivalences (or more generally morphisms or other kinds of relations between toposes)
- Arches of 'bridges': Characterizations for topos-theoretic invariants in terms of the two different representations

A typical 'bridge' between different site representations for the same topos looks as follows:



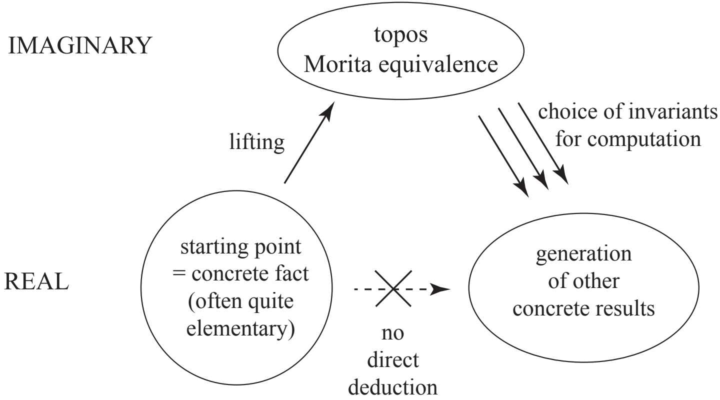
This 'bridge' yields a logical equivalence between the 'concrete' properties $P_{(\mathscr{C},J)}$ and $Q_{(\mathscr{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

A mathematical morphogenesis

- The essential ambiguity given by the fact that any topos is associated in general with an infinite number of theories or different sites allows to study the relations between different theories, and hence the theories themselves, by using toposes as 'bridges' between these different presentations.
- Every topos-theoretic invariant generates a veritable mathematical morphogenesis resulting from its expression in terms of different representations of toposes, which gives rise in general to connections between properties or notions that are completely different and apparently unrelated from each other.
- The mathematical exploration is therefore in a sense 'reversed' since it is guided by the Morita-equivalences and by topos-theoretic invariants, from which one proceeds to extract concrete information on the theories that one wishes to study.

A leap into the 'imaginary'

We can schematically represent the way of obtaining concrete results by applying the 'bridge' technique in the form of an ascent followed by a descent between two levels, the 'real' one of concrete mathematics and the 'imaginary' one of toposes:



The duality between 'real' and 'imaginary'

- The passage from a site (or a theory) to the associated topos can be regarded as a sort of 'completion' by the addition of 'imaginaries' (in the model-theoretic sense), which materializes the potential contained in the site (or theory).
- The duality between the (relatively) unstructured world of presentations of theories and the maximally strucured world of toposes is of great relevance as, on the one hand, the 'simplicity' and concreteness of theories or sites makes it easy to manipulate them, while, on the other hand, computations are much easier in the 'imaginary' world of toposes thanks to their very rich internal structure and the fact that invariants live at this level.

Toposes as 'bridges' and the Erlangen Program

The methodology 'toposes as bridges' is a vast extension of Felix Klein's Erlangen Program (A. Joyal)

More specifically:

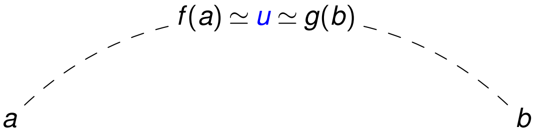
- Every group gives rise to a topos (namely, the category of actions of it), but the notion of topos is much more general.
- As Klein classified geometries by means of their automorphism groups, so we can study first-order geometric theories by studying the associated classifying toposes.
- As Klein established surprising connections between very different-looking geometries through the study of the algebraic properties of the associated automorphism groups, so the methodology 'toposes as bridges' allows to discover non-trivial connections between properties, concepts and results pertaining to different mathematical theories through the study of the categorical invariants of their classifying toposes.

Comparing different objects

- One is generally interested in comparing pairs of objects between which there is some kind of relation.
- In order to transfer information between objects related by a given relation, it is thus of fundamental importance to identify (and, possibly, classify) the properties of the objects that are invariant with respect to the relation.
- Depending on the cases, this can be an approachable task or an hopelessly difficult one.
- In fact, a relation between two given objects is in general an *abstract* entity, which lives in an ideal context which is generally different from that in which the two objects lie.
- Therefore, it becomes of crucial importance to identify more concrete entities which could act as 'bridges' connecting the two given objects.

Bridge objects

- We can think of a *bridge* object connecting two objects a and b as an object u which can be 'built' from any of the two objects and admits two different representations $f(a)$ and $g(b)$ related by some kind of equivalence $\simeq$, the former being in terms of the object a and the latter in terms of the object b:



- Transfers of information arise from the process of 'unraveling' properties of (resp. constructions on) the 'bridge object' u into properties of (resp. constructions on) the two objects a and b by using the two different representations $f(a)$ and $g(b)$ of u.

Invariants or dictionaries?

The method of bridges can be interpreted linguistically as a methodology for translating concepts from one context to another.

But which kind of translation is this?

In general, we can distinguish between two essentially different approaches to translation:

- The 'dictionary-oriented' or 'bottom-up' approach, consisting in a dictionary-based renaming of the single words composing the sentences;
- The 'invariant-oriented' or 'top-down' approach, consisting in the identification of appropriate concepts that should remain invariant in the translation, and in the subsequent analysis of how these invariants can be expressed in the two languages.

The 'bridge-based' translations are of the latter kind.

One or multiple?

- Any object can be thought of as the collection of all its presentations. A fundamental equivalence relation subsists between these presentations: that of presenting the same object.
- Any object can thus play the role of a 'bridge' across its different presentations.
- We 'access' an object by means of the multiplicity of its presentations, but the objects themselves are actually equivalence classes of presentations.

Generation from a source

- Any object, by virtue of its own existence, generates a sheaf of perceptions (or 'presentations') which are coherent with each other.

  Conversely, in the presence of coherence relations between different perceptions of the same phenomenon, it is scientifically reasonable (adopting a minimalist perspective) to suppose the existence of something that would 'generate' such perceptions and which could thus be considered ontologically responsible for the coherence relations existing between them.

- The language of category theory is particularly suitable for expressing coherence relations. In particular, the notion of sheaf expresses robust coherence relations of "local-global" type, which allow one to define 'global' entities starting from sets of compatible 'local' data.

- What is 'reality' if not a sheaf of coherent perceptions?

The Yoneda paradigm

An elementary but key result in category theory which well illustrates the power of categorical language to express philosophical ideas such as that of 'direction of observation' or of 'generation from a source' is the Yoneda lemma:

- Given an object c of a (small) category $\mathscr{C}$, a generalized element of c is an arrow in $\mathscr{C}$ to c.
- There is a functor

$$\mathrm{Hom}_{\mathscr{C}}(-, c) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$$

  canonically associated with c, called the functor represented by c, or the functor of generalized elements of c, which associates with any object a of $\mathscr{C}$ the set of the generalized elements of c whose domain is a.
- A functor $\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is said to be representable if it is isomorphic to a functor of the form $\mathrm{Hom}_{\mathscr{C}}(-, c)$ for some object c of $\mathscr{C}$.
- The Yoneda embeddding identifies an object c, up to isomorphism, with the functor $\mathrm{Hom}_{\mathscr{C}}(-, c)$ of its generalized elements.
- Remarkably, the Yoneda embedding induces an equivalence between a given topos and the category of sheaves on it (with respect to the canonical topology).

Unification and morphogenesis

- Bridges abound both in mathematics and in other scientific fields, and can be considered 'responsible' (at least abstractly) for the genesis of things and the nature of reality as we perceive it.
- Indeed, whenever we have an invariant, we can try to use it to build 'bridges' connecting its different manifestations.
- A 'bridge' is precisely the expression, and, in a sense, also the *explanation*, of the connection which exists between the different manifestations of a given invariant.
- Think, for instance, to the notion of *energy* in physics as an invariant: energy is in itself a very abstract concept, but the different forms in which it manifests itself can be very concrete (e.g., thermic energy, electromagnetic energy, mechanical energy, etc.); moreover, the possibility of transforming, as in a 'bridge', a form of energy into another is something very important.

Ideal = real?

- The idea of bridge is an abstraction (like that of invariant), but, interestingly, bridges arising in the experimental sciences can often be identified with actual physical objects (think, for instance, in biology, to the DNA, or, in astronomy, to the stars around which planets revolve).
- In fact, the most enlightening situations occur when these ideal objects admit 'concrete' representations, allowing us to contemplate the dynamics of 'differentiation from the unity' in a more direct and effective way.
- Topos theory allows us to materialize a tremendous number of ideal objects, and hence to establish effective bridges between a great variety of different contexts.
- In general, looking for 'concrete' representations of (or ways of realizing) imaginary concepts can lead to the discovery of more 'symmetric' environments in which phenomena can be described in natural and unified ways.

Contingent and universal

- Every language or point of view is partial (or 'holed'), and it is only through the integration of all possible points of view that one can capture the essence of things.
- There is no universal language that would be better (in an absolute sense) then all the others; every point of view enlightens certain aspects of a phenomenon by hiding others, and can be more or less convenient than others in relation to a certain goal.
- Universality should thus be researched not at the level of languages but at that of 'ideal' objects on which invariants are defined.
- It it therefore crucial to reason at two levels, that of invariants (and of objects on which they are defined) and that of their manifestations in the context of 'concrete' situations, and to study the duality between these two levels, a duality which can be thought of as that between a 'meaning' and the different ways to express it.

Completions and invariants

- To relate different languages or points of view with each other, we need in general to 'complete' them to objects which realize *explicitly* the *implicit* hidden in each of them.
- It is at the level of these completed objects that invariants, or symmetries, manifest themselves, and that we can understand the relations between our given objects thanks to the bridges induced by invariants.
- For example, the classifying topos of a theory is constructed by means of a process of completion of the theory itself, with respect, in a sense, to all the concepts that it is potentially capable to express.
- Thanks to the 'bridge' technique, different theories which describe the same mathematical content are put in relation with each other as if they were fragments of a unique object, partial languages which complete themselves by reflecting one into the other in the totality of points of view embodied by the classifying topos.

Future directions

The evidence provided by the results obtained so far shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories and for generating new equivalences, dualities and symmetries across different fields of Mathematics.

In fact, toposes have an authentic creative power in Mathematics, in the sense that their study naturally leads to the discovery of a great number of notions and 'concrete' results in different mathematical fields, which are pertinent but often unsuspected.

In the next years, we intend to continue pursuing the development of these general unifying methodologies both at the theoretical level and at the applied level, in order to continue developing the potential of toposes as fundamental tools in the study of mathematical theories and their relations, and as key concepts defining a new way of doing Mathematics liable to bring distinctly new insights in a great number of different subjects.

Future directions

Central themes in this programme will be:

- investigation of important dualities or correspondences in Mathematics from a topos-theoretic perspective (in particular, the theory of motives, class field theory and the Langlands programme)
- systematic study of invariants of toposes in terms of their presentations, and introduction of new invariants which capture important aspects of concrete mathematical problems
- interpretation and generalization of important parts of classical and modern model theory in terms of toposes and development of a functorial model theory
- introduction of new methodologies for generating Morita-equivalences
- development of general techniques for building spectra by using classifying toposes
- generalization of the 'bridge' technique to the setting of higher categories and toposes through the introduction of higher geometric logic
- development of a relative theory of classifying toposes

Future directions

A further development of the theory of toposes as 'bridges' is likely to inspire work in subjects other than mathematics, such as

- Physics (e.g., analysis and interpretation of dualities, relativity theory and its relationship with quantum mechanics)
- Computer science (e.g., semantics of programming languages and automated theorem proving)
- Linguistics (e.g., syntax and semantics of natural languages, comparative studies and the theory of translation)
- Philosophy (e.g., methodology of science, ontology of mathematical concepts)
- Music Theory (e.g., analysis of composition, interpretation and performance)

...

For further reading

- O. Caramello
  Grothendieck toposes as unifying ‘bridges’ : a mathematical morphogenesis,
  to appear in the Springer book *Philosophy of Mathematics. Objects, Structures, and Logics*.

- O. Caramello
  La “notion unificatrice” de topos,
  to appear in the proceedings volume of the *Lectures Grothendieckiennes* at the École Normale Superieure di Parigi,
  available from my website www.oliviacaramello.com.

- O. Caramello
  Grothendieck toposes as unifying ‘bridges’ in Mathematics,
  Mémoire d’habilitation à diriger des recherches,
  Université de Paris 7, 2016,
  available from my website www.oliviacaramello.com.

- O. Caramello.
  Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic ‘bridges’,
  Oxford University Press, 2017.