Grothendieck toposes as unifying 'bridges' in Mathematics

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The "unifying notion" of topos

"It is the topos theme which is this "bed" or "deep river" where come to be married geometry and algebra, topology and arithmetic, mathematical logic and category theory, the world of the "continuous" and that of "discontinuous" or discrete structures. It is what I have conceived of most broad to perceive with finesse, by the same language rich of geometric resonances, an "essence" which is common to situations most distant from each other coming from one region or another of the vast universe of mathematical things".

A. Grothendieck

Topos theory can be regarded as a unifying subject in Mathematics, with great relevance as a framework for systematically investigating the relationships between different mathematical theories and studying them by means of a multiplicity of different points of view. Its methods are transversal to the various fields and complementary to their own specialized techniques. In spite of their generality, the topos-theoretic techniques are liable to generate insights which would be hardly attainable otherwise and to establish deep connections that allow effective transfers of knowledge between different contexts.

The multifaceted nature of toposes

The role of toposes as unifying spaces is intimately tied to their multifaceted nature.

For instance, a topos can be seen as:

- a generalized space
- a mathematical universe
- a theory modulo 'Morita-equivalence'

We shall now review each of these classical points of view, and then present the more recent theory of topos-theoretic 'bridges', which combines all of them to provide tools for making toposes effective means for studying mathematical theories from multiple points of view, relating and unifying theories with each other and constructing 'bridges' across them.

Toposes as generalized spaces

- The notion of topos was introduced in the early sixties by A. Grothendieck with the aim of bringing a topological or geometric intuition also in areas where actual topological spaces do not occur.
- Grothendieck realized that many important properties of topological spaces X can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.
- He then defined toposes as more general categories of sheaves of sets, by replacing the topological space X by a pair $(\mathscr{C}, J)$, called a site, consisting of a (small) category $\mathscr{C}$ and a 'generalized notion of covering' J on it, and taking sheaves (in a generalized sense) over the pair:

$$\begin{array}{ccc} X & \dashrightarrow & \mathbf{Sh}(X) \\ \wr\downarrow & & \wr\downarrow \\ (\mathscr{C}, J) & \dashrightarrow & \mathbf{Sh}(\mathscr{C}, J) \end{array}$$

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Topos-theoretic invariants

- The notion of a geometric morphism of toposes has notably allowed to build general comology theories starting from the categories of internal abelian groups or modules in toposes. In particular, the topos-theoretic viewpoint has allowed Grothendieck to refine and enrich the study of cohomology, up to the so-called 'six-operation formalism'.
  The cohomological invariants have had a tremendous impact on the development of modern Algebraic Geometry and beyond.
- On the other hand, also homotopy-theoretic invariants such as the fundamental group and the higher homotopy groups can be defined as invarants of toposes.
- Still, these are by no means the only invariants that one can consider on toposes: indeed, there are infinitely many invariants of toposes (of algebraic, logical, geometric or whatever nature), the notion of identity for toposes being simply categorical equivalence.

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Toposes as mathematical universes

A decade later, W. Lawvere and M. Tierney discovered that a topos could not only be seen as a generalized space, but also as a mathematical universe in which one can do mathematics similarly to how one does it in the classical context of sets (with the only important exception that one must argue constructively).

Amongst other things, this discovery made it possible to:

- Exploit the inherent 'flexibility' of the notion of topos to construct 'new mathematical worlds' having particular properties.
- Consider models of any kind of (first-order) mathematical theory not just in the classical set-theoretic setting, but inside every topos, and hence 'relativise' Mathematics.

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Classifying toposes

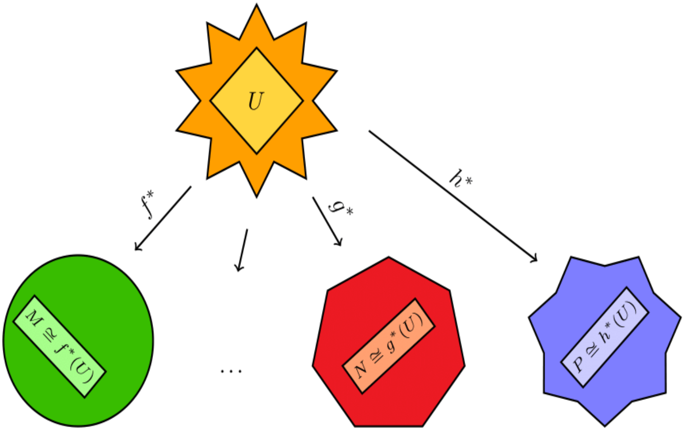
It was realized in the seventies (thanks to the work of several people, notably including W. Lawvere, A. Joyal, G. Reyes and M. Makkai) that:

- To any (geometric first-order) mathematical theory $\mathbb{T}$ one can canonically associate a topos $\mathscr{E}_{\mathbb{T}}$, called the classifying topos of the theory, which represents its 'semantical core'.
- The topos $\mathscr{E}_{\mathbb{T}}$ is characterized by the following representability property: for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

  natural in $\mathscr{E}$, where
  - $\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}})$ is the category of geometric morphisms $\mathscr{E} \to \mathscr{E}_{\mathbb{T}}$ and
  - $\mathbb{T}\text{-mod}(\mathscr{E})$ is the category of $\mathbb{T}$-models in $\mathscr{E}$.

Classifying toposes



Classifying topos

Toposes as theories up to 'Morita-equivalence'

- Two mathematical theories have the same classifying topos (up to equivalence) if and only if they have the same 'semantical core', that is if and only if they are indistinguishable from a semantic point of view; such theories are said to be Morita-equivalent.
- Conversely, every Grothendieck topos arises as the classifying topos of some theory.
- So a topos can be seen as a canonical representative of equivalence classes of theories modulo Morita-equivalence.

Toposes as *bridges*

- The notion of Morita-equivalence is ubiquitous in Mathematics; indeed, it formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods'.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted in terms of Morita-equivalences.
- On the other hand, Topos Theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.
- Any two theories which are bi-interpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.
- A mathematical theory alone gives rise to an infinite number of Morita-equivalences, through its 'internal dynamics'.

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Toposes as *bridges*

- In the topos-theoretic study of theories, the latter are represented by sites (of definition of their classifying topos or of some other topos naturally attached to them).
- The existence of theories which are Morita-equivalent to each other translates into the existence of different sites of definition (or, more generally, presentations) for the same Grothendieck topos.
- Grothendieck toposes can be effectively used as 'bridges' for transferring notions, properties and results across different Morita-equivalent theories:

- The transfer of information takes place by expressing topos-theoretic invariants in terms of the different sites of definition (or, more generally, presentations) for the given topos.
- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

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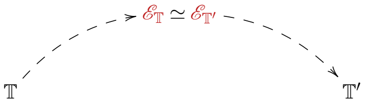

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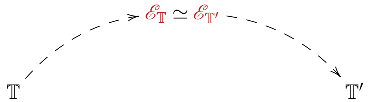

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$$\mathbb{T} \dashrightarrow \mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'} \dashrightarrow \mathbb{T}'$$

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Toposes as *bridges*

- This methodology is technically effective because the relationship between a topos and its representations is often very natural, enabling us to easily transfer invariants across different representations (and hence, between different theories).
- On the other hand, the 'bridge' technique is highly non-trivial, in the sense that it often yields deep and surprising results. This is due to the fact that a given invariant can manifest itself in significantly different ways in the context of different presentations.
- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories and constructions. Indeed, many important invariants of mathematical structures are actually invariants of toposes (think for instance of cohomology or homotopy groups) and topos-theoretic invariants considered on the classifying topos $\mathscr{E}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ often translate into interesting logical (i.e. syntactic or semantic) properties of $\mathbb{T}$.

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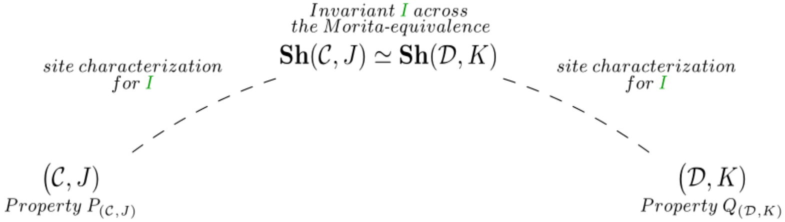
The 'bridge-building' technique

- Decks of 'bridges': Morita-equivalences (or more generally morphisms or other kinds of relations between toposes)
- Arches of 'bridges': Site characterizations for topos-theoretic invariants (or more generally 'unravelings' of topos-theoretic invariants in terms of concrete representations of the relevant topos)



The 'bridge' yields a logical equivalence (or an implication) between the 'concrete' properties $P_{(\mathscr{C},J)}$ and $Q_{(\mathscr{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

A few selected applications

Since the theory of topos-theoretic 'bridges' was introduced in 2010, several applications of it have been obtained in different fields of Mathematics, such as:

- Model theory (topos-theoretic Fraïssé theorem)
- Proof theory (various results for first-order theories)
- Algebra (topos-theoretic generalization of topological Galois theory)
- Topology (topos-theoretic interpretation/generation of Stone-type and Priestley-type dualities)
- Functional analysis (various results on Gelfand spectra and Wallman compactifications)
- Many-valued logics and lattice-ordered groups (three joint papers with A. C. Russo)
- Cyclic homology, as reinterpreted by A. Connes (work on *"cyclic theories"*, jointly with N. Wentzlaff)
- Algebraic geometry (logical analysis of (co)homological motives, cf. the paper *"Syntactic categories for Nori motives"* joint with L. Barbieri-Viale and L. Lafforgue)

Toposes as 'bridges' and the Erlangen Program

The methodology 'toposes as bridges' is a vast extension of Felix Klein's Erlangen Program (A. Joyal)

More specifically:

- Every group gives rise to a topos (namely, the category of actions of it), but the notion of topos is much more general.
- As Klein classified geometries by means of their automorphism groups, so we can study first-order geometric theories by studying the associated classifying toposes.
- As Klein established surprising connections between very different-looking geometries through the study of the algebraic properties of the associated automorphism groups, so the methodology 'toposes as bridges' allows to discover non-trivial connections between properties, concepts and results pertaining to different mathematical theories through the study of the categorical invariants of their classifying toposes.

Some examples of 'bridges'

We shall now discuss a few 'bridges' established in the context of the above-mentioned applications:

- Topological Galois theory
- Theories of presheaf type
- Topos-theoretic Fraïssé theorem
- Stone-type dualities

The results are completely *different*... but the methodology is always the *same*!

Topological Galois theory

Recall that classical topological Galois theory provides, given a Galois extension $F \subseteq L$, a bijective correspondence between the intermediate field extensions (resp. finite field extensions) $F \subseteq K \subseteq L$ and the closed (resp. open) subgroups of the Galois group $Aut_F(L)$.

This admits the following categorical reformulation: the functor $K \to \mathrm{Hom}(K, L)$ defines an equivalence of categories

$$(\mathscr{L}_F^L)^{\mathrm{op}} \simeq \mathbf{Cont}_t(Aut_F(L)),$$

where $\mathscr{L}_F^L$ is the category of finite intermediate field extensions and $\mathbf{Cont}_t(Aut_F(L))$ is the category of continuous non-empty transitive actions of $Aut_F(L)$ on discrete sets.

A natural question thus arises: can we characterize the categories $\mathscr{C}$ whose dual is equivalent to (or fully embeddable into) the category of (non-empty transitive) actions of a topological automorphism group?

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The topos-theoretic interpretation

Key observation: the above equivalence extends to an equivalence of toposes

$$\mathbf{Sh}({\mathscr{L}_F^L}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_F(L)),$$

where J_{at} is the atomic topology on ${\mathscr{L}_F^L}^{\mathrm{op}}$ and $\mathbf{Cont}(Aut_F(L))$ is the topos of continuous actions of $Aut_F(L)$ on discrete sets.

It is therefore natural to investigate our problem by using the methods of topos theory: more specifically, we shall look for conditions on a small category $\mathscr{C}$ and on an object u of its ind-completion for the existence of an equivalence of toposes of the form

$$\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u)) .$$

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The key notions I

- A category $\mathscr{C}$ is said to satisfy the amalgamation property (AP) if for every objects $a, b, c \in \mathscr{C}$ and morphisms $f : a \to b$, $g : a \to c$ in $\mathscr{C}$ there exists an object $d \in \mathscr{C}$ and morphisms $f' : b \to d$, $g' : c \to d$ in $\mathscr{C}$ such that $f' \circ f = g' \circ g$:

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ c & \xrightarrow[g']{} & d \end{array}$$

- A category $\mathscr{C}$ is said to satisfy the joint embedding property (JEP) if for every pair of objects $a, b \in \mathscr{C}$ there exists an object $c \in \mathscr{C}$ and morphisms $f : a \to c$, $g : b \to c$ in $\mathscr{C}$:

$$\begin{array}{ccc} & & a \\ & & \downarrow{\scriptstyle f} \\ b & \xrightarrow[g]{} & c \end{array}$$

The key notions II

- An object $u \in \text{Ind-}\mathscr{C}$ is said to be $\mathscr{C}$-universal if for every $a \in \mathscr{C}$ there exists an arrow $\chi : a \to u$ in $\text{Ind-}\mathscr{C}$:

$$a \overset{\chi}{\dashrightarrow} u$$

- An object $u \in \text{Ind-}\mathscr{C}$ is said to be $\mathscr{C}$-ultrahomogeneous if for any object $a \in \mathscr{C}$ and arrows $\chi_1 : a \to u$, $\chi_2 : a \to u$ in $\text{Ind-}\mathscr{C}$ there exists an automorphism $j : u \to u$ such that $j \circ \chi_1 = \chi_2$:

$$\begin{array}{ccc} a & \xrightarrow{\chi_1} & u \\ & \searrow^{\chi_2} & \downarrow j \\ & & u \end{array}$$

Topological Galois theory as a 'bridge'

Theorem

Let $\mathscr{C}$ be a small category satisfying the amalgamation and joint embedding properties, et let u be a $\mathscr{C}$-universal et $\mathscr{C}$-ultrahomogeneous object of the ind-completion Ind-$\mathscr{C}$ of $\mathscr{C}$. Then there is an equivalence of toposes

$$\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u)),$$

where $Aut(u)$ is endowed with the topology in which a basis of open neighbourhoods of the identity is given by the subgroups of the form $I_\chi = \{\alpha \in Aut(u) \mid \alpha \circ \chi = \chi\}$ for $\chi : c \to u$ an arrow in Ind-$\mathscr{C}$ from an object c of $\mathscr{C}$.

This equivalence is induced by the functor

$$F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cont}(Aut(u))$$

which sends any object c of $\mathscr{C}$ on the set $\mathrm{Hom}_{\text{Ind-}\mathscr{C}}(c, u)$ (endowed with the obvious action of $Aut(u)$) and any arrow $f : c \to d$ in $\mathscr{C}$ to the $Aut(u)$-equivariant map

$$- \circ f : \mathrm{Hom}_{\text{Ind-}\mathscr{C}}(d, u) \to \mathrm{Hom}_{\text{Ind-}\mathscr{C}}(c, u) .$$

Topological Galois theory as a ‘bridge’

The following result arises from two ‘bridges’, respectively obtained by considering the invariant notions of atom and of arrow between atoms.

Theorem

Under the hypotheses of the last theorem, the functor F is full and faithful if and only if every arrow of $\mathscr{C}$ is a strict monomorphism, and it is an equivalence on the full subcategory $\mathbf{Cont}_t(Aut(u))$ of $\mathbf{Cont}(Aut(u))$ on the non-empty transitive actions if $\mathscr{C}$ is moreover atomically complete.

$$\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u))$$

$$\mathscr{C}^{\mathrm{op}} \qquad\qquad\qquad \mathbf{Cont}_t(Aut(u))$$

This theorem generalizes Grothendieck’s theory of Galois categories and can be applied for generating Galois-type theories in different fields of Mathematics, for example that of finite groups and that of finite graphs.

Moreover, if a category $\mathscr{C}$ satisfies the first but not the second condition of the theorem, our topos-theoretic approach gives us a fully explicit way to complete it, by means of the addition of ‘imaginaries’, so that also the second condition gets satisfied.

Theories of presheaf type

Definition

A geometric theory is said to be of presheaf type if it is classified by a presheaf topos.

Theories of presheaf type are very important in that they constitute the basic 'building blocks' from which every geometric theory can be built. Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a 'quotient' of a theory of presheaf type.

These theories are, in a precise technical sense, the logical counterpart of small categories.

Most importantly, any theory of presheaf type $\mathbb{T}$ gives rise to two different representations of its classifying topos, which can be used to build 'bridges' connecting its syntax and semantics:

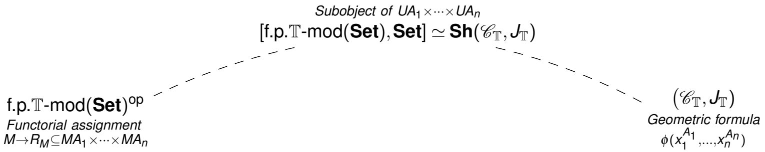
$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

$$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \qquad\qquad (\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

A definability theorem

Theorem

Let $\mathbb{T}$ be a theory of preshef type and suppose that we are given, for every finitely presentable **Set***-model $\mathscr{M}$ of $\mathbb{T}$, a subset $R_{\mathscr{M}}$ of $\mathscr{M}^n$ in such a way that every $\mathbb{T}$-model homomorphism $h : \mathscr{M} \to \mathscr{N}$ maps $R_{\mathscr{M}}$ into $R_{\mathscr{N}}$. Then there exists a geometric formula-in-context $\phi(x_1, \ldots, x_n)$ such that $R_{\mathscr{M}} = [[\vec{x} \,.\, \phi]]_{\mathscr{M}}$ for each finitely presentable $\mathbb{T}$-model $\mathscr{M}$.*



Topos-theoretic Fraïssé theorem

The following result, which generalizes Fraïssé's theorem in classical model theory, arises from a triple 'bridge'.

Definition

A set-based model M of a geometric theory $\mathbb{T}$ is said to be homogeneous if for any arrow $y : c \to M$ in $\mathbb{T}$-mod(**Set**) and any arrow f in f.p.$\mathbb{T}$-mod(**Set**) there exists an arrow u in $\mathbb{T}$-mod(**Set**) such that $u \circ f = y$:

$$\begin{array}{ccc} c & \xrightarrow{\;y\;} & M \\ {\scriptstyle f}\downarrow & \nearrow_{u} & \\ d & & \end{array}$$

Theorem

Let $\mathbb{T}$ be a theory of presheaf type such that the category f.p.$\mathbb{T}$-mod(**Set**) *is non-empty and has AP and JEP. Then the theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models is complete and atomic.*

Topos-theoretic Fraïssé theorem

Atomic topos
$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}, J_{at})$
Atomic site i.e. AP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Atomicity of $\mathbb{T}'$

Two-valued topos
$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}, J_{at})$
JEP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Completeness of $\mathbb{T}'$

Point of
$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}, J_{at})$
homogeneous $\mathbb{T}$-model in **Set**

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
$\mathbb{T}'$-model in **Set**

Stone-type dualities through 'bridges'

The 'bridge-building' technique allows one to unify all the classical Stone-type dualities between special kinds of preorders and partial orders, locales or topological spaces as instances of just one topos-theoretic phenomenon, and to generate many new such dualities.

More precisely, this machinery generates Stone-type dualities/equivalences by functorializing 'bridges' of the form

$$\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}}) \simeq \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}})$$

$$\mathscr{C} \qquad\qquad\qquad\qquad \mathscr{D}$$

where

- $\mathscr{C}$ is a preorder (regarded as a category),
- $J_{\mathscr{C}}$ is a (subcanonical) Grothendieck topology on $\mathscr{C}$,
- $\mathscr{C}$ is a $K_{\mathscr{D}}$-dense full subcategory of $\mathscr{D}$, and
- $J_{\mathscr{C}}$ is the induced Grothendieck topology $(K_{\mathscr{D}})|_{\mathscr{C}}$ on $\mathscr{C}$.

Future directions

The evidence provided by the results obtained so far shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories and for generating new equivalences, dualities and symmetries across different fields of Mathematics.

In fact, toposes have an authentic creative power in Mathematics, in the sense that their study naturally leads to the discovery of a great number of notions and 'concrete' results in different mathematical fields, which are pertinent but often unsuspected.

In the next years, we intend to continue pursuing the development of these general unifying methodologies both at the theoretical level and at the applied level, in order to continue developing the potential of toposes as fundamental tools in the study of mathematical theories and their relations, and as key concepts defining a new way of doing Mathematics liable to bring distinctly new insights in a great number of different subjects.

Future directions

Central themes in this programme will be:

- investigation of important dualities or correspondences in Mathematics from a topos-theoretic perspective (in particular, the theory of motives, class field theory and the Langlands programme)
- systematic study of invariants of toposes in terms of their presentations, and introduction of new invariants which capture important aspects of concrete mathematical problems
- interpretation and generalization of important parts of classical and modern model theory in terms of toposes and development of a functorial model theory
- introduction of new methodologies for generating Morita-equivalences
- development of general techniques for building spectra by using classifying toposes
- generalization of the 'bridge' technique to the setting of higher categories and toposes through the introduction of higher geometric logic
- development of a relative theory of classifying toposes

For further reading

O. Caramello
Grothendieck toposes as unifying 'bridges' in Mathematics,
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available from my website www.oliviacaramello.com.

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Oxford University Press (2017).