Toposic Fraïssé-Galois theory and motivic toposes

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Plan of the talk

- Grothendieck toposes as unifying 'bridges'
- Toposic Galois theory
- Toposic Fraïssé theory
- The unification of Galois and Fraïssé theory through toposes as 'bridges'
- Application to the construction of *motivic toposes*

The "unifying notion" of topos

In this talk the term 'topos' will always mean 'Grothendieck topos'.

> *"C'est le thème du topos qui est ce "lit", ou cette "rivière profonde" où viennent s'épouser la géométrie et l'algèbre, la topologie et l'arithmétique, la logique mathématique et la théorie des catégories, le monde du continu et celui des structures "discontinues" ou "discrètes". Il est ce que j'ai conçu de plus vaste, pour saisir avec finesse, par un même langage riche en résonances géométriques, une "essence" commune à des situations des plus éloignées les unes des autres provenant de telle région ou de telle autre du vaste univers des choses mathématiques".*
>
> *A. Grothendieck*

Since the times of my Ph.D. studies, I have developed a theory and a number of techniques allowing one to exploit the unifying potential of the notion of topos for establishing 'bridges' across different mathematical theories, by building in particular on the notion of classifying topos educed by categorical logicians.

Toposes as unifying 'bridges'

This theory, introduced in the programmatic paper "*The unification of Mathematics via Topos Theory*" of 2010, allows one to exploit the technical flexibility inherent to the concept of topos - most notably, the possibility of presenting a topos in a multitude of different ways - for building unifying 'bridges' useful for transferring notions, ideas and results across different mathematical contexts.

In the last years, besides leading to the solution of a number of long-standing problems in categorical logic, these techniques have generated several substantial applications in different mathematical fields. Still, much remains to be done so that toposes become a key tool universally used for investigating mathematical theories and their relations.

In fact, these 'bridges' have proved useful not only for connecting different mathematical theories with each other, but also for investigating a given theory from multiple points of view.

A few selected applications

Since the theory of topos-theoretic 'bridges' was introduced, several applications of it have been obtained in different fields of Mathematics, such as:

- Model theory (topos-theoretic Fraïssé theorem)
- Proof theory (various results for first-order theories)
- Algebra (topos-theoretic generalization of topological Galois theory)
- Topology (topos-theoretic interpretation/generation of Stone-type and Priestley-type dualities)
- Functional analysis (various results on Gelfand spectra and Wallman compactifications)
- Many-valued logics and lattice-ordered groups (two joint papers with A. C. Russo)
- Cyclic homology, as reinterpreted by A. Connes (work on *"cyclic theories"*, jointly with N. Wentzlaff)
- Algebraic geometry (logical analysis of (co)homological motives, cf. the paper *"Syntactic categories for Nori motives"* joint with L. Barbieri-Viale and L. Lafforgue)

The multifaceted nature of toposes

The role of toposes as unifying spaces is intimately tied to their multifaceted nature.

For instance, a Grothendieck topos can be seen as:

- a generalized space
- a mathematical universe
- a theory modulo Morita equivalence (a relation identifying two theories when they have, broadly speaking, the same mathematical content)

The theory of topos-theoretic ‘bridges’ combines all these different perspectives to provide tools for making toposes effective means for studying mathematical theories from multiple points of view, relating and unifying theories with each other and constructing ‘bridges’ across them.

Classifying toposes

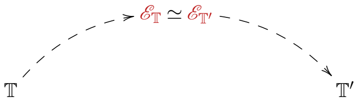
Grothendieck toposes are objects which are capable of capturing the essence of a great variety of different mathematical contexts. In particular, they can emboby the semantic content of a very wide class of theories:

Indeed, in the seventies, thanks to the work of a number of categorical logicians, notably including M. Makkai and G. Reyes, it was discovered that:

- With any mathematical theory $\mathbb{T}$ (of a very general form) one can canonically associate a topos $\mathscr{E}_{\mathbb{T}}$, called its classifying topos, which represents its 'semantical core'.
- Two given mathematical theories have the same classifying topos (up to equivalence) if and only if they have the same 'semantical core', that is, if and only if they are indistinguishable from a semantic viewpoint. Two such theories are said to be Morita-equivalent.
- Conversely, any topos is the classifying topos of some theory (in fact, of infinitely many theories).

The theory of topos-theoretic 'bridges'

- The notion of Morita-equivalence formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods', which explains its ubiquity in Mathematics.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted as arising from Morita-equivalences.
- Any two theories which are bi-interpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.
- The existence of different theories with the same classifying topos translates, at the technical level, into the existence of different presentations for the same topos.
- Topos-theoretic invariants, that is properties of (or construction on) toposes which are invariant with respect to their different presentations, can thus be used to transfer information from one theory to another:



- Transfers of information take place by expressing a given invariant in terms of the different presentations of the topos.

Toposic Galois theory

Recall that classical topological Galois theory provides, given a Galois extension $F \subseteq L$, a bijective correspondence between the intermediate field extensions (resp. finite field extensions) $F \subseteq K \subseteq L$ and the closed (resp. open) subgroups of the Galois group $Aut_F(L)$.

This admits the following categorical reformulation: the functor $K \to \mathrm{Hom}(K, L)$ defines an equivalence of categories

$$(\mathscr{L}_F^L)^{\mathrm{op}} \simeq \mathbf{Cont}_t(Aut_F(L)),$$

where $\mathscr{L}_F^L$ is the category of finite intermediate field extensions and $\mathbf{Cont}_t(Aut_F(L))$ is the category of continuous non-empty transitive actions of $Aut_F(L)$ on discrete sets.

A natural question thus arises: can we characterize the categories $\mathscr{C}$ whose dual is equivalent to (or fully embeddable into) the category of (non-empty transitive) actions of a topological automorphism group?

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The topos-theoretic interpretation

Key observation: the above equivalence extends to an equivalence of toposes

$$\mathbf{Sh}({\mathscr{L}_F^L}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_F(L)),$$

where J_{at} is the atomic topology on ${\mathscr{L}_F^L}^{\text{op}}$ and $\mathbf{Cont}(Aut_F(L))$ is the topos of continuous actions of $Aut_F(L)$ on discrete sets.

It is therefore natural to investigate our problem by using the methods of topos theory: more specifically, we shall look for conditions on a small category $\mathscr{C}$ and on an object u of its ind-completion for the existence of an equivalence of toposes of the form

$$\mathbf{Sh}(\mathscr{C}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u)) .$$

We will then be able to obtain, starting from such an equivalence, an answer to our question, and hence build Galois-type theories in a great variety of different mathematical contexts.

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The key notions I

- A category $\mathscr{C}$ is said to satisfy the amalgamation property (AP) if for every objects $a, b, c \in \mathscr{C}$ and morphisms $f : a \to b$, $g : a \to c$ in $\mathscr{C}$ there exists an object $d \in \mathscr{C}$ and morphisms $f' : b \to d$, $g' : c \to d$ in $\mathscr{C}$ such that $f' \circ f = g' \circ g$:

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ c & \underset{g'}{\dashrightarrow} & d \end{array}$$

- A category $\mathscr{C}$ is said to satisfy the joint embedding property (JEP) if for every pair of objects $a, b \in \mathscr{C}$ there exists an object $c \in \mathscr{C}$ and morphisms $f : a \to c$, $g : b \to c$ in $\mathscr{C}$:

$$\begin{array}{ccc} & & a \\ & & \downarrow{\scriptstyle f} \\ b & \underset{g}{\dashrightarrow} & c \end{array}$$

The key notions II

- An object $u \in \text{Ind-}\mathscr{C}$ is said to be $\mathscr{C}$-universal if for every $a \in \mathscr{C}$ there exists an arrow $\chi : a \to u$ in $\text{Ind-}\mathscr{C}$:

$$a \overset{\chi}{\dashrightarrow} u$$

- An object $u \in \text{Ind-}\mathscr{C}$ is said to be $\mathscr{C}$-ultrahomogeneous if for any object $a \in \mathscr{C}$ and arrows $\chi_1 : a \to u$, $\chi_2 : a \to u$ in $\text{Ind-}\mathscr{C}$ there exists an automorphism $j : u \to u$ such that $j \circ \chi_1 = \chi_2$:

$$\begin{array}{ccc} a & \xrightarrow{\chi_1} & u \\ & {\scriptstyle \chi_2} \searrow & \downarrow {\scriptstyle j} \\ & & u \end{array}$$

Topological Galois theory as a 'bridge'

Theorem

Let $\mathscr{C}$ be a small category satisfying the amalgamation and joint embedding properties, et let u be a $\mathscr{C}$-universal et $\mathscr{C}$-ultrahomogeneous object of the ind-completion Ind-$\mathscr{C}$ *of $\mathscr{C}$. Then there is an equivalence of toposes*

$$\mathbf{Sh}(\mathscr{C}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u)),$$

where $Aut(u)$ is endowed with the topology in which a basis of open neighbourhoods of the identity is given by the subgroups of the form $I_\chi = \{\alpha \in Aut(u) \mid \alpha \circ \chi = \chi\}$ for $\chi : c \to u$ an arrow in Ind-$\mathscr{C}$ *from an object c of $\mathscr{C}$.*
This equivalence is induced by the functor

$$F : \mathscr{C}^{\text{op}} \to \mathbf{Cont}(Aut(u))$$

which sends any object c of $\mathscr{C}$ on the set $\mathrm{Hom}_{\text{Ind-}\mathscr{C}}(c, u)$ (endowed with the obvious action of $Aut(u)$) and any arrow $f : c \to d$ in $\mathscr{C}$ to the $Aut(u)$-equivariant map

$$- \circ f : \mathrm{Hom}_{\text{Ind-}\mathscr{C}}(d, u) \to \mathrm{Hom}_{\text{Ind-}\mathscr{C}}(c, u) .$$

Topological Galois theory as a 'bridge'

The following result arises from two 'bridges', respectively obtained by considering the invariant notions of atom and of arrow between atoms.

Theorem

Under the hypotheses of the last theorem, the functor F is full and faithful if and only if every arrow of $\mathscr{C}$ is a strict monomorphism, and it is an equivalence on the full subcategory $\mathbf{Cont}_t(Aut(u))$ of $\mathbf{Cont}(Aut(u))$ on the non-empty transitive actions if $\mathscr{C}$ is moreover atomically complete.

$$\mathbf{Sh}(\mathscr{C}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u))$$

$\mathscr{C}^{\text{op}}$ $\qquad\qquad$ $\mathbf{Cont}_t(Aut(u))$

This theorem generalizes Grothendieck's theory of Galois categories and can be applied for generating Galois-type theories in different fields of Mathematics, for example that of finite groups and that of finite graphs.

Moreover, if a category $\mathscr{C}$ satisfies the first but not the second condition of the theorem, our topos-theoretic approach gives us a fully explicit way to complete it, by means of the addition of 'imaginaries', so that also the second condition gets satisfied.

Theories of presheaf type

- A geometric theory $\mathbb{T}$ over a signature Σ is said to be of presheaf type if it is classified by a presheaf topos.
- A model M of a theory of presheaf type $\mathbb{T}$ in the category **Set** is said to be finitely presentable if the functor $\text{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(M, -) : \mathbb{T}\text{-mod}(\mathbf{Set}) \to \mathbf{Set}$ preserves filtered colimits.

Theories of presheaf type are very important in that they constitute the basic 'building blocks' from which every geometric theory can be built. Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a 'quotient' of a theory of presheaf type. In fact, theories of presheaf type represent the logical equivalent of small categories.

Most importantly, any theory of presheaf type $\mathbb{T}$ gives rise to two different representations of its classifying topos, which can be used to build 'bridges' connecting its syntax and semantics:

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

$$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \qquad\qquad (\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

Topos-theoretic Fraïssé theorem

The following result, which generalizes Fraïssé's theorem in classical model theory, arises from a triple 'bridge'.

Definition

A set-based model M of a geometric theory $\mathbb{T}$ is said to be homogeneous if for any arrow $y : c \to M$ in $\mathbb{T}$-mod(**Set**) and any arrow f in f.p.$\mathbb{T}$-mod(**Set**) there exists an arrow u in $\mathbb{T}$-mod(**Set**) such that $u \circ f = y$:

$$\begin{array}{ccc} c & \xrightarrow{\;y\;} & M \\ {\scriptstyle f}\downarrow & \nearrow_{u} & \\ d & & \end{array}$$

Theorem

Let $\mathbb{T}$ be a theory of presheaf type such that the category f.p.$\mathbb{T}$-mod(**Set**) *is non-empty and has AP and JEP. Then the theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models is complete and atomic.*

Topos-theoretic Fraïssé theorem

Atomic topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
Atomic site i.e.
AP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Atomicity of $\mathbb{T}'$

Two-valued topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
JEP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Completeness of $\mathbb{T}'$

Point of

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
homogeneous $\mathbb{T}$-model in **Set**

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
$\mathbb{T}'$-model in **Set**

Sketch of the proof

Let $\mathbb{T}$ be a theory of presheaf type. If the category f.p.$\mathbb{T}$-mod(**Set**) satisfies AP then the subtopos

$$i_{at} : \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \hookrightarrow [\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$$

(where J_{at} is the atomic topology on f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$) transfers, via the syntax-semantics equivalence for the classifying topos for $\mathbb{T}$ considered above, to a subtopos of $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$, which can in turn be identified, via the duality theorem between quotients and subtoposes (proved in my Ph.D. thesis), with the canonical inclusion

$$i : \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \hookrightarrow \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

of the classifying topos of a unique quotient $\mathbb{T}'$ of $\mathbb{T}$ into the classifying topos of $\mathbb{T}$.

We thus obtain a commutative diagram

$$\begin{array}{ccc} [\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] & \xrightarrow{\simeq} & \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}}) \\ \uparrow i_{at} & & \uparrow i \\ \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) & \xrightarrow{\simeq} & \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \end{array} .$$

Sketch of the proof

- According to the method of 'toposes as bridges', the equivalence

$$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$$

  is taken as the starting point of the investigation.

- One proceeds to extract information about it by considering various topos-theoretic invariants from the points of view of the two sites of definition of the given classifying topos.

- The different properties involved in Fraïssé's theorem are interpreted as different manifestations of a unique property lying at the topos-theoretic level.

Atomicity

Definition

A Grothendieck topos is said to be atomic if all its subobject lattices are atomic complete Boolean algebras.

Theorem

Let $(\mathscr{C}, J)$ be a site. The topos $\mathbf{Sh}(\mathscr{C}, J)$ is atomic if and only if for every $c \in \mathscr{C}$ there exists a J-covering sieve on c generated by arrows f with the property that $\emptyset \notin J(dom(f))$ and for every arrow g which factors through f, either $\{k : dom(k) \to dom(f) \mid f \circ k \text{ factors through } g\} \in J(dom(f))$ or $\emptyset \in J(dom(g))$. In particular:

- *If $\mathscr{C}^{\text{op}}$ satisfies AP and J is the atomic topology on $\mathscr{C}$ then $\mathbf{Sh}(\mathscr{C}, J)$ is atomic;*
- *If $(\mathscr{C}, J)$ is the syntactic site of a geometric theory $\mathbb{T}$ then $\mathbf{Sh}(\mathscr{C}, J)$ is atomic if and only if $\mathbb{T}$ is atomic, i.e. for every context $\vec{x}$ over the signature of $\mathbb{T}$, there is a set $B_{\vec{x}}$ of $\mathbb{T}$-complete geometric formulae in that context such that $\top \vdash_{x} \bigvee_{\phi \in B_{\vec{x}}} \phi$ is provable in $\mathbb{T}$ (where by $\mathbb{T}$-complete formula we mean a geometric formula $\phi(\vec{x})$ such that the sequent $\phi \vdash_{\vec{x}} \bot$ is not provable in $\mathbb{T}$, but for every geometric formula ψ in the same context either $\psi \wedge \phi \vdash_{\vec{x}} \bot$ or $\phi \vdash_{\vec{x}} \psi$ is provable in $\mathbb{T}$).*

Two-valuedness

Definition

A Grothendieck topos is said to be two-valued if the only subobjects of the terminal object are the zero one and the identity one, and they are distinct from each other.

Theorem

Let $(\mathscr{C}, J)$ be a site. Then the topos $\mathbf{Sh}(\mathscr{C}, J)$ is two-valued if and only if the only J-ideals on $\mathscr{C}$ are the trivial ones, and they are distinct from each other. In particular:

- *If $\mathscr{C}^{\text{op}}$ is non-empty and satisfies AP and J is the atomic topology on $\mathscr{C}$ then $\mathbf{Sh}(\mathscr{C}, J)$ is two-valued if and only if $\mathscr{C}^{\text{op}}$ satisfies JEP.*
- *If $(\mathscr{C}, J)$ is the syntactic site of a geometric theory $\mathbb{T}$ then $\mathbf{Sh}(\mathscr{C}, J)$ is two-valued if and only if $\mathbb{T}$ is complete, i.e. for any geometric sentence ϕ over the signature of $\mathbb{T}$, either ϕ is $\mathbb{T}$-provably equivalent to $\bot$ or to $\top$, but not both.*

Theorem

Let $\mathbb{T}$ be a geometric theory. If $\mathbb{T}$ is complete and atomic then $\mathbb{T}$ is countably categorical, i.e. any two countable models of $\mathbb{T}$ in $\mathbf{Set}$ are isomorphic.

The theory of homogeneous models

Definition

A point of a Grothendieck topos $\mathscr{E}$ is a geometric morphism $\mathbf{Set} \to \mathscr{E}$.

Theorem

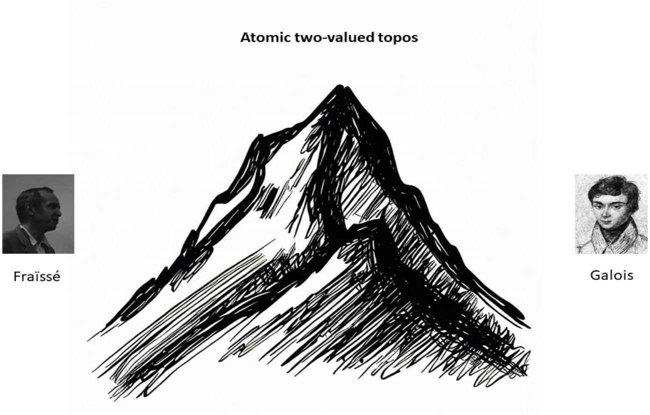
Let $(\mathscr{C}, J)$ be a site. Then the points of the topos $\mathbf{Sh}(\mathscr{C}, J)$ correspond to the J-continuous flat functors on $\mathscr{C}$. In particular:

- *If $(\mathscr{C}, J)$ is the syntactic site of a geometric theory $\mathbb{T}$ then the points of the topos $\mathbf{Sh}(\mathscr{C}, J)$ correspond to the models of $\mathbb{T}$ in $\mathbf{Set}$.*
- *If $\mathscr{C}$ is the opposite of the category f.p.$\mathbb{T}$-mod$(\mathbf{Set})$ (for a theory of presheaf type $\mathbb{T}$) and J is the atomic topology on $\mathscr{C}$ then the points of the topos $\mathbf{Sh}(\mathscr{C}, J)$ correspond to the homogeneous $\mathbb{T}$-models in $\mathbf{Set}$, i.e. to the models $M \in \mathbb{T}\text{-mod}(\mathbf{Set})$ such that for each arrow $f : c \to d$ in $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ and arrow $y : c \to M$ in $\mathbb{T}\text{-mod}(\mathbf{Set})$ there exists an arrow $u_f : d \to M$ in $\mathbb{T}\text{-mod}(\mathbf{Set})$ such that $y = u_f \circ f$:*

$$\begin{array}{ccc} c & \xrightarrow{\;y\;} & M \\ {\scriptstyle f}\downarrow & \nearrow_{u_f} & \\ d & & \end{array}$$

A unified Fraïssé-Galois theory

From the topos-theoretic perspective that we have adopted, Fraïssé theory and Galois theory appear as two independent ways of investigating the same mathematical content, embodied by an atomic two-valued topos. Indeed, Fraïssé explores it from the point of view of atomic sites, while Galois from the point of view of groups:



Motivic toposes

It is natural to wonder whether the ℓ-adic cohomological functors can be viewed as points of an atomic two-valued topos as above. If so, all the known independence from ℓ properties would follow from this structural result.

More specifically, the paper *Motivic toposes* proposes to take, for each cohomology theory T, $\mathbb{T}_T$ equal to a presheaf completion of a theory consisting of all the Horn sequents over a first-order language for schemes which are satisfied by T, and shows that the following two properties entail the fact that the ℓ-adic cohomological functors satisfy the same first-order properties written in the language of schemes:

- ℓ-adic cohomology is homogeneous with respect to its finitely generated substructures (i.e., it is homogeneous as a model of the associated theory of presheaf type);
- the theories of presheaf type $\mathbb{T}_T$ do not depend from the ℓ-adic cohomological functor T.

Quite remarkably, the homogeneity condition subsumes all the usual exactness conditions that are known to hold for cohomological functors.

Future directions

The evidence provided by the results obtained so far shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories and for generating new equivalences, dualities and symmetries across different fields of Mathematics.

In fact, toposes have an authentic creative power in Mathematics, in the sense that their study naturally leads to the discovery of a great number of notions and 'concrete' results in different mathematical fields, which are pertinent but often unsuspected.

In the next years, we intend to continue pursuing the development of these general unifying methodologies both at the theoretical level and at the applied level, in order to continue developing the potential of toposes as fundamental tools in the study of mathematical theories and their relations, and as key concepts defining a new way of doing Mathematics liable to bring distinctly new insights in a great number of different subjects.

Future directions

Central themes in this programme will be:

- investigation of important dualities or correspondences in Mathematics from a topos-theoretic perspective (in particular, the theory of motives, class field theory and the Langlands programme)
- systematic study of invariants of toposes in terms of their presentations, and introduction of new invariants which capture important aspects of concrete mathematical problems
- interpretation and generalization of important parts of classical and modern model theory in terms of toposes and development of a functorial model theory
- introduction of new methodologies for generating Morita-equivalences
- development of general techniques for building spectra by using classifying toposes
- generalization of the 'bridge' technique to the setting of higher categories and toposes through the introduction of higher geometric logic
- development of a relative theory of classifying toposes

A new institute for interdisciplinary mathematics

These research directions are notably pursued at a new institute (legally, a private foundation) that we have recently founded in Italy:

ISTITUTO GROTHENDIECK

www.igrothendieck.org

For further reading

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