# Relative topos theory via stacks

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# Plan of the talk

- Using stacks for relative topos theory
- Some preliminaries
- The big picture
- Relative toposes
  - Relative presheaf toposes
  - The fundamental adjunction
  - Relative sheaf toposes
- Future directions

# Topos theory over an arbitrary base topos

In this talk we shall present new foundations for relative topos theory (i.e. topos theory over an arbitrary base topos) based on stacks.

The approach of category theorists (Lawevere, Diasconescu, Johnstone, etc.) to this subject is chiefly based on the notions of internal category and of internal site.

The problem with these notions is that they are too rigid to naturally capture relative topos-theoretic phenomena, as well as for making computations and formalizing 'parametric reasoning'.

We shall resort to the more general and technically flexible notion of stack, developing the point of view originally introduced by J. Giraud in his paper *Classifying topos*.

# Stacks over a site

The role of stacks in our approach to relative topos theory is two-fold:

- On the one hand, the notion of stack represents a higher-order categorical generalization of the notion of sheaf. Accordingly, categories of stacks on a site represent higher-categorical analogues of Grothendieck toposes. One can thus expect to be able to lift a number of notions and constructions pertaining to sheaves (resp. Grothendieck toposes) to stacks (resp. categories of stacks on a site).
- On the other hand, stacks on a site $(\mathcal{C}, J)$ generalize internal categories in the topos $\mathbf{Sh}(\mathcal{C}, J)$. Since (usual) categories can be endowed with Grothendieck topologies, so stacks on a site can also be endowed with suitable analogues of Grothendieck topologies. This leads to the notion of *site relative to a base topos*, which is crucial for developing relative topos theory.

## Remark

*Every stack is equivalent to a split stack, that is to an internal category, but most stacks naturally arising in the mathematical practice are not split (think, for instance, of the canonical site of a topos).*

# The big picture

Our theory is based on a network of 2-adjunctions, as follows:

$$
\begin{tikzcd}
\mathbf{Ind}_{\mathcal{C}} \arrow[rr, shift left, "\Lambda"] \arrow[rr, phantom, "\perp"] \arrow[d, shift left, "s_J"] \arrow[d, phantom, "\vdash"] & & \mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J)^{co} \arrow[ll, shift left, "\Gamma"] \\
\mathbf{St}(\mathcal{C}, J) \arrow[u, hook, shift left] \arrow[rr, shift left, "\Lambda'"] \arrow[rr, phantom, "\perp"] \arrow[d, shift left, "E \circ \Lambda'"] \arrow[d, phantom, "\vdash"] & & \mathbf{EssTopos}/\mathbf{Sh}(\mathcal{C}, J)^{co} \arrow[ll, shift left, "\Gamma'"] \arrow[u, hook] \arrow[dll, shift right, "E"'] \arrow[dll, phantom, "\dashv"] \\
\mathbf{Sh}(\mathcal{C}, J) \arrow[u, hook, shift left] \arrow[urr, shift right, "L"'] & &
\end{tikzcd}
$$

In this diagram $\mathbf{Ind}_{\mathcal{C}}$ denotes the category of $\mathcal{C}$-indexed categories, $\mathbf{St}(\mathcal{C}, J)$ the category of $J$-stacks on $\mathcal{C}$ (where $(\mathcal{C}, J)$ is a small-generated site), $s_J$ the stackification functor, **Topos** the category of Grothendieck toposes and geometric morphisms and **EssTopos** the full subcategory on the essential geometric morphisms.

The functor $E$ sends an essential geometric morphism $f : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$ to the object $f_!(1_{\mathscr{E}})$ (where $f_!$ is the left adjoint to the inverse image $f^*$ of $f$) and the functor $L$ sends an object $P$ of $\mathbf{Sh}(\mathcal{C}, J)$ to the canonical local homeomorphism $\mathbf{Sh}(\mathcal{C}, J)/P \to \mathbf{Sh}(\mathcal{C}, J)$.

# Morphisms and comorphisms of sites

Let us recall that:

## Definition

- A morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that the composite $l' \circ F$, where $l'$ is the canonical functor $\mathcal{D} \to \mathbf{Sh}(\mathcal{D}, K)$, is flat and sends $J$-covering sieves to epimorphic families. If $\mathcal{C}$ and $\mathcal{D}$ have finite limits then $F$ is a morphism of sites if and only if it preserves finite limits.

- A comorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ is a functor $\pi : \mathcal{D} \to \mathcal{C}$ which has the covering-lifting property (in the sense that for any $d \in \mathcal{D}$ and any $J$-covering sieve $S$ on $\pi(d)$ there is a $K$-covering sieve $R$ on $d$ such that $\pi(R) \subseteq S$).

- Given sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, a functor $A : \mathcal{C} \to \mathcal{D}$ is said to be continuous if the functor

$$D_A := (- \circ A^{\text{op}}) : [\mathcal{D}^{\text{op}}, \mathbf{Set}] \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$$

restricts to a functor $\mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.

# Fibrations as comorphisms of sites

Recall that, given a functor $A : \mathcal{C} \to \mathcal{D}$ and a Grothendieck topology $K$ in $\mathcal{D}$, there is a smallest Grothendieck topology $M_K^A$ on $\mathcal{C}$ which makes $A$ a comorphism of sites to $(\mathcal{D}, K)$.

## Proposition (O.C. and R.Z.)

*If $A$ is a fibration, the topology $M_K^A$ admits the following simple description: a sieve $R$ is $M_K^A$-covering if and only if the collection of cartesian arrows in $R$ is sent by $A$ to a $K$-covering family.*

We shall call $M_K^A$ the Giraud topology induced by $K$, in honour of Jean Giraud, who used it for constructing the classifying topos $\mathbf{Sh}(\mathcal{C}, M_K^A)$ of a stack $A$ on $(\mathcal{D}, K)$.

## Proposition (O.C.)

*For any Grothendieck topology $K$ on $\mathcal{D}$, every morphism of fibrations $(A : \mathcal{C} \to \mathcal{D}) \to (A' : \mathcal{C}' \to \mathcal{D})$ yields a continuous comorphism of sites $(\mathcal{C}, M_K^A) \to (\mathcal{C}', M_K^{A'})$.*

*In particular, a fibration $A : \mathcal{C} \to \mathcal{D}$ yields a continuous comorphism of sites $(\mathcal{C}, M_K^A) \to (\mathcal{D}, K)$ for any Grothendieck topology $K$ on $\mathcal{D}$.*

# Giraud topologies

The study of the Giraud topology can provide insights on the given fibration. As a basic example of this, under the assumption that $J$ is subcanonical, the property of being a prestack can be checked directly by analysing the Giraud topology:

## Proposition (O.C. and R.Z.)

*Consider a subcanonical site $(\mathcal{C}, J)$ and a cloven fibration $p : \mathcal{D} \to \mathcal{C}$: then $p$ is a prestack if and only if the Giraud topology $M_J^p$ is subcanonical.*

We actually have a Giraud topology functor

$$\mathfrak{G} : \mathbf{Cat}/\mathcal{C} \to \mathbf{Com}/(\mathcal{C}, J),$$

mapping $[p : \mathcal{E} \to \mathcal{C}]$ to $p : (\mathcal{E}, M_J^p) \to (\mathcal{C}, J)$.

By the above results, this functor actually takes values in the subcategory of continuous comorphisms of sites.

# Morphisms induced by functors between sites

As is well-known, morphisms and comorphisms of sites induce geometric morphisms, as follows:

## Theorem

- *Every morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ induces a geometric morphism* $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.
- *Every comorphism of sites $\pi : (\mathcal{D}, K) \to (\mathcal{C}, J)$ induces a geometric morphism $C_\pi$* : $\mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.
  - *If $\pi$ is continuous then $C_\pi$ is moreover essential.*
  - *(O.C.) If $\pi$ is a fibration then $\pi$ is continuous and $C_\pi$ is even locally connected.*

## Remark

*For any geometric morphism $f : \mathcal{F} \to \mathcal{E}$, $f^*$ is a morphism of sites $(\mathcal{E}, J_{\mathcal{E}}^{\text{can}}) \to (\mathcal{F}, J_{\mathcal{F}}^{\text{can}})$ such that* $f = \mathbf{Sh}(f^*)$.

# From morphisms to comorphisms of sites

## Theorem (O.C.)

*Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Let $i_F$ be the functor $\mathcal{C} \to (1_\mathcal{D} \downarrow F)$ sending any object $c$ of $\mathcal{C}$ to the triplet $(F(c), c, 1_{F(c)})$ (and acting on arrows in the obvious way), and $\pi_\mathcal{C} : (1_\mathcal{D} \downarrow F) \to \mathcal{C}$ and $\pi_\mathcal{D} : (1_\mathcal{D} \downarrow F) \to \mathcal{D}$ the canonical projection functors. Let $\tilde{K}$ be the Grothendieck topology on $(1_\mathcal{D} \downarrow F)$ whose covering sieves are those whose image under $\pi_\mathcal{D}$ is $K$-covering. Then*

(i) *$\pi_\mathcal{C} \dashv i_F$, $\pi_\mathcal{D} \circ i_F = F$, $i_F$ is a morphism of sites $(\mathcal{C}, J) \to ((1_\mathcal{D} \downarrow F), \tilde{K})$ and $c_F := \pi_\mathcal{C}$ is a comorphism of sites $((1_\mathcal{D} \downarrow F), \tilde{K}) \to (\mathcal{C}, J)$;*

(ii) *$\pi_\mathcal{D} : ((1_\mathcal{D} \downarrow F), \tilde{K}) \to (\mathcal{D}, K)$ is both a morphism of sites and a comorphism of sites inducing equivalences*

$$C_{\pi_\mathcal{D}} : \mathbf{Sh}((1_\mathcal{D} \downarrow F), \tilde{K}) \to \mathbf{Sh}(\mathcal{D}, K)$$

*and*

$$\mathbf{Sh}(\pi_\mathcal{D}) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}((1_\mathcal{D} \downarrow F), \tilde{K})$$

*which are quasi-inverse to each other and make the following triangle commute:*

$$\begin{tikzcd} \mathbf{Sh}((1_\mathcal{D} \downarrow F), \tilde{K}) \arrow[rr, shift left, "C_{\pi_\mathcal{D}}"] \arrow[dr, "C_{\pi_\mathcal{C}} \cong \mathbf{Sh}(i_F)"'] & \sim & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left, "\mathbf{Sh}(\pi_\mathcal{D})"] \arrow[dl, "\mathbf{Sh}(F)"] \\ & \mathbf{Sh}(\mathcal{C}, J) & \end{tikzcd}$$

# The canonical stack of a geometric morphism

## Corollary (O.C.)

*Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then the canonical projection functor*

$$\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$$

*is a comorphism of sites $((1_{\mathcal{F}} \downarrow f^*), \widetilde{J_{\mathcal{F}}^{\text{can}}}) \to (\mathcal{E}, J_{\mathcal{E}}^{\text{can}})$ such that $f = C_{\pi_{\mathcal{E}}}$.*

The functor $\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$ is actually a stack on $\mathcal{E}$, which we call the canonical stack of $f$: from an indexed point of view, this stack sends any object $E$ of $\mathcal{E}$ to the topos $\mathcal{F}/f^*(E)$ and any arrow $u : E' \to E$ to the pullback functor $u^* : \mathcal{F}/f^*(E) \to \mathcal{F}/f^*(E')$.

By taking $f$ to be the identity, and choosing a site of definition $(\mathcal{C}, J)$ for $\mathcal{E}$, we obtain the canonical stack $\mathcal{S}_{(\mathcal{C},J)}$ on $(\mathcal{C}, J)$, which sends any object $c$ of $\mathcal{C}$ to the topos $\mathbf{Sh}(\mathcal{C}, J)/l(c)$. The above corollary thus specializes to an equivalence

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{S}_{(\mathcal{C},J)}, \widetilde{J^{\text{can}}_{\mathbf{Sh}(\mathcal{C},J)}}),$$

which represents a 'thickening' of the usual representation of a Grothendieck topos as the topos of sheaves over itself with respect to the canonical topology.

# Relative 'presheaf toposes'

Given a $\mathcal{C}$-indexed category $\mathbb{D}$, we denote by $\mathcal{G}(\mathbb{D})$ the fibration on $\mathcal{C}$ associated with it (through the Grothendieck construction) and by $p_{\mathbb{D}}$ the canonical projection functor $\mathcal{G}(\mathbb{D}) \to \mathcal{C}$.

## Proposition (O.C. and R.Z.)

*Let $(\mathcal{C}, J)$ be a small-generated site, $\mathbb{D}$ a $\mathcal{C}$-indexed category and $\mathbb{D}^V$ be the opposite indexed category of $\mathbb{D}$ (defined by setting, for each $c \in \mathcal{C}$, $\mathbb{D}^V(c) = \mathbb{D}(c)^{\mathrm{op}}$). Then we have a natural equivalence*

$$\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}}) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}^V, \mathcal{S}_{(\mathcal{C},J)}) .$$

This proposition shows that, if $\mathbb{D}$ is a stack, the classifying topos $\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}})$ of $\mathbb{D}$, can indeed be seen as the "topos of relative presheaves on $\mathbb{D}$".

We will see that, for any $\mathbb{D}$, the Giraud topos $C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}}) \to \mathbf{Sh}(\mathcal{C}, J)$ can be naturally seen as a weighted colimit of a diagram of étale toposes over $\mathbf{Sh}(\mathcal{C}, J)$.

# Weighted colimits

Consider two weak 2-categories $\mathcal{C}$ and $\mathcal{K}$, a pseudofunctor $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ and a pseudofunctor $R : \mathcal{C} \to \mathcal{K}$: the $\mathbb{D}$-weighted pseudocolimit of $R$ is an object $L$ of $\mathcal{K}$, usually denoted by $\mathrm{colim}^{\mathbb{D}}_{ps} R$, such that there a pseudonatural equivalence

$$\mathcal{K}(\mathrm{colim}^{\mathbb{D}}_{ps} R, K) \simeq [\mathcal{C}^{op}, \mathbf{CAT}]_{ps}(\mathbb{D}, \mathcal{K}(R(-), K)) :$$

A pseudococone $F$ on $R$ weighted by $\mathbb{D}$ can be visualized as follows: for any $y : Y \to X$ in $\mathcal{C}$ and $a : U \to V$ in $\mathbb{D}(X)$, we have

$$\begin{tikzcd}
R(X) \arrow[d, bend right=50, "F_X(V)"'] \arrow[d, bend left=50, "F_X(U)"] \arrow[d, phantom, "\overset{F_X(a)}{\Leftarrow}"] & & R(Y) \arrow[ll, "R(y)"'] \arrow[dll, bend left, "F_Y(\mathbb{D}(y)(U))"] \arrow[dll, phantom, "\overset{F_y(U)}{\Rightarrow}"] \\
K & &
\end{tikzcd}$$

where all the arrows satisfy natural conditions and the $F_y(U)$ are all isomorphisms.

# Giraud toposes as weighted colimits

## Theorem

*Given a small-generated site $(\mathcal{C}, J)$ and a cloven fibration $p : \mathcal{D} \to \mathcal{C}$ with corresponding $\mathcal{C}$-indexed category $\mathbb{D}$, the topos of sheaves $\mathrm{Gir}_J(p) := \mathbf{Sh}(\mathcal{D}, M_J^p)$ is the $\mathbb{D}$-weighted pseudocolimit of the diagram*

$$L : \mathcal{C} \xrightarrow{\mathcal{C}/-} \mathbf{cFib}_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}/(\mathcal{C}, J) \xrightarrow{C_{(-)}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) :$$

*That is, for any $\mathbf{Sh}(\mathcal{C}, J)$-topos $\mathscr{E}$, there is an equivalence between*

$$\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)\,(\mathrm{Gir}_J(p), \mathscr{E})$$

*and*

$$\mathbf{Ind}_{\mathcal{C}}\left(\mathbb{D}, \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)\left(\mathbf{Sh}(\mathcal{C}/(-), J_{(-)}), \mathscr{E}\right)\right),$$

*which moreover is pseudonatural in $\mathscr{E}$.*

# Giraud toposes as weighted colimits

In other words, the Giraud topos $\mathrm{Gir}_J(p) := \mathbf{Sh}(\mathcal{D}, M_J^p)$ is a universal $\mathbb{D}$-weighted pseudococone on the diagram $L$:

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}/X, J_X) \arrow[d, bend right=50, "\lambda_{(X,V)}"'] \arrow[d, bend left=50, "\lambda_{(X,U)}"] \arrow[d, phantom, "\overset{\lambda_{(X,a)}}{\Leftarrow}"] \arrow[dr, phantom, "\cong"] & & \mathbf{Sh}(\mathcal{C}/Y, J_X) \arrow[ll, "C_{\Sigma_y}"'] \arrow[dll, bend left, "\lambda_{(Y,(\mathbb{D}(y)(U)))}"] \\
\mathbf{Sh}(\mathcal{D}, M_J^p) & &
\end{tikzcd}$$

where $y : Y \to X$ and $a : U \to V$ are arrows respectively in $\mathcal{C}$ and in $\mathbb{D}(X)$, the legs $\lambda_{(X,U)} : \mathbf{Sh}(\mathcal{C}/X, J_X) \to \mathbf{Sh}(\mathcal{D}, M_J^p)$ of the cocone are the morphisms $C_{\xi_{(X,U)}}$ induced by the morphisms of fibrations $\xi_{(X,U)} : \mathcal{C}/X \to \mathcal{D}$ over $\mathcal{C}$ given by the fibered Yoneda lemma, and the functor $\Sigma_y : \mathcal{C}/Y \to \mathcal{C}/X$ are given by composition with $y$.

In fact, this weighted colimit already exists at the level of categories over $\mathcal{C}$, as well as at that of continuous comorphisms of sites over $\mathcal{C}$.

# The fundamental adjunction

The universal property of the above weighted colimit yields a fundamental 2-adjunction between cloven fibrations over $\mathcal{C}$ and toposes over $\mathbf{Sh}(\mathcal{C}, J)$:

## Theorem (O.C and R.Z.)

*For any small-generated site* $(\mathcal{C}, J)$*, the two pseudofunctors*

$$\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{cFib}_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}/(\mathcal{C}, J) \xrightarrow{C_{(-)}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J),$$

$$\left[[p : \mathcal{D} \to \mathcal{C}] \xrightarrow{(F,\phi)} [q : \mathcal{E} \to \mathcal{C}]\right] \mapsto \left[[\mathrm{Gir}_J(p)] \xrightarrow{(C_F, C_\phi)} [\mathrm{Gir}_J(q)]\right],$$

*and*

$$\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Ind}_{\mathcal{C}} \simeq \mathbf{cFib}_{\mathcal{C}},$$

*which acts by mapping a geometric morphism* $E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$ *to*

$$\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), [E]) : \mathcal{C}^{op} \to \mathbf{CAT},$$

*are the two components of a 2-adjunction*

$$\begin{tikzcd} \mathbf{cFib}_{\mathcal{C}} \arrow[r, bend left, "\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}"] \arrow[r, phantom, "\perp"] & \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \arrow[l, bend left, "\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}"] \end{tikzcd}$$

## Remark

*Since* $\mathrm{Gir}_J(p) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathcal{D}^V, \mathcal{S}_{(\mathcal{C},J)})$*, the canonical stack* $\mathcal{S}_{(\mathcal{C},J)}$ *has a similar behavior to that of a dualizing object for the adjunction* $\Lambda \dashv \Gamma$.

# The discrete setting

Let us now consider the restriction of our fundamental adjunction in the setting of presheaves (that is, discrete fibrations). This will yield a generalization to the context of arbitrary sites of the classical adjunction

$$\begin{tikzcd} \mathbf{Psh}(X) \arrow[r, bend left, "\Lambda"] \arrow[r, phantom, "\perp"] & \mathbf{Top}/X \arrow[l, bend left, "\Gamma"] \end{tikzcd} .$$

between presheaves on a topological space $X$ and bundles over it. [Recall that $\Lambda$ maps a presheaf $P$ to its bundle of germs $\pi_P : E_P = \coprod_{x \in X} P_x \to X$, while $\Gamma$ is the global sections functor.]

For this, we need the following

## Definition

We call a geometric morphism $F : \mathscr{F} \to \mathbf{Sh}(\mathcal{C}, J)$ small relative to $\mathbf{Sh}(\mathcal{C}, J)$ if for any $J$-sheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ the geometric morphisms $\mathbf{Sh}(\mathcal{C}, J)/P \to \mathscr{F}$ over $\mathbf{Sh}(\mathcal{C}, J)$ form a set (up to equivalence of geometric morphisms), that is, if the category

$$\mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}, J)/P, \mathscr{F})$$

is small.

We denote by $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ the full subcategory of the 1-category $\mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)$ whose objects are the small geometric morphisms relative to $\mathbf{Sh}(\mathcal{C}, J)$.

# The discrete setting

## Proposition (O.C. and R.Z.)

*Consider a small-generated site* $(\mathcal{C}, J)$*:*

- *There is an adjunction of 1-categories*

$$\begin{tikzcd} {[\mathcal{C}^{op}, \mathbf{Set}]} \arrow[r, bend left, "\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}"] \arrow[r, phantom, "\perp"] & \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J) \arrow[l, bend left, "\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}"] \end{tikzcd} .$$

  - *The functor* $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ *maps a presheaf* $P$ *to* $\prod_{a_J(P)} : \mathbf{Sh}(\mathcal{C}, J)/a_J(P) \to \mathbf{Sh}(\mathcal{C}, J)$ *or, in terms of comorphisms of sites, to* $\Lambda(P) := [C_{p_P} : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\mathcal{C}, J)]$ *and* $\Lambda(g) := C_{\int g} : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\int Q, J_Q)$.
  - *The functor* $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ *acts like a Hom-functor by mapping an object* $[F : \mathscr{F} \to \mathbf{Sh}(\mathcal{C}, J)]$ *of* $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ *to the presheaf*

$$\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}, J)/\ell_J(-), \mathscr{F}) : \mathcal{C}^{op} \to \mathbf{Set} .$$

# The general presheaf-étale adjunction

- The image of $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ factors through $\mathbf{Topos}^{\acute{e}tale}/\mathbf{Sh}(\mathcal{C},J)$, and the image of $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ factors through $\mathbf{Sh}(\mathcal{C},J)$;
- The fixed points of $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)$ are precisely the étale geometric morphisms, while those of $[\mathcal{C}^{op},\mathbf{Set}]$ are $J$-sheaves.
- The adjunction $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)} \dashv \Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ restricts to an equivalence

$$\mathbf{Sh}(\mathcal{C},J) \simeq \mathbf{Topos}^{\acute{e}tale}/_1\mathbf{Sh}(\mathcal{C},J)\ .$$

- The composite functor $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ is naturally isomorphic to the sheafification functor

$$i_J a_J : [\mathcal{C}^{op},\mathbf{Set}] \to \mathbf{Sh}(\mathcal{C},J) \to [\mathcal{C}^{op},\mathbf{Set}];$$

# Some applications

The presheaf-bundle adjunction for topological spaces is useful mostly becase it provides a geometric interpretation of several fundamental constructions on (pre)sheaves, such as direct and inverse images, as well as the sheafification process, in the language of fibrations.

Thanks to our site-theoretic generalization, we can extend these techniques to arbitrary presheaves. In particular, we obtain the following results:

- For any $c \in \mathcal{C}$, the elements $a_J(P)(c)$ of the $J$-sheafification of a given presheaf $P$ can be identified with the geometric morphisms over $\mathbf{Sh}(\mathcal{C}, J)$ from $\mathbf{Sh}(\mathcal{C}/c, J_c)$ to $\mathbf{Sh}(\int P, J_P)$, all of which can be locally represented as being induced by morphisms of fibrations.

  This is strictly related to the construction of $a_J(P)(c)$ in terms of locally matching families of elements of $P$.

# Direct and inverse images in terms of fibrations

- Given a functor $F : \mathcal{C} \to \mathcal{D}$ and two presheaves $P : \mathcal{C}^{op} \to \mathbf{Set}$ and $Q : \mathcal{D}^{op} \to \mathbf{Set}$ with associated fibrations $\pi_P : \int P \to \mathcal{C}$ and $\pi_Q : \int Q \to \mathcal{D}$,
  - the fibration corresponding to the direct image presheaf $Q \circ F^{\mathrm{op}}$ is computed as the strict pullback of $\pi_Q$ along $F$:

$$\begin{tikzcd} \int(F^*(Q)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int Q \arrow[d, "\pi_Q"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

  - If $F$ is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ then, for any $J$-sheaf $P$ on $\mathcal{C}$, the inverse image $\mathbf{Sh}(F)^*(P)$ coincides with the discrete part of the $K$-comprehensive factorization (in the sense of O.C.) of the composite functor $F \circ \pi_P$.

We have also established natural analogues of these results in the context of stacks.

# Relative sheaf toposes

As any Grothendieck topos is a subtopos of a presheaf topos, so any relative topos should be a subtopos of a relative presheaf topos. This motivates the following

## Definition

Let $(\mathcal{C}, J)$ be a small-generated site. A site relative to $(\mathcal{C}, J)$ is a pair consisting of a $\mathcal{C}$-indexed category $\mathbb{D}$ and a Grothendieck topology $K$ on $\mathcal{G}(\mathbb{D})$ which contains the Giraud topology $M_J^{p_\mathbb{D}}$.

The topos of sheaves on such a relative site $(\mathbb{D}, K)$ is defined to be the geometric morphism

$$C_{p_\mathbb{D}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), K) \to \mathbf{Sh}(\mathcal{C}, J)$$

induced by the comorphism of sites $p_\mathbb{D} : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$.

## Remark

*Not every Grothendieck topology on $K$ can be generated starting by horizontal or vertical data (that is, by sieves generated by cartesian arrows or entirely lying in some fiber), but many important relative topologies naturally arising in practice are of this form.*

# Examples of relative topologies

- The Giraud topology is an example of a relative topology generated by horizontal data.
- The total topology of a fibered site, in the sense of Grothendieck, is generated by vertical data.
- The topology presenting the over-topos at a model (introduced in a joint work with Axel Osmond), defined on the stack of its generalized elements, is an example of a 'mixed' relative topology.

We have shown that, for a wide class of relative topologies generated by horizontal and vertical data, one can describe bases for them consisting of multicompositions of horizontal families with vertical families, thus generalizing the description of bases provided in the context of the over-topos construction.

# Local morphisms

Recall that a (weak) geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be local if $f_*$ has a fully faithful right adjoint.

## Theorem (O.C.)

*Let $F : \mathcal{D} \to \mathcal{C}$ be a continuous comorphism of sites (also regarded as a weak morphism of sites) $(\mathcal{D}, K) \to (\mathcal{C}, J)$. Then:*

- (i) *The geometric morphism $C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ is essential, and*

$$(C_F)_! \cong \mathbf{Sh}(F)^* \dashv \mathbf{Sh}(F)_* \cong (C_F)^* = D_F := (- \circ F^{\mathrm{op}}) \dashv (C_F)_*$$

- (ii) *The weak morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ is local if and only if $C_F$ is an inclusion, that is, if and only if $F$ is $K$-faithful and $K$-full.*
- (iii) *The canonical geometric transformation*

$$1_{\mathbf{Sh}(\mathcal{D},K)} \to \mathbf{Sh}(F) \circ C_F$$

*(given by the unit of the adjunction between $\mathbf{Sh}(F)$ and $C_F$) is an isomorphism if (and only if) $F$ is $K$-faithful and $K$-full. In this case, if $F$ is moreover a morphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$, the morphisms $C_F$ and $\mathbf{Sh}(F)$ realize the topos $\mathbf{Sh}(\mathcal{D}, K)$ as a (coadjoint) retract of $\mathbf{Sh}(\mathcal{C}, J)$ in $\mathbf{Topos}$.*

# *Gros* and *petit* toposes

The above result can be notably applied to construct pairs of *gros* and *petit* toposes starting from a ($K$-)full and ($K$-)faithful morphism and comorphism of sites

$$(\mathcal{D}, K) \to (\mathcal{T}/T_{\mathcal{D}}, E_{T_{\mathcal{D}}}),$$

where $\mathcal{T}$ is a category endowed with a Grothendieck topology $E$, $T_{\mathcal{D}}$ is an object of $\mathcal{T}$ and $E_{T_{\mathcal{D}}}$ is the Grothendieck topology induced on $(\mathcal{T}/T_{\mathcal{D}})$ by $E$.

Pairs of gros and petit toposes are important for several reasons. Morally, a *petit* topos is thought of as a generalized space, while a *gros* topos is conceived as a category of spaces.

In fact, one advantage of *gros* toposes is that they are associated with sites which tend to have better categorical properties than those of the site presenting the *petit* topos.

Still, *gros* and *petit* toposes in a given pair are homotopically equivalent (as they are related by a local retraction), whence they share the same cohomological invariants.

# A question of Grothendieck

As recently brought to the public attention by Colin McLarty, Grothendieck expressed, in his 1973 Buffalo lectures, the aspiration of viewing any object of a topos geometrically as an étale space over the terminal object:

> *The intuition is the following: viewing objects of a topos as being something like étalé spaces over the final object of the topos, and the induced topos over an object as just the object itself. That is I think the way one should handle the situation.*
> *It's a funny situation because in strict terms, you see, the language which I want to push through doesn't make sense. But of course there are a number of mathematical statements which substantiate it.*

Given his conception of *gros* and *petit* toposes, we can more broadly interpret his wish as that for a framework allowing one to think geometrically about any topos, that is, as it were a 'petit' topos related to a 'gros' topos by a local retraction.

# Every Grothendieck topos is a 'small topos'

We define a Grothendieck topology $J^{\text{ét}}$ on **Topos**, which we call the étale cover topology, by postulating that a sieve on a topos $\mathcal{E}$ is $J^{\text{ét}}$-covering if and only if it contains a family $\{\mathcal{E}/A_i \to \mathcal{E} \mid i \in I\}$ of canonical local homeomorphisms such that the family of arrows $\{!_{A_i} : A_i \to 1_{\mathcal{E}} \mid i \in I\}$ is epimorphic in $\mathcal{E}$.

We thus have a 'big' topos $\mathbf{Sh}(\mathbf{Topos}, J^{\text{ét}})$[1] with a canonical functor $l : \mathbf{Topos} \to \mathbf{Sh}(\mathbf{Topos}, J^{\text{ét}})$, and for any Grothendieck topos $\mathcal{E}$ we can consider the slice topos

$$\mathbf{Sh}(\mathbf{Topos}, J^{\text{ét}})/l(\mathcal{E}) \simeq \mathbf{Sh}(\mathbf{Topos}/\mathcal{E}, J^{\text{ét}}_{\mathcal{E}}),$$

where $J^{\text{ét}}_{\mathcal{E}}$ is the Grothendieck topology whose covering sieves are those which are sent by the forgetful functor $\mathbf{Topos}/\mathcal{E} \to \mathbf{Topos}$ to $J^{\text{ét}}$-covering families. We call this topos the *big topos* associated with $\mathcal{E}$.

[1] in a suitable Grothendieck universe

# Every Grothendieck topos is a 'small topos'

The functor $L$ is a $J$-full and $J$-faithful morphism as well as comorphism of sites from $(\mathcal{C}, J)$ to $(\mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J), J^{\text{ét}}_{\mathbf{Sh}(\mathcal{C},J)})$.

So, by the above result, the 'petit' topos $\mathbf{Sh}(\mathcal{C}, J)$ identifies with a coadjoint retract of the 'big' topos $\mathbf{Sh}(\mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J), J^{\text{ét}}_{\mathbf{Sh}(\mathcal{C},J)}) \simeq \mathbf{Sh}(\mathbf{Topos}, J^{\text{ét}})/l(\mathbf{Sh}(\mathcal{C}, J))$ via the geometric morphisms

$$C_L : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J), J^{\text{ét}}_{\mathbf{Sh}(\mathcal{C},J)})$$

induced by $L$ as a comorphism of sites and

$$\mathbf{Sh}(L) : \mathbf{Sh}(\mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J), J^{\text{ét}}_{\mathbf{Sh}(\mathcal{C},J)}) \to \mathbf{Sh}(\mathcal{C}, J)$$

induced by $L$ as a morphism of sites; moreover, $\mathbf{Sh}(L)$ is local and $C_L$ is an essential inclusion.

This shows that *every* Grothendieck topos can be naturally regarded as a 'petit' topos embedded in an associated 'gros' topos, and that this embedding allows one to view any object of the original topos as an étale morphism to the terminal object in the associated 'gros' topos, providing a solution to Grothendieck's problem.

# For further reading


- O. Caramello,
  *Denseness conditions, morphisms and equivalences of toposes*,
  monograph draft available as arxiv:math.CT/1906.08737v3, (2020).

- O. Caramello and R. Zanfa,
  *Relative topos theory via stacks*,
  to be shortly available on the Mathematics Arxiv (2021).

# International school and conference on topos theory


Everyone is welcome!