Toposes for the working mathematician

Lecture II: Sheaves on a site
Lecture III: Classifying toposes

Olivia Caramello

University of Insubria (Como), Grothendieck Institute and MICS - CentraleSupélec

Laboratoire MICS, CentraleSupélec
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Presheaves on a topological space

Definition

Let X be a topological space. A presheaf $\mathcal{F}$ on X consists of the data:

(i) for every open subset U of X, a set $\mathcal{F}(U)$ and
(ii) for every inclusion $V \subseteq U$ of open subsets of X, a function $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ subject to the conditions
  - $\rho_{U,U}$ is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$ and
  - if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

The maps $\rho_{U,V}$ are called restriction maps, and we sometimes write $s|_V$ instead of $\rho_{U,V}(s)$, if $s \in \mathcal{F}(U)$.
A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ on a topological space X is a collection of maps $\mathcal{F}(U) \to \mathcal{G}(U)$ which is compatible with respect to restriction maps.

Remark

*Categorically, a presheaf $\mathcal{F}$ on X is a functor $\mathcal{F} : \mathcal{O}(X)^{op} \to$ **Set**, where $\mathcal{O}(X)$ is the poset category corresponding to the lattice of open sets of the topological space X (with respect to the inclusion relation).*
A morphism of presheaves is then just a natural transformation between the corresponding functors.
So we have a category $[\mathcal{O}(X)^{op}, \textbf{Set}]$ of presheaves on X.

Sheaves on a topological space

Definition

A sheaf $\mathcal{F}$ on a topological space X is a presheaf on X satisfying the additional conditions

(i) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if $s, t \in \mathcal{F}(U)$ are elements such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$;

(ii) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i, with the property that for each $i, j \in I$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ (necessarily unique by (i)) such that $s|_{V_i} = s_i$ for each i.

A morphism of sheaves is defined as a morphism of the underlying presheaves.

Examples

- the sheaf of continuous real-valued functions on any topological space
- the sheaf of regular functions on a variety
- the sheaf of differentiable functions on a differentiable manifold
- the sheaf of holomorphic functions on a complex manifold

In each of the above examples, the restriction maps of the sheaf are the usual set-theoretic restrictions of functions to a subset.

Sheaves from a categorical point of view

Sheaves arising in Mathematics are often equipped with more structure than the mere set-theoretic one; for example, one may wish to consider sheaves of modules (resp. rings, abelian groups, ...) on a topological space X.

The natural categorical way of looking at these notions is to consider them as models of certain (geometric) theories in a category $\mathbf{Sh}(X)$ of sheaves of sets.

Remarks

- *Categorically, a sheaf is a functor $\mathcal{O}(X)^{op} \to \mathbf{Set}$ which satisfies certain conditions expressible in categorical language entirely in terms of the poset category $\mathcal{O}(X)$ and of the usual notion of covering on it. The category $\mathbf{Sh}(X)$ of sheaves on a topological space X is thus a full subcategory of the category $[\mathcal{O}(X)^{op}, \mathbf{Set}]$ of presheaves on X.*
- *Many important properties of topological spaces X can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.*

These remarks led Grothendieck to introduce a significant categorical generalization of the notion of sheaf, and hence the notion of Grothendieck topos.

Adjunctions induced by points

Let x be a point of a topological space X.

Definition

Let A be a set. Then the skyscraper sheaf $\text{Sky}_x(A)$ of A at x is the sheaf on X defined as

- $\text{Sky}_x(A)(U) = A$ if $x \in U$
- $\text{Sky}_x(A)(U) = 1 = \{*\}$ if $x \notin U$

and in the obvious way on arrows.

The assignment $A \to \text{Sky}_x(A)$ is clearly functorial.

Theorem

The stalk functor $\text{Stalk}_x : \mathbf{Sh}(X) \to \mathbf{Set}$ *at* x *is left adjoint to the skyscraper functor* $\text{Sky}_x : \mathbf{Set} \to \mathbf{Sh}(X)$.

In fact, as we shall see later in the course, points in topos theory are defined as suitable kinds of functors (more precisely, colimit and finite-limit preserving ones).

Open sets as subterminal objects

Since limits in a category $\mathbf{Sh}(X)$ are computed as in the category of presheaves $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$, a subobject of a sheaf F in $\mathbf{Sh}(X)$ is just a subsheaf, that is a subfunctor which is a sheaf.
Notice that a subfunctor $S \subseteq F$ is a sheaf if and only if for every open covering $\{U_i \subseteq U \mid i \in I\}$ and every element $x \in F(U)$, $x \in S(U)$ if and only if $x|_{U_i} \in S(U_i)$.

Definition

In a category with a terminal object, a subterminal object is an object whose unique arrow to the terminal object is a monomorphism.

Theorem

Let X be a topological space. Then we have a frame isomorphism

$$\mathrm{Sub}_{\mathbf{Sh}(X)}(1) \cong \mathcal{O}(X)\ .$$

between the subterminal objects of $\mathbf{Sh}(X)$ and the open sets of X.

Sieves

In order to 'categorify' the notion of sheaf of a topological space, the first step is to introduce an abstract notion of covering (of an object by a family of arrows to it) in a category.

Definition

- Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a presieve P in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c.
- Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a sieve S in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c such that

$$f \in S \Rightarrow f \circ g \in S$$

whenever this composition makes sense.
- We say that a sieve S is generated by a presieve P on an object c if it is the smallest sieve containing it, that is if it is the collection of arrows to c which factor through an arrow in P.

If S is a sieve on c and $h : d \to c$ is any arrow to c, then

$$h^*(S) := \{g \mid \mathrm{cod}(g) = d,\ h \circ g \in S\}$$

is a sieve on d.

Grothendieck topologies

Definition

- A Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each object c of $\mathcal{C}$ a collection $J(c)$ of sieves on c in such a way that
  1. (maximality axiom) the maximal sieve $M_c = \{f \mid \mathrm{cod}(f) = c\}$ is in $J(c)$;
  2. (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;
  3. (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

  The sieves S which belong to $J(c)$ for some object c of $\mathcal{C}$ are said to be J-covering.

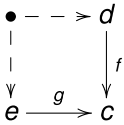
- A site is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a small category and J is a Grothendieck topology on $\mathcal{C}$.

Examples of Grothendieck topologies

- For any (small) category $\mathcal{C}$, the trivial topology on $\mathcal{C}$ is the Grothendieck topology in which the only sieve covering an object c is the maximal sieve M_c.
- The dense topology D on a category $\mathcal{C}$ is defined by: for a sieve S,

$$S \in D(c) \quad \text{if and only if} \quad \text{for any } f : d \to c \text{ there exists } g : e \to d \text{ such that } f \circ g \in S\,.$$

If $\mathcal{C}$ satisfies the right Ore condition i.e. the property that any two arrows $f : d \to c$ and $g : e \to c$ with a common codomain c can be completed to a commutative square



then the dense topology on $\mathcal{C}$ specializes to the atomic topology on $\mathcal{C}$ i.e. the topology J_{at} defined by: for a sieve S,

$$S \in J_{at}(c) \text{ if and only if } S \neq \emptyset\,.$$

Examples of Grothendieck topologies

- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathcal{O}(X)}$ on the poset category $\mathcal{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathcal{O}(X))$,

$$S \in J_{\mathcal{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U \, .$$

- More generally, given a frame (or complete Heyting algebra) H, we can define a Grothendieck topology J_H, called the *canonical topology on H*, by:

$$\{a_i \mid i \in I\} \in J_H(a) \text{ if and only if } \bigvee_{i \in I} a_i = a \, .$$

- Given a small category of topological spaces which is closed under finite limits and under taking open subspaces, one may define the open cover topology on it by specifying as basis the collection of open embeddings $\{Y_i \hookrightarrow X \mid i \in I\}$ such that $\bigcup_{i \in I} Y_i = X$.
- The Zariski topology on the opposite of the category $\mathbf{Rng}_{\text{f.g.}}$ of finitely generated commutative rings with unit is defined by: for any cosieve S in $\mathbf{Rng}_{\text{f.g.}}$ on an object A, $S \in Z(A)$ if and only if S contains a finite family $\{\xi_i : A \to A_{f_i} \mid 1 \leq i \leq n\}$ of canonical maps $\xi_i : A \to A_{f_i}$ in $\mathbf{Rng}_{\text{f.g.}}$ where $\{f_1, \ldots, f_n\}$ is a set of elements of A which is not contained in any proper ideal of A.

Sheaves on a site

Definition

- A presheaf on a (small) category $\mathcal{C}$ is a functor $P : \mathcal{C}^{op} \to \mathbf{Set}$.
- Let $P : \mathcal{C}^{op} \to \mathbf{Set}$ be a presheaf on $\mathcal{C}$ and S be a sieve on an object c of $\mathcal{C}$.

  A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

  $$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d\,.$$

  An amalgamation for such a family is a single element $x \in P(c)$ such that

  $$P(f)(x) = x_f \quad \text{for all } f \text{ in } S\,.$$

- Given a site $(\mathcal{C}, J)$, a presheaf on $\mathcal{C}$ is a J-sheaf if every matching family for any J-covering sieve on any object of $\mathcal{C}$ has a unique amalgamation.
- The category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on the site $(\mathcal{C}, J)$ is the full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$ on the presheaves which are J-sheaves.
- A Grothendieck topos is any category equivalent to the category of sheaves on a site.

Examples of toposes

The following examples show that toposes can be naturally attached to mathematical notions as different as (small) categories, topological spaces, or groups.

Examples

- For any (small) category $\mathcal{C}$, $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is the category of sheaves $\mathbf{Sh}(\mathcal{C}, T)$ where T is the trivial topology on $\mathcal{C}$.
- For any topological space X, $\mathbf{Sh}(\mathcal{O}(X), J_{\mathcal{O}(X)})$ is equivalent to the usual category $\mathbf{Sh}(X)$ of sheaves on X.
- For any (topological) group G, the category $BG = \mathbf{Cont}(G)$ of continuous actions of G on discrete sets is a Grothendieck topos (equivalent, as we shall see, to the category $\mathbf{Sh}(\mathbf{Cont}_t(G), J_{at})$ of sheaves on the full subcategory $\mathbf{Cont}_t(G)$ on the non-empty transitive actions with respect to the atomic topology).

Basic properties of Grothendieck toposes

Grothendieck toposes satisfy all the categorical properties that one might hope for:

Theorem

Let $(\mathcal{C}, J)$ be a site. Then

- *the inclusion* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ *has a left adjoint* $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ *(called the associated sheaf functor), which preserves finite limits.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has all (small) limits, which are preserved by the inclusion functor* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$*; in particular, limits are computed pointwise and the terminal object* $1_{\mathbf{Sh}(\mathcal{C},J)}$ *of* $\mathbf{Sh}(\mathcal{C}, J)$ *is the functor* $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ *sending each object* $c \in Ob(\mathcal{C})$ *to the singleton* $\{*\}$.
- *The associated sheaf functor* $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ *preserves colimits; in particular,* $\mathbf{Sh}(\mathcal{C}, J)$ *has all (small) colimits.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has exponentials, which are constructed as in the topos* $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has a subobject classifier.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has a separating set of objects.*

Subobjects in a Grothendieck topos

Since limits in a topos $\mathbf{Sh}(\mathcal{C}, J)$ are computed as in the presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$, a subobject of a sheaf F in $\mathbf{Sh}(\mathcal{C}, J)$ is just a subsheaf, that is a subfunctor which is a sheaf.

Notice that a subfunctor $F' \subseteq F$ is a sheaf if and only if for every J-covering sieve S and every element $x \in F(c)$, $x \in F'(c)$ if and only if $F(f)(x) \in F'(\text{dom}(f))$ for every $f \in S$.

Theorem

- *For any Grothendieck topos $\mathcal{E}$ and any object a of $\mathcal{E}$, the poset $\text{Sub}_{\mathcal{E}}(a)$ of all subobjects of a in $\mathcal{E}$ is a complete Heyting algebra.*
- *For any arrow $f : a \to b$ in a Grothendieck topos $\mathcal{E}$, the pullback functor $f^* : \text{Sub}_{\mathcal{E}}(b) \to \text{Sub}_{\mathcal{E}}(a)$ has both a left adjoint $\exists_f : \text{Sub}_{\mathcal{E}}(a) \to \text{Sub}_{\mathcal{E}}(b)$ and a right adjoint $\forall_f : \text{Sub}_{\mathcal{E}}(a) \to \text{Sub}_{\mathcal{E}}(b)$.*

The Heyting operations on subobjects

Proposition

The collection $\mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(E)$ *of subobjects of an object* E *in* $\mathbf{Sh}(\mathcal{C},J)$ *has the structure of a complete* Heyting algebra *with respect to the natural ordering* $A \leq B$ *if and only if for every* $c \in \mathcal{C}$, $A(c) \subseteq B(c)$. *We have that*

- $(A \wedge B)(c) = A(c) \cap B(c)$ *for any* $c \in \mathcal{C}$;
- $(A \vee B)(c) = \{x \in E(c) \mid \{f : d \to c \mid E(f)(x) \in A(d) \cup B(d)\} \in J(c)\}$ *for any* $c \in \mathcal{C}$;

  (the infinitary analogue of this holds)
- $(A \Rightarrow B)(c) = \{x \in E(c) \mid \text{ for every } f : d \to c, E(f)(x) \in A(d) \text{ implies } E(f)(x) \in B(d)\}$ *for any* $c \in \mathcal{C}$.
- *the bottom subobject* $0 \rightarrowtail E$ *is given by the embedding into* E *of the initial object* 0 *of* $\mathbf{Sh}(\mathcal{C},J)$ *(given by:* $0(c) = \emptyset$ *if* $\emptyset \notin J(c)$ *and* $0(c) = \{*\}$ *if* $\emptyset \in J(c)$*);*
- *the top subobject is the identity one.*

The interpretation of quantifiers

Let $\phi : E \to F$ be a morphism in $\mathbf{Sh}(\mathcal{C}, J)$.

- The pullback functor

$$\phi^* : \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(F) \to \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(E)$$

  is given by: $\phi^*(B)(c) = \phi(c)^{-1}(B(c))$ for any subobject $B \rightarrowtail F$ and any $c \in \mathcal{C}$.

- The left adjoint

$$\exists_\phi : \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(E) \to \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(F)$$

  is given by: $\exists_\phi(A)(c) = \{y \in E(c) \mid \{f : d \to c \mid (\exists a \in A(d))(\phi(d)(a) = E(f)(y))\} \in J(c)\}$
  for any subobject $A \rightarrowtail E$ and any $c \in \mathcal{C}$.

- The right adjoint

$$\forall_\phi : \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(E) \to \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(F)$$

  is given by $\forall_\phi(A)(c) = \{y \in E(c) \mid \text{ for all } f : d \to c, \phi(d)^{-1}(E(f)(y)) \subseteq A(d)\}$
  for any subobject $A \rightarrowtail E$ and any $c \in \mathcal{C}$.

Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism. The natural notion of morphism of geometric morphisms if that of geometric transformation.

Definition

(i) Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of a pair of functors $f_* : \mathcal{E} \to \mathcal{F}$ (the direct image of f) and $f^* : \mathcal{F} \to \mathcal{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathcal{E} \to \mathcal{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

(iii) A point of a topos $\mathcal{E}$ is a geometric morphism $\mathbf{Set} \to \mathcal{E}$.

- Grothendieck toposes and geometric morphisms between them form a 2-category.
- Given two toposes $\mathcal{E}$ and $\mathcal{F}$, geometric morphisms from $\mathcal{E}$ to $\mathcal{F}$ and geometric transformations between them form a category, denoted by $\mathbf{Geom}(\mathcal{E}, \mathcal{F})$.

Examples of geometric morphisms

- A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The direct image $\mathbf{Sh}(f)_*$ sends a sheaf $F \in Ob(\mathbf{Sh}(X))$ to the sheaf $\mathbf{Sh}(f)_*(F)$ defined by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for any open subset V of Y. The inverse image $\mathbf{Sh}(f)^*$ acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.
- Every Grothendieck topos $\mathcal{E}$ has a unique geometric morphism $\mathcal{E} \to \mathbf{Set}$. The direct image is the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$, sending an object $e \in \mathcal{E}$ to the set $\mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, e)$, while the inverse image functor $\Delta : \mathbf{Set} \to \mathcal{E}$ sends a set S to the coproduct $\coprod_{s \in S} 1_{\mathcal{E}}$.
- For any site $(\mathcal{C}, J)$, the pair of functors formed by the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and the associated sheaf functor $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ yields a geometric morphism $i : \mathbf{Sh}(\mathcal{C}, J) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

Geometric morphisms to $\mathbf{Sh}(\mathcal{C}, J)$ I

Given a cartesian category (i.e. a category with all finite limits) $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{E}$ is said to be cartesian if it preserves finite limits. We shall denote by $\mathbf{Cart}(\mathcal{C}, \mathcal{E})$ the category of cartesian functors $\mathcal{C} \to \mathcal{E}$ and natural transformations between them.

Definition

Let $\mathcal{E}$ be a Grothendieck topos.

- A family $\{f_i : a_i \to a \mid i \in I\}$ of arrows in $\mathcal{E}$ with common codomain is said to be epimorphic if for any pair of arrows $g, h : a \to b$ with domain a, $g = h$ if and only if $g \circ f_i = h \circ f_i$ for all $i \in I$.
- If $(\mathcal{C}, J)$ is a site, a functor $F : \mathcal{C} \to \mathcal{E}$ is said to be J-continuous if it sends J-covering sieves to epimorphic families.

The full subcategory of $\mathbf{Cart}(\mathcal{C}, \mathcal{E})$ on the J-continuous flat functors will be denoted by $\mathbf{Cart}_J(\mathcal{C}, \mathcal{E})$.

Geometric morphisms to $\mathbf{Sh}(\mathcal{C}, J)$ II

Theorem

For any cartesian site $(\mathcal{C}, J)$ and Grothendieck topos $\mathcal{E}$, we have an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Cart}_J(\mathcal{C}, \mathcal{E})$$

natural in $\mathcal{E}$.

This equivalence sends a geometric morphism $f : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ to the functor given by the composite $f^* \circ l$ of $f^* : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}$ with the canonical functor $l : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$.

Fact

This theorem generalizes to the case of an arbitrary site $(\mathcal{C}, J)$, replacing the notion of cartesian functor by that of flat functor.

Model theory in toposes

We can consider models of arbitrary first-order theories in any Grothendieck topos, thanks to the rich categorical structure present on it.

The notion of model of a first-order theory in a topos is a natural generalization of the usual Taskian definition of a (set-based) model of the theory.

Let Σ be a (possibly multi-sorted) first-order signature. A *structure* M over Σ in a category $\mathscr{E}$ with finite products is specified by the following data:

- any sort A of Σ is interpreted by an *object* MA of $\mathscr{E}$
- any function symbol $f : A_1, \ldots, A_n \to B$ of Σ is interpreted as an *arrow* $Mf : MA_1 \times \cdots \times MA_n \to MB$ in $\mathscr{E}$
- any relation symbol $R \rightarrowtail A_1, \ldots, A_n$ of Σ is interpreted as a *subobject* $MR \rightarrowtail MA_1 \times \cdots \times MA_n$ in $\mathscr{E}$

Any formula $\{\vec{x} \,.\, \phi\}$ in a given context $\vec{x}$ over Σ is interpreted as a subobject $[[\vec{x} \,.\, \phi]]_M \rightarrowtail MA_1 \times \cdots \times MA_n$ defined recursively on the structure of the formula.

A model of a theory $\mathbb{T}$ over a first-order signature Σ is a structure over Σ in which all the axioms of $\mathbb{T}$ are satisfied.

Geometric theories

Definition

A geometric theory $\mathbb{T}$ is a theory over a first-order signature Σ whose axioms can be presented in the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are *geometric formulae*, that is formulae in the context $\vec{x}$ built up from atomic formulae over Σ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.

Remark

Inverse image functors of geometric morphisms of toposes always preserve models of a geometric theory (but in general not those of an arbitrary first-order theory).

Most of the first-order theories naturally arising in Mathematics are geometric; anyway, if a finitary first-order theory is not geometric, one can always canonically associate with it a geometric theory, called its *Morleyization*, having the same set-based models.

Classifying toposes

It was realized in the seventies (thanks to the work of several people, notably including W. Lawvere, A. Joyal, G. Reyes and M. Makkai) that:

- Every geometric theory $\mathbb{T}$ has a classifying topos $\mathscr{E}_{\mathbb{T}}$ which is characterized by the following representability property: for any Grothendieck topos $\mathcal{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathcal{E})$$

  natural in $\mathcal{E}$, where
  - **Geom**$(\mathcal{E}, \mathcal{E}_{\mathbb{T}})$ is the category of geometric morphisms $\mathcal{E} \to \mathcal{E}_{\mathbb{T}}$ and
  - $\mathbb{T}$-mod$(\mathcal{E})$ is the category of $\mathbb{T}$-models in $\mathcal{E}$.

- The classifying topos of a geometric theory $\mathbb{T}$ can be canonically built as the category **Sh**$(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of sheaves on the syntactic site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$.

The syntactic category of a geometric theory

Definition (Makkai and Reyes 1977)

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. The syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the 'renaming'-equivalence classes of geometric formulae-in-context $\{\vec{x} \,.\, \phi\}$ over Σ and as arrows $\{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, \psi\}$ (where the contexts $\vec{x}$ and $\vec{y}$ are supposed to be disjoint without loss of generality) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\vec{x}, \vec{y})$ which are $\mathbb{T}$-provably functional i.e. such that the sequents

$$(\phi \vdash_{\vec{x}} (\exists y)\theta),$$
$$(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi), \text{ and}$$
$$((\theta \wedge \theta[\vec{z}/\vec{y}]) \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{y} = \vec{z}))$$

  are provable in $\mathbb{T}$.
- The composite of two arrows

$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\theta]} \{\vec{y} \,.\, \psi\} \xrightarrow{[\gamma]} \{\vec{z} \,.\, \chi\}$$

  is defined as the $\mathbb{T}$-provable-equivalence class of the formula $(\exists \vec{y})\theta \wedge \gamma$.
- The identity arrow on an object $\{\vec{x} \,.\, \phi\}$ is the arrow

$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\phi \wedge \vec{x'} = \vec{x}]} \{\vec{x'} \,.\, \phi[\vec{x'}/\vec{x}]\}$$

The syntactic site

On the syntactic category of a geometric theory it is natural to put the Grothendieck topology defined as follows:

Definition

The syntactic topology $J_{\mathbb{T}}$ on the syntactic category $\mathcal{C}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ is given by:

a small family $\{[\theta_i] : \{\vec{x_i} \,.\, \phi_i\} \to \{\vec{y} \,.\, \psi\}\}$ in $\mathcal{C}_{\mathbb{T}}$ is $J_{\mathbb{T}}$-covering

if and only if

the sequent $(\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x_i})\theta_i)$ is provable in $\mathbb{T}$.

This notion is instrumental for identifying the models of the theory $\mathbb{T}$ in any geometric category $\mathcal{C}$ (and in particular in any Grothendieck topos) as suitable functors defined on the syntactic category $\mathcal{C}_{\mathbb{T}}$ with values in $\mathcal{C}$; indeed, these are precisely the $J_{\mathbb{T}}$-continuous cartesian functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$. So if $\mathcal{C}$ is a Grothendieck topos they correspond precisely to the geometric morphisms from $\mathcal{C}$ to $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$. This topos therefore classifies $\mathbb{T}$.

Morita equivalence

- Two mathematical theories are said to be Morita-equivalent if have the same classifying topos (up to equivalence): this means that they have equivalent categories of models in every Grothendieck topos $\mathcal{E}$, naturally in $\mathcal{E}$.
- Every Grothendieck topos is the classifying topos of *some* geometric theory (and in fact, of infinitely many theories).
- So a Grothendieck topos can be seen as a canonical representative of equivalence classes of theories modulo Morita-equivalence.

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Toposes for the working mathematician

Lecture IV: Toposes as 'bridges'

Olivia Caramello

University of Insubria (Como), Grothendieck Institute and MICS - CentraleSupélec

Laboratoire MICS, CentraleSupélec
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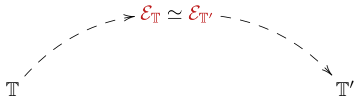
Toposes as *bridges*

- The notion of Morita-equivalence is ubiquitous in Mathematics; indeed, it formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods'.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted in terms of Morita-equivalences.
- On the other hand, Topos Theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.
- Any two theories which are bi-interpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.
- We can expect most of the categorical equivalences between categories of set-based models of geometric theories to lift to Morita equivalences.

Toposes as *bridges*

- In the topos-theoretic study of theories, the latter are represented by sites (of definition of their classifying topos or of some other topos naturally attached to them).
- The existence of theories which are Morita-equivalent to each other translates into the existence of different sites of definition (or, more generally, presentations) for the same Grothendieck topos.
- Grothendieck toposes can be effectively used as 'bridges' for transferring notions, properties and results across different Morita-equivalent theories:



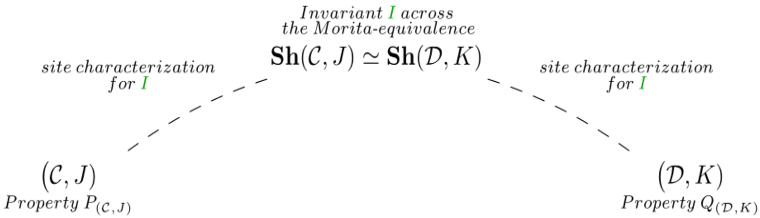
- The transfer of information takes place by expressing topos-theoretic invariants in terms of the different sites of definition (or, more generally, presentations) for the given topos.
- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

Toposes as *bridges*

- This methodology is technically effective because the relationship between a topos and its representations is often very natural, enabling us to easily transfer invariants across different representations (and hence, between different theories).
- On the other hand, the 'bridge' technique is highly non-trivial, in the sense that it often yields deep and surprising results. This is due to the fact that a given invariant can manifest itself in significanly different ways in the context of different presentations.
- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories and constructions.

The 'bridge-building' technique

- Decks of 'bridges': Morita-equivalences (or more generally morphisms or other kinds of relations between toposes)
- Arches of 'bridges': Site characterizations for topos-theoretic invariants (or more generally 'unravelings' of topos-theoretic invariants in terms of concrete representations of the relevant topos)



The 'bridge' yields a logical equivalence (or an implication) between the 'concrete' properties $P_{(\mathcal{C},J)}$ and $Q_{(\mathcal{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

A few selected applications

Since the theory of topos-theoretic 'bridges' was introduced in 2010, several applications of it have been obtained in different fields of Mathematics, such as:

- Model theory (topos-theoretic Fraïssé theorem)
- Proof theory (various results for first-order theories)
- Algebra (topos-theoretic generalization of topological Galois theory)
- Topology (topos-theoretic interpretation/generation of Stone-type and Priestley-type dualities)
- Functional analysis (various results on Gelfand spectra and Wallman compactifications)
- Many-valued logics and lattice-ordered groups (two joint papers with A. C. Russo)
- Cyclic homology, as reinterpreted by A. Connes (work on *"cyclic theories"*, jointly with N. Wentzlaff)
- Algebraic geometry (logical analysis of (co)homological motives, cf. the paper *"Syntactic categories for Nori motives"* joint with L. Barbieri-Viale and L. Lafforgue)

Some examples of 'bridges'

We shall now discuss a few 'bridges' established in the context of the above-mentioned applications:

- Topological Galois theory
- Topos-theoretic Fraïssé theorem
- Stone-type dualities

The results are completely *different*... but the methodology is always the *same*!

Topological Galois theory

Recall that classical topological Galois theory provides, given a Galois extension $F \subseteq L$, a bijective correspondence between the intermediate field extensions (resp. finite field extensions) $F \subseteq K \subseteq L$ and the closed (resp. open) subgroups of the Galois group $\mathrm{Aut}_F(L)$.

This admits the following categorical reformulation: the functor $K \to \mathrm{Hom}(K, L)$ defines an equivalence of categories

$$(\mathcal{L}_F^L)^{\mathrm{op}} \simeq \mathbf{Cont}_t(\mathrm{Aut}_F(L)),$$

where $\mathcal{L}_F^L$ is the category of finite intermediate field extensions and $\mathbf{Cont}_t(\mathrm{Aut}_F(L))$ is the category of continuous non-empty transitive actions of $\mathrm{Aut}_F(L)$ on discrete sets.

A natural question thus arises: can we characterize the categories $\mathcal{C}$ whose dual is equivalent to (or fully embeddable into) the category of (non-empty transitive) actions of a topological automorphism group?

The topos-theoretic interpretation

Key observation: the above equivalence extends to an equivalence of toposes

$$\mathbf{Sh}({\mathcal{L}_F^L}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}_F(L)),$$

where J_{at} is the atomic topology on ${\mathcal{L}_F^L}^{\text{op}}$ and $\mathbf{Cont}(\mathrm{Aut}_F(L))$ is the topos of continuous actions of $\mathrm{Aut}_F(L)$ on discrete sets.

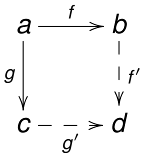
It is therefore natural to investigate our problem by using the methods of topos theory: more specifically, we shall look for conditions on a small category $\mathcal{C}$ and on an object u of its ind-completion for the existence of an equivalence of toposes of the form

$$\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}(u)) \ .$$

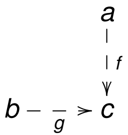
We will then be able to obtain, starting from such an equivalence, an answer to our question, and hence build Galois-type theories in a great variety of different mathematical contexts.

The key notions I

- A category $\mathcal{C}$ is said to satisfy the amalgamation property (AP) if for every objects $a, b, c \in \mathcal{C}$ and morphisms $f : a \to b$, $g : a \to c$ in $\mathcal{C}$ there exists an object $d \in \mathcal{C}$ and morphisms $f' : b \to d$, $g' : c \to d$ in $\mathcal{C}$ such that $f' \circ f = g' \circ g$:



- A category $\mathcal{C}$ is said to satisfy the joint embedding property (JEP) if for every pair of objects $a, b \in \mathcal{C}$ there exists an object $c \in \mathcal{C}$ and morphisms $f : a \to c$, $g : b \to c$ in $\mathcal{C}$:



The key notions II

- An object $u \in \text{Ind-}\mathcal{C}$ is said to be $\mathcal{C}$-universal if for every $a \in \mathcal{C}$ there exists an arrow $\chi : a \to u$ in $\text{Ind-}\mathcal{C}$:

$$a \overset{\chi}{\dashrightarrow} u$$

- An object $u \in \text{Ind-}\mathcal{C}$ is said to be $\mathcal{C}$-ultrahomogeneous if for any object $a \in \mathcal{C}$ and arrows $\chi_1 : a \to u$, $\chi_2 : a \to u$ in $\text{Ind-}\mathcal{C}$ there exists an automorphism $j : u \to u$ such that $j \circ \chi_1 = \chi_2$:

$$\begin{array}{ccc} a & \xrightarrow{\chi_1} & u \\ & \underset{\chi_2}{\searrow} & \downarrow j \\ & & u \end{array}$$

Topological Galois theory as a 'bridge'

Theorem

Let $\mathcal{C}$ be a small category satisfying the amalgamation and joint embedding properties, et let u be a $\mathcal{C}$-universal et $\mathcal{C}$-ultrahomogeneous object of the ind-completion Ind-$\mathcal{C}$ of $\mathcal{C}$. Then there is an equivalence of toposes

$$\mathbf{Sh}(\mathcal{C}^{op}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}(u)),$$

where $\mathrm{Aut}(u)$ is endowed with the topology in which a basis of open neighbourhoods of the identity is given by the subgroups of the form $I_\chi = \{\alpha \in \mathrm{Aut}(u) \mid \alpha \circ \chi = \chi\}$ for $\chi : c \to u$ an arrow in Ind-$\mathcal{C}$ from an object c of $\mathcal{C}$.
This equivalence is induced by the functor

$$F : \mathcal{C}^{op} \to \mathbf{Cont}(\mathrm{Aut}(u))$$

which sends any object c of $\mathcal{C}$ on the set $\mathrm{Hom}_{\text{Ind-}\mathcal{C}}(c, u)$ (endowed with the obvious action of $\mathrm{Aut}(u)$) and any arrow $f : c \to d$ in $\mathcal{C}$ to the $\mathrm{Aut}(u)$-equivariant map

$$- \circ f : \mathrm{Hom}_{\text{Ind-}\mathcal{C}}(d, u) \to \mathrm{Hom}_{\text{Ind-}\mathcal{C}}(c, u) .$$

Topological Galois theory as a 'bridge'

The following result arises from two 'bridges', respectively obtained by considering the invariant notions of atom and of arrow between atoms.

Theorem

Under the hypotheses of the last theorem, the functor F is full and faithful if and only if every arrow of $\mathcal{C}$ is a strict monomorphism, and it is an equivalence on the full subcategory $\mathbf{Cont}_t(\mathrm{Aut}(u))$ of $\mathbf{Cont}(\mathrm{Aut}(u))$ on the non-empty transitive actions if $\mathcal{C}$ is moreover atomically complete.

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}(u))$$

$$\mathcal{C}^{\mathrm{op}} \qquad\qquad \mathbf{Cont}_t(\mathrm{Aut}(u))$$

This theorem generalizes Grothendieck's theory of Galois categories and can be applied for generating Galois-type theories in different fields of Mathematics, for example that of finite groups and that of finite graphs.

Moreover, if a category $\mathcal{C}$ satisfies the first but not the second condition of the theorem, our topos-theoretic approach gives us a fully explicit way to complete it, by means of the addition of 'imaginaries', so that also the second condition gets satisfied.

Theories of presheaf type

Definition

A geometric theory is said to be of presheaf type if it is classified by a presheaf topos.

Theories of presheaf type are very important in that they constitute the basic 'building blocks' from which every geometric theory can be built. Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a 'quotient' of a theory of presheaf type.

These theories are the logical counterpart of small categories, in the sense that:

- For any theory of presheaf type $\mathbb{T}$, its category $\mathbb{T}$-mod(**Set**) of (set-based) models is equivalent to the ind-completion of the full subcategory f.p.$\mathbb{T}$-mod(**Set**) on the finitely presentable models.
- Any small category $\mathcal{C}$ is, up to idempotent splitting completion, equivalent to the category f.p.$\mathbb{T}$-mod(**Set**) for some theory of presheaf type $\mathbb{T}$.

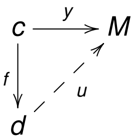
Moreover, any geometric theory $\mathbb{T}$ can be expanded to a theory classified by the topos $[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$.

Topos-theoretic Fraïssé theorem

The following result, which generalizes Fraïssé's theorem in classical model theory, arises from a triple 'bridge'.

Definition

A set-based model M of a geometric theory $\mathbb{T}$ is said to be homogeneous if for any arrow $y : c \to M$ in $\mathbb{T}$-mod(**Set**) and any arrow f in f.p.$\mathbb{T}$-mod(**Set**) there exists an arrow u in $\mathbb{T}$-mod(**Set**) such that $u \circ f = y$:



Theorem

Let $\mathbb{T}$ be a theory of presheaf type such that the category f.p.$\mathbb{T}$-mod(**Set**) *is non-empty and has AP and JEP. Then the theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models is complete and atomic.*

Topos-theoretic Fraïssé theorem

Atomic topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
Atomic site i.e.
AP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Atomicity of $\mathbb{T}'$

Two-valued topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
JEP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Completeness of $\mathbb{T}'$

Point of

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
homogeneous $\mathbb{T}$-model in **Set**

$(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
$\mathbb{T}'$-model in **Set**

Stone-type dualities through 'bridges'

The 'bridge-building' technique allows one to unify all the classical Stone-type dualities between special kinds of preorders and partial orders, locales or topological spaces as instances of just one topos-theoretic phenomenon, and to generate many new such dualities.

More precisely, this machinery generates Stone-type dualities/equivalences by functorializing 'bridges' of the form

$$\mathbf{Sh}(\mathcal{C}, J_{\mathcal{C}}) \simeq \mathbf{Sh}(\mathcal{D}, K_{\mathcal{D}})$$

$\mathcal{C}$ $\qquad\qquad$ $\mathcal{D}$

where

- $\mathcal{C}$ is a preorder (regarded as a category),
- $J_{\mathcal{C}}$ is a (subcanonical) Grothendieck topology on $\mathcal{C}$,
- $\mathcal{C}$ is a $K_{\mathcal{D}}$-dense full subcategory of $\mathcal{D}$, and
- $J_{\mathcal{C}}$ is the induced Grothendieck topology $(K_{\mathcal{D}})|_{\mathcal{C}}$ on $\mathcal{C}$.

Stone-type dualities through 'bridges'

Our machinery relies on the following key points:

- The possibility of defining Grothendieck topologies on posets in an intrinsic way which exploits the lattice-theoretic structure present on them.
- The possibility of functorializing the assignments $\mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J_{\mathcal{C}})$ and $\mathcal{D} \to \mathbf{Sh}(\mathcal{D}, K_{\mathcal{D}})$ by means of the (covariant or contravariant) theory of morphisms of sites.
- The possibility of recovering (under suitable hypotheses which are satisfied in a great number of cases) a given preordered structure from the associated topos by means of a topos-theoretic invariant.
  More precisely, if the topologies $K_{\mathcal{D}}$ (resp. $J_{\mathcal{C}}$) can be 'uniformly described through an invariant C of families of subterminals in a topos' then the elements of $\mathcal{D}$ (resp. of $\mathcal{C}$) can be recovered as the subterminal objects of the topos $\mathbf{Sh}(\mathcal{D}, K_{\mathcal{D}})$ (resp. $\mathbf{Sh}(\mathcal{C}, J_{\mathcal{C}})$) which satisfy a condition of C-compactness.

Future directions

The evidence provided by the results obtained so far shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories and for generating new equivalences, dualities and symmetries across different fields of Mathematics.

In fact, toposes have an authentic creative power in Mathematics, in the sense that their study naturally leads to the discovery of a great number of notions and 'concrete' results in different mathematical fields, which are pertinent but often unsuspected.

In the next years, we intend to continue pursuing the development of these general unifying methodologies both at the theoretical level and at the applied level, in order to continue developing the potential of toposes as fundamental tools in the study of mathematical theories and their relations, and as key concepts defining a new way of doing Mathematics liable to bring distinctly new insights in a great number of different subjects.

Future directions

Central themes in this programme will be:

- investigation of important dualities or correspondences in Mathematics from a topos-theoretic perspective (in particular, the theory of motives, class field theory and the Langlands programme)
- systematic study of invariants of toposes in terms of their presentations, and introduction of new invariants which capture important aspects of concrete mathematical problems
- interpretation and generalization of important parts of classical and modern model theory in terms of toposes and development of a functorial model theory
- introduction of new methodologies for generating Morita-equivalences
- development of general techniques for building spectra by using classifying toposes
- generalization of the 'bridge' technique to the setting of higher categories and toposes through the introduction of higher geometric logic
- development of a relative theory of classifying toposes

For further reading

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