Topos-empowered automatic theorem generation

Olivia Caramello

(University of Insubria - Como, and Grothendieck Institute)

CRI, Mines Paris, 17 September 2024

Plan of the talk

- The theory of topos-theoretic 'bridges'
- Topos-empowered automatic theorem generation
- Central themes and subprojects
- Other applications to computer science and artificial intelligence
- The Grothendieck Institute

The "unifying notion" of topos

"C'est le thème du topos qui est ce "lit", ou cette "rivière profonde" où viennent s'épouser la géométrie et l'algèbre, la topologie et l'arithmétique, la logique mathématique et la théorie des catégories, le monde du continu et celui des structures "discontinues" ou "discrètes". Il est ce que j'ai conçu de plus vaste, pour saisir avec finesse, par un même langage riche en résonances géométriques, une "essence" commune à des situations des plus éloignées les unes des autres provenant de telle région ou de telle autre du vaste univers des choses mathématiques".

A. Grothendieck

Since the times of my Ph.D. studies, I have developed a theory and a number of techniques allowing one to exploit the unifying potential of the notion of topos for establishing 'bridges' across different mathematical theories, by building in particular on the notion of classifying topos educed by categorical logicians.

Toposes as unifying 'bridges'

This theory, introduced in the programmatic paper "*The unification of Mathematics via Topos Theory*" of 2010, allows one to exploit the technical flexibility inherent to the concept of topos - most notably, the possibility of presenting a topos in a multitude of different ways - for building unifying 'bridges' useful for transferring notions, ideas and results across different mathematical contexts.

In the last years, besides leading to the solution of a number of long-standing problems in categorical logic, these techniques have generated several substantial applications in different mathematical fields. Still, much remains to be done so that toposes become a key tool universally used for investigating mathematical theories and their relations.

In fact, these 'bridges' have proved useful not only for connecting different mathematical theories with each other, but also for investigating a given theory from multiple points of view.

Moreover, the construction of 'bridges' can be automatized in many cases, leading to significant applications to computer science (notably, in connection with ATP) and AI.

Bridge objects

- Given two objects a and b, imagine to be able to associate with a an object $f(a)$ through a certain 'construction' f and with b an object $g(b)$ through a 'construction' g, in such a way that $f(a)$ and $g(b)$ be related by a certain equivalence relation $\simeq$. Then a and b can be seen as two distinct representations of a unique object u, equivalent on the one hand to $f(a)$ and on the other hand to $g(b)$, which can be used as a 'bridge' for transferring invariants across its different presentations:

Invariant I

$$f(a) \simeq u \simeq g(b)$$

characterization for I   *characterization for I*

a — *Property* P_a b — *Property* Q_b

- In the topos-theoretic implementation of the 'bridge' technique, the objects a and b are formalized contexts or theories while $f(a)$ and $g(b)$ are toposes associated with them. Note that a and b may also be seen as different 'points of view' on the bridge object u.

Toposes as *bridges*

- The existence of theories which are sematically equivalent to each other translates into the existence of different presentations for the same Grothendieck topos.
- Grothendieck toposes can be effectively used as 'bridges' for transferring notions, properties and results across different sematically equivalent theories:

$$\begin{tikzcd} & \mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'} \arrow[dr, dashed] & \\ \mathbb{T} \arrow[ur, dashed] & & \mathbb{T}' \end{tikzcd}$$

- The transfer of information takes place by expressing topos-theoretic invariants in terms of different presentations for the given topos.
- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

A new way of doing mathematics

As a matter of fact, the method of 'bridges' defines a new way of doing Mathematics which is, in a sense, 'upside-down' compared with the 'usual' ones.

Indeed, instead of starting with simple ingredients and combining them to build more complicated structures, one assumes as primitive ingredients rich and sophisticated mathematical entities, namely topos equivalences and topos-theoretic invariants, and extracts from them a huge amount of information relevant for classical mathematics.

Due to the strong element of automatism inherent to these techniques (the computation of topos-theoretic invariants is essentially canonical), one can generate new mathematical results without really making any creative effort: indeed, in many cases one can just readily apply the general characterizations connecting properties of sites and topos-theoretic invariants to the particular equivalence under consideration.

A new way of doing mathematics

The results generated in this way are in general non-trivial, they can be rather ‘weird’ according to the usual mathematical standards (in spite of being technically quite deep) but, with a careful choice of topos equivalences and invariants, one can easily get interesting and naturalmathematical results.

The level of mathematical depth of the results thus generated varies enourmously, depending on the complexity of the chosen topos-theoretic invariants and of the given equivalence of toposes. It can range from trivialities to very deep results (think, for instance, of the complexity inherent to the computation of cohomological invariants).

A lot of information that is not visible with the usual ‘glasses’ is revealed by the application of this machinery.

The range of applicability of these methods is boundless within Mathematics, by the very generality of the notion of topos.

A universal calculus on mathematical theories

Topos theory notably provides a very broad and effective setting for studying any aspects of mathematical theories, both in themselves and in relation to each other.

It yields a universal calculus on theories; in fact, topos theory naturally leads to the introduction of several important operations on theories, and also provides effective methods for computing them. By way of analogy, as in arithmetic one makes computations with numbers, so in topos theory one can make computations with theories.

For example, in my 2017 book I have shown that the collection of all mathematical theories (presented in geometric form) in a given language has the structure of a lattice (in fact, of a Heyting algebra); in particular, one can take the intersection, the union, the pseudocomplement of theories, *etc.* By using these methods, I have obtained effective computations for these operations, and even managed to prove a very general deduction theorem for geometric logic without any creative effort, that is, by mechanically calculating a certain Grothendieck topology!

Automatic generation of theorems

We aim at implementing the theory of topos-theoretic 'bridges' on a computer, so as to obtain a proof assistant capable to generate new results in any field of mathematics in an automatic way:

- Once an equivalence between two different presentations of the same topos is established, the calculation of how invariants express in terms of the two presentations is essentially canonical and can be automatized in many cases: descriptions of classes of invariants, of logical or geometric nature, for which such calculations can be performed in a semi-automatic way are provided in my papers.
- In order to obtain insights on the equivalence of toposes under consideration, in many cases one can just readily apply to it general characterizations connecting properties of sites and topos-theoretic invariants. Still, the results generated in this way are in general non-trivial.
- This means that a computer could well be programmed in order to generate a huge amount of new (non-trivial) results in different mathematical fields by implementing these techniques.

An automatic theorem prover

More specifically, such a proof assistant should take as inputs:

(1) a database of topos equivalences, that is of pairs of different presentations for a given topos;

(2) a database of characterizations of topos-theoretic invariants in terms of different presentations for toposes (what we called above the 'encyclopedia of invariants and their characterizations').

Starting from them, it could combine (1) (that is, the 'decks' of bridges) with (2) (that is, the 'arches' of bridges) to generate concrete correspondences between 'unravelings' of a given invariant in terms of different presentations for a given topos.

The system should ideally also automatically update the two databases generating by itself characterizations of topos-theoretic invariants and equivalences of toposes.

Central themes and subprojects

The project will involve a number of different subprojects, on each of which we expect an intense collaboration with computer scientists with different skills:

1. Preliminary work on the formalisation of the necessary topos-theoretic background in a proof assistant (notably, Lean)
2. Geometrisation of theories
3. Representation of objects in toposes
4. An 'encyclopedia of invariants and their characterizations'
5. A database of topos equivalences
6. An important case study: the duality between quotients and subtoposes
7. Tactics and rewriting rules
8. Integration of AI tools and 'syntactic learning' techniques

Preliminary formalisation work

- The plan is to formalise the relevant topos-theoretic notions and results in the proof assistant Lean.
- It is essential to dispose of a complete formalisation of the basic notions of geometric theory, of site, of Grothendieck topos, of classifying topos, of invariant, etc. before undertaking the more advanced stages of the project.
- Lean appears to be the most suitable system for our needs, mainly because of the existence of a large amount of already formalized mathematical results in different fields, as well as of tactics and rewriting systems (e.g. Aesop), which could be further developed and improved.

Geometrization of theories

- In order to compute the classifying topos of a (first-order) mathematical theory, the latter must be *geometrized*. This can be done in a fully canonical through the Morleyization construction.
- However, this construction is not economical at all (it actually requires the addition of a infinite number of new predicates), so it is not apt for a computer implementation.
- There is scope for the development of algorithms for geometrizing theories in a most economical way (e.g. by replacing only the non-geometric elements arising in the syntax of the theory with geometric ones).

Representation of objects in toposes

- It will be crucial, in connection with the generation of 'arches' of topos-theoretic 'bridges', to develop a computer representation of objects of a Grothendieck topos in terms of generators and relations.
- More specifically, any object of the topos $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a site $(\mathscr{C}, J)$ can be represented as a definable quotient of a coproduct of objects coming from $\mathscr{C}$.
- These 'codings' make it possible to describe whatever happens inside a topos in combinatorial terms, so they can be useful, in particular, for obtaining site-characterizations for topos-theoretic invariants.

New proof systems for geometric theories

We have showed that the classical proof system of geometric logic over a given geometric theory is equivalent to new proof systems based on the notion of Grothendieck topology.

These equivalences result from a proof-theoretic interpretation of a duality between the quotients (i.e. geometric theory extensions over the same signature) of a given geometric theory and the subtoposes of its classifying topos.

Interestingly, these alternative proof systems turn out to be computationally better-behaved than the classical one for many purposes, as shown in my 2017 book.

More generally, we plan to further investigate the proof-theoretic equivalences arising from different presentations of the classifying topos of a theory.

Tactics and rewriting rules

- It will be indispensable for a successful realisation of the project to dispose of rewriting systems allowing to turn machine generated results into human readable statements.
- Some of such systems already exist, but a great amount of further work should be done in this direction in order to make, at least, whatever is 'routine' for a mathematician also 'routine' for the proof system.
- We expect advanced AI tools (including LLMs), combined with syntactic learning techniques, to be very relevant in this connection, in particular for rewriting mathematical statements in a way which is most readable or meaningful to the human mind, as well as for recognizing new patterns in the way in which mathematicians generate proofs and express their findings.

Towards a theory of semantic information

- As we explained above, toposes are objects which embody the semantic content of a wide class of theories, or the essence of a great number of mathematical contexts (pertaining to different areas of mathematics).

- It is therefore natural to develop a theory of semantic information based on toposes and on the invariants that one can define on them.

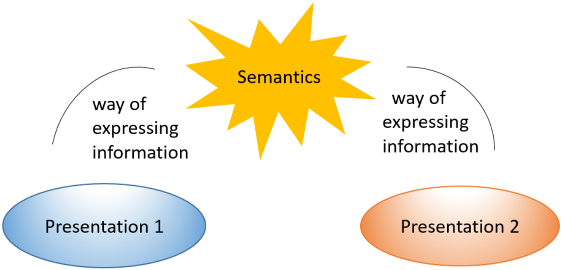
- As a matter of fact, the fundamental invariants of mathematical structures are actually invariants of toposes associated with these structures.

- Indeed, it is at the topos-theoretic level that invariants naturally live. This is due to the fact that toposes, unlike ordinary mathematical structures, have a very rich internal structure, actually being completions of concrete theories or structures with respect to all the natural operations that one might want to perform on them.

Communication through 'bridges'

Topos-theoretic 'bridges' have proved to be very effective in acting as 'universal translators', that is, as tools for unifying different presentations of a given semantic content and for transferring knowledge between them.

We thus expect communication between different intelligent agents to be profitably understandable in terms of 'bridges' induced by equivalences (or more general relations) between toposes which describe their functioning. More generally, toposes embodying a given semantic content can act as 'bridges' across different knowledge representations:



Modelling of learning processes via proofs

We plan to explore the possibility of modelling the functioning of (artificial) learning processes, in particular of deep neural networks, by building on the notion of mathematical proof:

- Note that any intelligent agent must, in order to get an effective understanding of (aspects of) the world, derive knowledge starting from certain 'sensory' inputs, which play a similar role to that of axioms for a mathematical theory, by following certain dynamical rules, which correspond to the inference rules of the logical system inside which the mathematical theory is formulated.
- As every mathematical theory can be enriched by the addition of new axioms, so the functioning of an agent can be updated by the integration of new information which becomes available to it.
- The functioning of a learning system can thus be modelled by a sequence of a mathematical theories, each of which more refined (that is, with more axioms, or fewer models) than the previous ones.

Modelling of learning processes via proofs

- In principle, any learning sequence, as any process 'approximating truth', is infinite, but for practical purposes one is normally satisfied by the result of a learning process when the last theory in the sequence leaves a degree of ambiguity which is sufficiently low for the desired applications. (For example, for a car with an object detection system, it is important to be able to identify the kind of animal that might cross the road, but not necessarily the color of its hair!).
- All the theories in the sequence should extend (that is, be defined over) the basic theory of the agent formalizing its essential features. More generally, every theory in the sequence can be seen as theory defined *over* each of the preceding ones.
- Note that any constraints embedded in the logical formalism (or integrated at some step of the sequence of theories) will allow to significantly reduce the space of parameters that the agent has to explore and hence correspondingly decrease the computational complexity of the learning process.

Geometric logic

Among the possible formalisms for modelling the functioning of intelligent agents, we plan to focus in particular on (relative) geometric logic.

Geometric logic is the logic underlying Grothendieck toposes. Indeed, any theory formulated within geometric logic admits a classifying topos and, conversely, any Grothendieck topos is the classifying topos of some (actually, of infinitely many) geometric theories.

In fact, Grothendieck toposes geometrically embody, in a 'maximally structured' way, the logical information contained in geometric theories.

Geometric logic is a particular kind of (infinitary) first-order logic (actually not of inferior expressiveness). It is widely considered as the "logic of finite observations", and is particularly amenable to computation and automated theorem proving.

Indeed, geometric logic is inherently constructive. This is very relevant from a computer science perspective, by the well-known paradigm identifying programs with constructive proofs.

Geometric logic

Whilst higher-order logic is clearly more expressive than first-order logic, the model theory of geometric theories is much better-behaved and more amenable to computation, as witnessed for instance by the existence theorem for classifying toposes and universal models.

The development of relative geometric logic (in the context of our joint work with Riccardo Zanfa on relative toposes) will allow us to formalize a great number of higher-order notions whilst preserving geometricity and the computational advantages arising from it. (A nice illustration of the expressive power of relative theories is provided by the ongoing work on a refoundation of functional analysis (which is second-order) as algebra (which is first-order) over a suitable base topos by Fields Medalist Peter Scholze and his collaborator Dustin Clausen.)

Thus our sequence of theories formalizing the functioning, say, of a neural network, will correspond to a sequence of relative toposes, the base topos being the one formalizing the architecture of the network.

Computational power of Grothendieck topologies

One of the reasons for the computational effectiveness of geometric logic is the connection with Grothendieck toposes, given by the classifying topos construction.

In my 2017 book it is shown that the classical proof system of geometric logic over a given geometric theory is equivalent to new proof systems whose inference rules correspond to the axioms of Grothendieck topologies. These equivalences actually result from topos-theoretic 'bridges' between different presentations of the classifying topos of the theory.

Interestingly, these alternative proof systems turn out to be computationally much better-behaved than the classical one. In fact, instead of having several axioms and inference rules, they only have *two* inference rules, making it much more manegeable to compute inside them. In fact, there is even a *formula characterizing the 'theorems' provable in such systems*!

'Bridges' involving Grothendieck topologies have also proved very useful for building a great variety of structures presented by generators and relations in most explicit ways.

Syntactic learning

In order to practically implement the above ideas, one should, first of all, empower any learning system with large formal vocabularies that will serve for expressing the concepts that the system will learn from data.

The idea is to enforce the learning to take place at the abstract level of syntax, rather than at that of a particular semantics, as it currently happens.

While the vocabulary should be given at the outset, we could let the system discover any (non-already embedded) logical rules expressible in that vocabulary by itself, by using the usual techniques. Also, we should let it suggest enrichments of the vocabulary on the basis of invariances empirically discovered in the data, thereby achieving a form of 'emergence of concepts'.

In this way, we could obtain systems capable of inferring all sorts of 'syntactic rules' from data (e.g. the grammar rules of a language from a great amount of samples of texts, or the rules of a game from a big collection of matches, etc.): these rules could, of course, be of different nature and complexity (e.g., propositional, first-order, higher-order, etc.).

Making AI systems 'speak'

All of this goes in the direction of constructing logical languages for AI agents, allowing them, in a sense, to 'speak'. After all, how can we expect a system to learn in a robust way if we do not give it the possibility of reasoning linguistically? Think about how our learning, as human beings, would be impaired without the possibility of expressing, testing and communicating our ideas by using languages.

Think also about the process of learning of a natural language. Little babies have no other possibility to learn a language than to merely rely on data (note, however, that their brain has already a lot of *a priori structures* which are used to organize and categorize the knowledge that they gather from the environment). Still, their knowledge of the language remains fragile until they are brought into contact, notably at school, with grammar, which represents syntax in this context.

The role of grammar is crucial in making *explicit* a lot of the *implicit* which had been accumulated in the previous 'bottop-up' learning process, and in bringing the understanding to a higher level, notably in terms of explainability.

Uncovering hidden structures and rules

Syntax represents the 'skeleton' of structures, whose concrete manifestations we access through data. Trying to 'lift' from a particular concrete setting to the level of syntax represents an abstract form of understanding which enjoys much more relisience and adaptability than the usual forms of statistically-based learning.

The deep reasons for the regularities that we may observe in concrete contexts actually live at the syntactic level (think, for instance, of motives in Algebraic Geometry, or to definability and preservation theorems in Logic).

Philosophically speaking, we need to teach learning systems to lift from the phenomenological level (of concrete manifestations of topos invariants) to the ontological one (of toposes themselves, regarded as classifying toposes of logical theories).

In conclusion, we advocate for an integration of a 'bottop-up' approach to artificial learning, such as the one which is dominant today, with a 'top-down' one based on logic and topos invariants.

A topos-theoretic analysis of images

We also plan to investigate images from a topos-theoretic perspective:

- Logical axiomatisations of the objects (in the three-dimensional space) that images are meant to represent, together with a study of their two-dimensional projections, would greatly improve the systems for image recognition.
- On the other hand, images can also be profitably understood from a geometric, sheaf-theoretic, perspective, as arising from the glueing of local regions admitting simpler descriptions.
- The integration between logic and geometry provided by topos theory would allow one to switch from the logical point of view to the geometric one, thus taking advantage of both. In particular, a topos-theoretic treatment would allow a swift passage between different scales.

Structural approximation theory

The classical theory of neural networks is based on the approximation of real-valued functions and hence, ultimately, on a very particular metric space: $\mathbb{R}$ (with the usual Euclidean metric).

Numbers are not semantically meaningful by themselves; this makes any theory of deep learning which is based merely on them necessarily *fragile*, i.e. exposed to the risk of overfitting. In order to make a learning process *robust* and capable of *generalisation*, we need to make it structural.

As observed by Laurent Lafforgue, numbers should be thought of as 'traces' of geometric structures. Accordingly, we should develop an approximation theory for structures rather than for mere numerical functions. These structures should result from logical constraints imposed at the outset (analogously to the *a priori* structures of our brain, which make us organize the data that we infer from our sensory inputs in a certain way) as well as from symmetries of the object of the learning process.

Transformers from a topos-theoretic perspective

As observed by Michael Robinson, trasformers should be formalized and investigated through a sheaf-theoretic perspective.

Indeed, the process of building a global view from a family of compatible local views is akin to the construction of elements of a sheaf as amalgamations of matching families of local data.

In order to formalize semantic information collected from different sets of sensory inputs, each of which using its own knowledge representation, it is more sensible, in order to have a common language in which all the data coming from the different inputs are formulated, to formalize the local view of each of the sensors by a topos, and to think of a transformer as a stack of such toposes.

Note that the use of toposes of local views as opposed to sets (as in Robinson's proposal) allows one to formalize non-trivial symmetries in the views of sensory inputs.

The Grothendieck Institute

- The Grothendieck Institute is an international foundation devoted to cutting-edge research in mathematics and its interactions with other disciplines.
- Established in 2022 and based in Mondovì (Italy), the Institute is named after great mathematician Alexander Grothendieck, whose work it is committed to valorize and disseminate.
- Its main focus is interdisciplinarity, with particular reference to the development of unifying methods, both within mathematics and in relation to other areas of knowledge.
- The Institute pursues its mission notably through its research centres, and by collaborating with academic institutions and scientific associations which share its interests.
- It also offers doctoral studentships and research fellowships to outstanding young scholars selected by its Scientific Council, which includes among its members three Fields Medalists.

Toposes in Mondovì

In September 2024, the fourth edition of the international topos theory school and conference, organized by the Grothendieck Institute, has taken place in Mondovì:

ISTITUTO GROTHENDIECK

3 Fields Medals

121 participants from
- **5** continents
- **30** countries

7 days of activities:
- **4** of school
- **3** of conference

30 study grants

Città di Mondovì

8 courses given by internationally renowned lecturers

10 invited talks given by specialists

14 contributed talks by young researchers

2 public events of scientific dissemination

For further reading

- O. Caramello
  Grothendieck toposes as unifying ‘bridges’: a mathematical morphogenesis,
  in the Springer book *Philosophy of Mathematics. Objects, Structures, and Logics*.

- O. Caramello
  Grothendieck toposes as unifying ‘bridges’ in Mathematics,
  Mémoire d’habilitation à diriger des recherches,
  Université de Paris 7, 2016,
  available from my website www.oliviacaramello.com.

- O. Caramello
  Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic ‘bridges’,
  Oxford University Press, 2017.

Grothendieck topologies

Definition

A Grothendieck topology on a small category $\mathscr{C}$ is a function J which assigns to each object c of $\mathscr{C}$ a collection $J(c)$ of sieves on c in such a way that

(i) (maximality axiom) the maximal sieve $M_c = \{f \mid cod(f) = c\}$ is in $J(c)$;

(ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;

(iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

The sieves S which belong to $J(c)$ for some object c of $\mathscr{C}$ are said to be J-covering.

A site is a pair $(\mathscr{C}, J)$ consisting of a category $\mathscr{C}$ and a Grothendieck topology J on $\mathscr{C}$.

The syntactic category of a geometric theory

Definition (Makkai and Reyes 1977)

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. The syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the 'renaming'-equivalence classes of geometric formulae-in-context $\{\vec{x} \,.\, \phi\}$ over Σ and as arrows $\{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, \psi\}$ (where the contexts $\vec{x}$ and $\vec{y}$ are supposed to be disjoint without loss of generality) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\vec{x}, \vec{y})$ which are $\mathbb{T}$-provably functional i.e. such that the sequents
$$(\phi \vdash_{\vec{x}} (\exists y)\theta),$$
$$(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi), \text{ and}$$
$$((\theta \wedge \theta[\vec{z}/\vec{y}]) \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{y} = \vec{z}))$$
are provable in $\mathbb{T}$.
- The composite of two arrows
$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\theta]} \{\vec{y} \,.\, \psi\} \xrightarrow{[\gamma]} \{\vec{z} \,.\, \chi\}$$
is defined as the $\mathbb{T}$-provable-equivalence class of the formula $(\exists \vec{y})\theta \wedge \gamma$.
- The identity arrow on an object $\{\vec{x} \,.\, \phi\}$ is the arrow
$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\phi \wedge \vec{x}' = \vec{x}]} \{\vec{x}' \,.\, \phi[\vec{x}'/\vec{x}]\}$$

Syntactic sites

- For any geometric theory $\mathbb{T}$, its syntactic category $\mathscr{C}_{\mathbb{T}}$ is a geometric category, i.e. a well-powered cartesian category in which images of morphisms and arbitrary unions of subobjects exist and are stable under pullback.
- For a geometric theory $\mathbb{T}$, the geometric topology on $\mathscr{C}_{\mathbb{T}}$ is the Grothendieck topology $J_{\mathbb{T}}$ whose covering sieves are those which contain small covering families.

Definition

The syntactic topology $J_{\mathbb{T}}$ on the syntactic category $\mathscr{C}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ is the geometric topology on it; in particular,

a small family $\{[\theta_i] : \{\vec{x}_i . \phi_i\} \to \{\vec{y} . \psi\}\}$ in $\mathscr{C}_{\mathbb{T}}$ is $J_{\mathbb{T}}$-covering

if and only if

the sequent $(\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x}_i)\theta_i)$ is provable in $\mathbb{T}$.

Grothendieck toposes

One can define sheaves on an arbitrary site in a formally analogous way to how one defines sheaves on a topological space. This leads to the following

Definition

- A Grothendieck topos is a category (equivalent to the category) $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a (small-generated) site $(\mathscr{C}, J)$.
- A geometric morphism of toposes $f : \mathscr{E} \to \mathscr{F}$ is a pair of adjoint functors whose left adjoint (called the inverse image functor) $f^* : \mathscr{F} \to \mathscr{E}$ preserves finite limits.

  For instance, the inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ of a Grothendieck topos $\mathbf{Sh}(\mathscr{C}, J)$ in the corresponding presheaf topos $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ yields a geometric morphism between these toposes (whose inverse image is the associated sheaf functor).

Subtoposes

Definition

A subtopos of a topos $\mathscr{E}$ is an equivalence class of geometric inclusions to $\mathscr{E}$.

Fact

- *The notion of subtopos is a topos-theoretic invariant.*
- *If $\mathscr{E}$ is the topos $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a site $(\mathscr{C}, J)$, the subtoposes of $\mathscr{E}$ are in bijective correspondence with the Grothendieck topologies J' on $\mathscr{C}$ which contain J (i.e. such that every J-covering sieve is J'-covering).*

A duality theorem

Definition

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. A quotient of $\mathbb{T}$ is a geometric theory $\mathbb{T}'$ over Σ such that every axiom of $\mathbb{T}$ is provable in $\mathbb{T}'$.
- Let $\mathbb{T}$ and $\mathbb{T}'$ be geometric theories over a signature Σ. We say that $\mathbb{T}$ and $\mathbb{T}'$ are syntactically equivalent, and we write $\mathbb{T} \equiv_s \mathbb{T}'$, if for every geometric sequent σ over Σ, σ is provable in $\mathbb{T}$ if and only if σ is provable in $\mathbb{T}'$.

Theorem

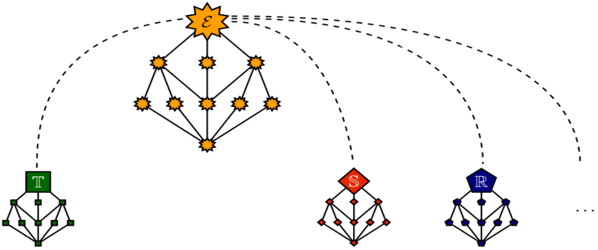
Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the $\equiv_s$-equivalence classes of quotients of $\mathbb{T}$ and the subtoposes of the classifying topos **Set**$[\mathbb{T}]$ *of $\mathbb{T}$.*

Some consequences

This duality theorem has several implications; in particular, it allows one to import many notions and results from topos theory into the realm of geometric logic. For instance, one can deduce from it that

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the collection $\mathfrak{Th}_{\Sigma}^{\mathbb{T}}$ of (syntactic-equivalence classes of) geometric theories over Σ which are quotients of $\mathbb{T}$, endowed with the order defined by $\mathbb{T}' \leq \mathbb{T}''$ if and only if all the axioms of $\mathbb{T}'$ are provable in $\mathbb{T}''$', is a Heyting algebra.



Lattices of theories

‘Bridges’ between quotients and topologies

This duality also allows one to establish ‘bridges’ of the following form:

Subtopos of
$\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Set}[\mathbb{T}]$

Grothendieck topology on $\mathscr{C}$ containing J

Quotient of $\mathbb{T}$

That is, if the classifying topos of a geometric theory $\mathbb{T}$ can be represented as the category $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a (small) site $(\mathscr{C}, J)$ then we have a natural, order-preserving bijection

$$\text{quotients of } \mathbb{T}$$
$$|||$$
$$\text{Grothendieck topologies on } \mathscr{C} \text{ which contain } J$$

Two notable cases

We shall focus on two particular cases of this result:

1. $(\mathscr{C}, J)$ is the syntactic site $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$
2. - $\mathbb{T}$ is a theory of presheaf type (e.g. a finitary algebraic, or more generally cartesian, theory),
   - $\mathscr{C}$ is the opposite of its category f.p.$\mathbb{T}$-mod(**Set**) of finitely presentable models, and
   - J is the trivial topology on it.

In the first case, we obtain an order-preserving bijective correspondence between the quotients of $\mathbb{T}$ and the Grothendieck topologies on $\mathscr{C}_{\mathbb{T}}$ which contain $J_{\mathbb{T}}$.

In the second case, we obtain an order-preserving bijective correspondence between the quotients of $\mathbb{T}$ and the Grothendieck topologies on f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{\mathrm{op}}$.

In both cases, these correspondences can be naturally interpreted as proof-theoretic equivalences between the classical proof system of geometric logic over $\mathbb{T}$ and new proof systems for sieves whose inference rules correspond to the axioms of Grothendieck topologies.

Proof systems for sieves

Given a collection $\mathscr{A}$ of sieves on a given category $\mathscr{C}$, the notion of Grothendieck topology on $\mathscr{C}$ naturally gives rise to a proof system $\mathscr{T}_{\mathscr{C}}^{\mathscr{A}}$, as follows: the axioms of $\mathscr{T}_{\mathscr{C}}^{\mathscr{A}}$ are the sieves in $\mathscr{A}$ plus all the maximal sieves, while the inference rules of $\mathscr{T}_{\mathscr{C}}^{\mathscr{A}}$ are the proof-theoretic versions of the well-known axioms for Grothendieck topologies, i.e. the following rules:

- *Stability rule:*

$$\frac{R}{f^*(R)}$$

where R is any sieve on an object c of $\mathscr{C}$ and f is any arrow in $\mathscr{C}$ with codomain c.

- *Transitivity rule:*

$$\frac{Z \quad \{f^*(R) \mid f \in Z\}}{R}$$

where R and Z are sieves in $\mathscr{C}$ on a given object of $\mathscr{C}$.

N.B. The 'closed theories' of this proof system are precisely the Grothendieck topologies on $\mathscr{C}$ which contain the sieves in $\mathscr{A}$ as covering sieves. The closure of a 'theory' in $\mathscr{T}_{\mathscr{C}}^{\mathscr{A}}$, i.e. of a collection $\mathscr{U}$ of sieves in $\mathscr{C}$, is exactly the Grothendieck topology on $\mathscr{C}$ generated by $\mathscr{A}$ and $\mathscr{U}$.

The first correspondence

Let $\mathbb{T}$ be a geometric theory over a signature Σ, $\mathscr{S}$ the collection of geometric sequents over Σ and $S(\mathscr{C}_{\mathbb{T}})$ the collection of (small-generated) sieves in the syntactic category $\mathscr{C}_{\mathbb{T}}$.

- Given a geometric sequent $\sigma \equiv (\phi \vdash_{\vec{x}} \psi)$ over Σ, we set $\mathscr{F}(\sigma)$ equal to the principal sieve in $\mathscr{C}_{\mathbb{T}}$ generated by the monomorphism

$$\{\vec{x'} \, . \, \phi \wedge \psi\} \xrightarrow{[(\phi \wedge \psi \wedge \vec{x'} = \vec{x})]} \{\vec{x} \, . \, \phi\} \, .$$

- Given a small-generated sieve $R = \{[\theta_i] : \{\vec{x_i} \, . \, \phi_i\} \to \{\vec{y} \, . \, \psi\}\}$ in $\mathscr{C}_{\mathbb{T}}$, we set $\mathscr{G}(R)$ equal to the sequent $(\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x_i})\theta_i)$.

The first equivalence

Let $V \to \overline{V}^{\mathscr{T}}$ and $U \to \overline{U}^{\mathbb{T}}$ respectively be the operations consisting in taking the Grothendieck topology $\overline{V}^{\mathscr{T}}$ generated by $J_{\mathbb{T}}$ plus the sieves in V and in taking the collection $\overline{U}^{\mathbb{T}}$ of geometric sequents provable in $\mathbb{T} \cup U$ by using geometric logic. Let $F : \mathscr{P}(\mathscr{S}) \to \mathscr{P}(S(\mathscr{C}_{\mathbb{T}}))$ be the composite $\overline{(-)}^{\mathscr{T}} \circ \mathscr{P}(\mathscr{F})$ and $G : \mathscr{P}(S(\mathscr{C}_{\mathbb{T}})) \to \mathscr{P}(\mathscr{S})$ be the composite $\overline{(-)}^{\mathbb{T}}_{\tilde{\mathscr{G}}} \circ \mathscr{P}(\mathscr{G})$. Then

Theorem

1. *For any $U \in \mathscr{P}(\mathscr{S})$, $\mathscr{F}(\overline{U}^{\mathbb{T}}) \subseteq \overline{\mathscr{F}(U)}^{\mathscr{T}}$.*
2. *For any $V \in \mathscr{P}(S(\mathscr{C}_{\mathbb{T}}))$, $\mathscr{G}(\overline{V}^{\mathscr{T}}) \subseteq \overline{\mathscr{G}(V)}^{\mathbb{T}}$.*
3. *For any $U \in \mathscr{P}(\mathscr{S})$, $G(F(U)) = \overline{U}^{\mathbb{T}}$.*
4. *For any $V \in \mathscr{P}(S(\mathscr{C}_{\mathbb{T}}))$, $F(G(V)) = \overline{V}^{\mathscr{T}}$.*

In other words, the maps F and G define a proof-theoretic equivalence between the classical deduction system for geometric logic over $\mathbb{T}$ and the proof system $\mathscr{T}^{J_{\mathbb{T}}}_{\mathscr{C}_{\mathbb{T}}}$.

Describing the second equivalence

Recall that a geometric theory is said to be of presheaf type if it is classified by a presheaf topos (equivalently, by the topos $[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$). Theories of presheaf type are very important in that they constitute the basic 'building blocks' from which every geometric theory can be built. Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a 'quotient' of a theory of presheaf type.

Every finitary algebraic (or more generally any cartesian) theory is of presheaf type, but this class also contain many other interesting mathematical theories.

Definition

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then a geometric formula $\phi(\vec{x})$ over Σ is said to be $\mathbb{T}$-irreducible if, regarded as an object of the syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$, it does not admit any non-trivial $J_{\mathbb{T}}$-covering sieves.

Theorem

A geometric theory $\mathbb{T}$ over a signature Σ is of presheaf type if and only if every geometric formula $\phi(\vec{x})$ over Σ, when regarded as an object of $\mathscr{C}_{\mathbb{T}}$, is $J_{\mathbb{T}}$-covered by $\mathbb{T}$-irreducible formulae over Σ.

Irreducible formulae and finitely presentable models

Theorem

Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ. Then

(i) *Any finitely presentable $\mathbb{T}$-model in* **Set** *is presented by a $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ;*

(ii) *Conversely, any $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ presents a $\mathbb{T}$-model.*

In fact, the category $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$ *is equivalent to the full subcategory* $\mathscr{C}_{\mathbb{T}}^{\text{irr}}$ *of* $\mathscr{C}_{\mathbb{T}}$ *on the $\mathbb{T}$-irreducible formulae.*

Irreducible object

$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$

Every object

$(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

$\mathbb{T}$-irreducible formula

Sequents and sieves on f.p. models

- By using the fact that every geometric formula over Σ can be $J_{\mathbb{T}}$-covered in $\mathscr{C}_{\mathbb{T}}$ by $\mathbb{T}$-irreducible formulae, one can show that every geometric sequent over Σ is provably equivalent in $\mathbb{T}$ to a collection of sequents σ of the form $(\phi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{y}_i)\theta_i)$ where, for each $i \in I$, $[\theta_i] : \{\vec{y}_i \,.\, \psi_i\} \to \{\vec{x} \,.\, \phi\}$ is an arrow in $\mathscr{C}_{\mathbb{T}}$ and $\phi(\vec{x})$, $\psi(\vec{y}_i)$ are geometric formulae over Σ presenting respectively $\mathbb{T}$-models $M_{\{\vec{x} \,.\, \phi\}}$ and $M_{\{\vec{y}_i \,.\, \psi_i\}}$.
- To such a sequent σ, we can associate the cosieve S_σ on $M_{\{\vec{x} \,.\, \phi\}}$ in f.p.$\mathbb{T}$-mod(**Set**) defined as follows. For each $i \in I$, $[[\theta_i]]_{M_{\{\vec{y}_i \,.\, \psi_i\}}}$ is the graph of a morphism $[[\vec{y}_i \,.\, \psi_i]]_{M_{\{\vec{y}_i \,.\, \psi_i\}}} \to [[\vec{x} \,.\, \phi]]_{M_{\{\vec{y}_i \,.\, \psi_i\}}}$; then the image of the generators of $M_{\{\vec{y}_i \,.\, \psi_i\}}$ via this morphism is an element of $[[\vec{x} \,.\, \phi]]_{M_{\{\vec{y}_i \,.\, \psi_i\}}}$ and this in turn determines, by definition of $M_{\{\vec{x} \,.\, \phi\}}$, a unique arrow $s_i : M_{\{\vec{x} \,.\, \phi\}} \to M_{\{\vec{y}_i \,.\, \psi_i\}}$ in $\mathbb{T}$-mod(**Set**). We set S_σ equal to the sieve in f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{\mathrm{op}}$ on M_ϕ generated by the arrows s_i as i varies in I.

Sequents and sieves on f.p. models

Conversely, by the equivalence $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \simeq \mathscr{C}_{\mathbb{T}}^{\text{irr}}$, every sieve in $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$ is clearly of the form S_σ for such a sequent σ.

These correspondences define, similarly to above, a proof-theoretic equivalence between the classical deduction system for geometric logic over $\mathbb{T}$ and the proof system $\mathscr{T}^{T}_{\text{f.p.T-mod}(\mathbf{Set})^{\text{op}}}$ (where T is the trivial Grothendieck topology).

In particular, the Grothendieck topology J on $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$ associated with a quotient $\mathbb{T}'$ of $\mathbb{T}$ is generated by the sieves S_σ, where σ varies among the sequents associated with the axioms of $\mathbb{T}'$ as above.

Moreover, for any σ of the above form, σ is provable in $\mathbb{T}'$ if and only if S_σ belongs to J.

This generalizes Coste-Lombardi-Roy's correspondence between dynamical theories (viewed as coherent quotients of universal Horn theories) and the coherent Grothendieck topologies associated with them.

Why are these equivalences interesting?

These equivalences are useful in that they allow us to study (the proof theory of) geometric theories through the associated Grothendieck topologies: the condition of provability of a sequent in a geometric theory gets transformed in the requirement for a sieve (or a family of sieves) to belong to a certain Grothendieck topology, something which is often much easier to investigate.

Indeed, we have shown that Grothendieck topologies are particularly amenable to computation by deriving

- An explicit formula for the Grothendieck topology generated by a given family of sieves
- Explicit descriptions of the lattice operations on Grothendieck topologies on a given category which refine a certain topology (recall that these correspond to the lattice operations on quotients via the above duality).

Formulas for Grothendieck topologies

Meet of Grothendieck topologies

If J_1 and J_2 are Grothendieck topologies on a category $\mathscr{C}$ respectively generated by bases K_1 and K_2, the meet $J_1 \wedge J_2$ is generated by the collection of sieves which are unions of sieves in K_1 with sieves in K_2.

Grothendieck topology generated by a family of sieves

The Grothendieck topology G_D generated by a family D of sieves in $\mathscr{C}$ which is stable under pullback is given by

$$\begin{array}{rcl} G_D(c) & = & \{S \text{ sieve on } c \mid \text{for any sieve } T \text{ on } c, \\ & & [(\text{for any arrow } d \stackrel{g}{\to} c \text{ and sieve } Z \text{ on } d \\ & & (Z \in D(d) \text{ and } Z \subseteq g^*(T)) \text{ implies } g \in T) \text{ and } (S \subseteq T)] \\ & & \text{implies } T = M_c\} \end{array}$$

for any object $c \in \mathscr{C}$.

Heyting implication of Grothendieck topologies

$$\begin{array}{rcl} (J_1 \Rightarrow J_2)(c) & = & \{S \text{ sieve on } c \mid \text{for any arrow } d \stackrel{f}{\to} c \text{ and sieve } Z \text{ on } d, \\ & & [Z \text{ is } J_1\text{-covering and } J_2\text{-closed and } f^*(S) \subseteq Z] \\ & & \text{implies } Z = M_d\}\ . \end{array}$$

Some applications

Theorem (A deduction theorem for geometric logic)

Let $\mathbb{T}$ be a geometric theory over a signature Σ and ϕ, ψ geometric sentences over Σ such that the sequent $(\top \vdash_{[]} \psi)$ is provable in the theory $\mathbb{T} \cup \{(\top \vdash_{[]} \phi)\}$. Then the sequent $(\phi \vdash_{[]} \psi)$ is provable in the theory $\mathbb{T}$.

We have proved this theorem by showing (using the above-mentioned formula for the Grothendieck topology generated by a given family of sieves) that if the principal sieve in $\mathscr{C}_{\mathbb{T}}$ generated by the arrow $\{[] \, . \, \psi\} \overset{[\psi]}{\rightarrowtail} \{[] \, . \, \top\}$ belongs to the Grothendieck topology on $\mathscr{C}_{\mathbb{T}}$ generated over $J_{\mathbb{T}}$ by the principal sieve generated by the arrow $\{[] \, . \, \phi\} \overset{[\phi]}{\rightarrowtail} \{[] \, . \, \top\}$, then $[\phi] \leq [\psi]$ in $\mathrm{Sub}_{\mathscr{C}_{\mathbb{T}}}(\{[] \, . \, \top\})$.

Some applications

Theorem

The meet of the theory of local rings and that of integral domains in the lattice of quotients of the theory of commutative rings with unit is obtained from the latter theory by adding the sequents

$$(0 = 1 \vdash_{[]} \bot)$$

and

$$(\bigwedge_{1\leq s\leq m} P_s(\vec{x}) = 0 \vdash_{\vec{x}} \bigvee_{1\leq i\leq k} (\exists y)(G_i(\vec{x}) \cdot y = 1) \vee \bigvee_{1\leq j\leq l} H_j(\vec{x}) = 0)$$

where for each $1 \leq s \leq m$, $1 \leq i \leq k$ and $1 \leq j \leq l$ the P_s's, G_i's and H_j's are any polynomials in a finite string $\vec{x} = (x_1, \dots, x_n)$ of variables with the property that $\{P_1, \dots, P_s, G_1, \dots, G_k\}$ is a set of elements of $\mathbb{Z}[x_1, \dots, x_n]$ which is not contained in any proper ideal of $\mathbb{Z}[x_1, \dots, x_n]$ and $\prod_{1\leq j\leq l} H_j \in (P_1, \dots, P_s)$ in $\mathbb{Z}[x_1, \dots, x_n]$.

We have derived this result by calculating the meet of the Grothendieck topologies associated with the two quotients by using suitable bases for them.

Some applications

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ and $\mathbb{T}_1, \mathbb{T}_2$ two quotients of $\mathbb{T}$. Then the Heyting implication $\mathbb{T}_1 \Rightarrow \mathbb{T}_2$ in $\mathfrak{Th}_{\Sigma}^{\mathbb{T}}$ is the theory obtained from $\mathbb{T}$ by adding all the geometric sequents $(\psi \vdash_{\vec{y}} \psi')$ over Σ with the property that $(\psi' \vdash_{\vec{y}} \psi)$ is provable in $\mathbb{T}$ and for any $\mathbb{T}$-provably functional geometric formula $\theta(\vec{x}, \vec{y})$ from a geometric formula-in-context $\{\vec{x} \,.\, \phi\}$ to $\{\vec{y} \,.\, \psi\}$ and any geometric formula χ in the context $\vec{x}$ such that $(\chi \vdash_{\vec{x}} \phi)$ is provable in $\mathbb{T}$, the conjunction of the facts

1. $(\phi \vdash_{\vec{x}} \chi)$ *is provable in* $\mathbb{T}_1$
2. $((\exists \vec{y})(\theta(\vec{x}, \vec{y}) \wedge \psi'(\vec{y})) \vdash_{\vec{x}} \chi)$ *is provable in* $\mathbb{T}$

implies that $(\phi \vdash_{\vec{x}} \chi)$ is provable in $\mathbb{T}_2$.

Relevant references

- The papers *Topologies for intermediate logics* and *Site characterizations for geometric invariants of toposes* provide general recipes for automatically computing large classes of logical (resp. geometric) invariants of toposes.
- A machinery for automatically building Stone-type dualities has been built in the paper *A topos-theoretic approach to Stone-type dualities*.
- An infinite class of new dualities for MV-algebras has been generated in the paper *On the geometric theory of local MV-algebras* (joint with A.C. Russo) by using the 'bridge' technique.
- The book *Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'* contains the above-mentioned results on quotients and subtoposes.
- A forthcoming paper with Laurent Lafforgue treats the problem of generation of Grothendieck topologies from arbitrary families of sieves in a systematic way, deriving a number of new applications.
- The paper *Densenss conditions, morphisms and equivalences of toposes* provides several criteria that can be implemented to test whether a given functor between sites induces an equivalence between the corresponding toposes.