Introduction

Toposes as generalized spaces

Toposes as mathematical universes

Categorical logic

Classifying toposes

Toposes as bridges

Topological Galois theory

Theories of presheaf type

Topos-theoretic Fraïssé theorem

Quotients of theories of presheaf type

Future directions

An invitation to topos-theoretic model theory

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Topos-theoretic model theory

- We can call *topos-theretic model theory* the study of the syntax and semantics of (first-order) mathematical theories through the lenses of topos theory.

- Topos theory can be regarded as a unifying subject in Mathematics, with great relevance as a framework for systematically investigating the relationships between different mathematical theories and studying them by means of a multiplicity of different points of view. In particular, the 'bridge' technique allows one to generate insights which would be hardly attainable otherwise and to establish deep connections that allow effective transfers of knowledge between different contexts.

- As we shall see in the course, the topos-theoretic study of the semantics of first-order mathematical theories presents several advantages, notably including the greater generality, functorial nature, and constructiveness of the methods and the presence of classification results unavailable in the restricted setting of set theory.

Plan of the course

- Background on toposes and categorical semantics
- Classifying toposes and the ‘bridge’ technique
- Some examples of model-theoretic results proved through topos theory
- Future perspectives

The multifaceted nature of toposes

The role of toposes as unifying spaces is intimately tied to their multifaceted nature.

For instance, a topos can be seen as:

- a generalized space
- a mathematical universe
- a theory modulo 'Morita-equivalence'

We shall now briefly review each of these classical points of view, and then present the more recent theory of topos-theoretic 'bridges', which combines all of them to provide tools for making toposes effective means for studying mathematical theories from multiple points of view, relating and unifying theories with each other and constructing 'bridges' across them.

Presheaves on a topological space

Definition

Let X be a topological space. A presheaf $\mathscr{F}$ on X consists of the data:

(i) for every open subset U of X, a set $\mathscr{F}(U)$ and

(ii) for every inclusion $V \subseteq U$ of open subsets of X, a function $\rho_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ subject to the conditions

- $\rho_{U,U}$ is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$ and
- if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

The maps $\rho_{U,V}$ are called restriction maps, and we sometimes write $s|_V$ instead of $\rho_{U,V}(s)$, if $s \in \mathscr{F}(U)$.

A morphism of presheaves $\mathscr{F} \to \mathscr{G}$ on a topological space X is a collection of maps $\mathscr{F}(U) \to \mathscr{G}(U)$ which is compatible with respect to restriction maps.

Remark

*Categorically, a presheaf $\mathscr{F}$ on X is a functor $\mathscr{F} : \mathscr{O}(X)^{op} \to$ **Set**, where $\mathscr{O}(X)$ is the poset category corresponding to the lattice of open sets of the topological space X (with respect to the inclusion relation).*

A morphism of presheaves is then just a natural transformation between the corresponding functors.

So we have a category $[\mathscr{O}(X)^{op}, \mathbf{Set}]$ of presheaves on X.

Sheaves on a topological space

Definition

A sheaf $\mathscr{F}$ on a topological space X is a presheaf on X satisfying the additional conditions

(i) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if $s, t \in \mathscr{F}(U)$ are elements such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$;

(ii) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each $i, j \in I$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathscr{F}(U)$ (necessarily unique by (i)) such that $s|_{V_i} = s_i$ for each i.

A morphism of sheaves is defined as a morphism of the underlying presheaves.

Examples

- the sheaf of continuous real-valued functions on any topological space
- the sheaf of regular functions on a variety
- the sheaf of differentiable functions on a differentiable manifold
- the sheaf of holomorphic functions on a complex manifold

In each of the above examples, the restriction maps of the sheaf are the usual set-theoretic restrictions of functions to a subset.

Sheaves from a categorical point of view

Sheaves arising in Mathematics are often equipped with more structure than the mere set-theoretic one; for example, one may wish to consider sheaves of modules (resp. rings, abelian groups, ...) on a topological space X.

The natural categorical way of looking at these notions is to consider them as models of certain (geometric) theories in a category $\mathbf{Sh}(X)$ of sheaves of sets.

Remarks

- *Categorically, a sheaf is a functor $\mathscr{O}(X)^{op} \to \mathbf{Set}$ which satisfies certain conditions expressible in categorical language entirely in terms of the poset category $\mathscr{O}(X)$ and of the usual notion of covering on it. The category $\mathbf{Sh}(X)$ of sheaves on a topological space X is thus a full subcategory of the category $[\mathscr{O}(X)^{op}, \mathbf{Set}]$ of presheaves on X.*
- *Many important properties of topological spaces X can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.*

These remarks led Grothendieck to introduce a significant categorical generalization of the notion of sheaf, and hence the notion of Grothendieck topos.

Limits and colimits in $\mathbf{Sh}(X)$

Theorem

(i) *The category $\mathbf{Sh}(X)$ is closed in $[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ under arbitrary (small) limits.*

(ii) *The associated sheaf functor $a : [\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(X)$ (having a right adjoint) preserves all (small) colimits.*

- *Part (i) follows from the fact that limits commute with limits, in light of the characterization of sheaves in terms of limits.*
- *From part (ii) it follows that $\mathbf{Sh}(X)$ has all small colimits, which are computed by applying the associated sheaf functor to the colimit of the diagram considered with values in $[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$.*

Adjunctions induced by points

Let x be a point of a topological space X.

Definition

Let A be a set. Then the skyscraper sheaf $\mathrm{Sky}_x(A)$ of A at x is the sheaf on X defined as

- $\mathrm{Sky}_x(A)(U) = A$ if $x \in U$
- $\mathrm{Sky}_x(A)(U) = 1 = \{*\}$ if $x \notin U$

and in the obvious way on arrows.

The assignment $A \to \mathrm{Sky}_x(A)$ is clearly functorial.

Theorem

The stalk functor $\mathrm{Stalk}_x : \mathbf{Sh}(X) \to \mathbf{Set}$ *at* x *is left adjoint to the skyscraper functor* $\mathrm{Sky}_x : \mathbf{Set} \to \mathbf{Sh}(X)$.

In fact, as we shall see later in the course, points in topos theory are defined as suitable kinds of functors (more precisely, colimit and finite-limit preserving ones).

Open sets as subterminal objects

Since limits in a category $\mathbf{Sh}(X)$ are computed as in the category of presheaves $[\mathscr{O}(X)^{\text{op}}, \mathbf{Set}]$, a subobject of a sheaf F in $\mathbf{Sh}(X)$ is just a subsheaf, that is a subfunctor which is a sheaf.
Notice that a subfunctor $S \subseteq F$ is a sheaf if and only if for every open covering $\{U_i \subseteq U \mid i \in I\}$ and every element $x \in F(U)$, $x \in S(U)$ if and only if $x|_{U_i} \in S(U_i)$.

Definition

In a category with a terminal object, a subterminal object is an object whose unique arrow to the terminal object is a monomorphism.

Theorem

Let X be a topological space. Then we have a frame isomorphism

$$\mathrm{Sub}_{\mathbf{Sh}(X)}(1) \cong \mathscr{O}(X) .$$

between the subterminal objects of $\mathbf{Sh}(X)$ and the open sets of X.

Sieves

In order to 'categorify' the notion of sheaf of a topological space, the first step is to introduce an abstract notion of covering (of an object by a family of arrows to it) in a category.

Definition

- Given a category $\mathscr{C}$ and an object $c \in \mathrm{Ob}(\mathscr{C})$, a presieve P in $\mathscr{C}$ on c is a collection of arrows in $\mathscr{C}$ with codomain c.
- Given a category $\mathscr{C}$ and an object $c \in \mathrm{Ob}(\mathscr{C})$, a sieve S in $\mathscr{C}$ on c is a collection of arrows in $\mathscr{C}$ with codomain c such that

  $$f \in S \Rightarrow f \circ g \in S$$

  whenever this composition makes sense.
- We say that a sieve S is generated by a presieve P on an object c if it is the smallest sieve containing it, that is if it is the collection of arrows to c which factor through an arrow in P.

If S is a sieve on c and $h : d \to c$ is any arrow to c, then

$$h^*(S) := \{g \mid \mathrm{cod}(g) = d,\ h \circ g \in S\}$$

is a sieve on d.

Grothendieck topologies

Definition

- A Grothendieck topology on a category $\mathscr{C}$ is a function J which assigns to each object c of $\mathscr{C}$ a collection $J(c)$ of sieves on c in such a way that
  - (i) (maximality axiom) the maximal sieve $M_c = \{f \mid \mathrm{cod}(f) = c\}$ is in $J(c)$;
  - (ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;
  - (iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

  The sieves S which belong to $J(c)$ for some object c of $\mathscr{C}$ are said to be J-covering.
- A site is a pair $(\mathscr{C}, J)$ where $\mathscr{C}$ is a small category and J is a Grothendieck topology on $\mathscr{C}$.

Examples of Grothendieck topologies

- For any (small) category $\mathscr{C}$, the trivial topology on $\mathscr{C}$ is the Grothendieck topology in which the only sieve covering an object c is the maximal sieve M_c.
- The dense topology D on a category $\mathscr{C}$ is defined by: for a sieve S,

$$S \in D(c) \quad \text{if and only if} \quad \text{for any } f : d \to c \text{ there exists } g : e \to d \text{ such that } f \circ g \in S\,.$$

If $\mathscr{C}$ satisfies the right Ore condition i.e. the property that any two arrows $f : d \to c$ and $g : e \to c$ with a common codomain c can be completed to a commutative square

$$\begin{tikzcd} \bullet \arrow[r, dashed] \arrow[d, dashed] & d \arrow[d, "f"] \\ e \arrow[r, "g"] & c \end{tikzcd}$$

then the dense topology on $\mathscr{C}$ specializes to the atomic topology on $\mathscr{C}$ i.e. the topology J_{at} defined by: for a sieve S,

$$S \in J_{at}(c) \text{ if and only if } S \neq \emptyset\,.$$

Examples of Grothendieck topologies

- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathscr{O}(X)}$ on the poset category $\mathscr{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathscr{O}(X))$,

$$S \in J_{\mathscr{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U\,.$$

- More generally, given a frame (or complete Heyting algebra) H, we can define a Grothendieck topology J_H, called the *canonical topology on H*, by:

$$\{a_i \mid i \in I\} \in J_H(a) \text{ if and only if } \bigvee_{i \in I} a_i = a\,.$$

- Given a small category of topological spaces which is closed under finite limits and under taking open subspaces, one may define the open cover topology on it by specifying as basis the collection of open embeddings $\{Y_i \hookrightarrow X \mid i \in I\}$ such that $\bigcup_{i \in I} Y_i = X$.
- The Zariski topology on the opposite of the category $\mathbf{Rng}_{\text{f.g.}}$ of finitely generated commutative rings with unit is defined by: for any cosieve S in $\mathbf{Rng}_{\text{f.g.}}$ on an object A, $S \in Z(A)$ if and only if S contains a finite family $\{\xi_i : A \to A_{f_i} \mid 1 \leq i \leq n\}$ of canonical maps $\xi_i : A \to A_{f_i}$ in $\mathbf{Rng}_{\text{f.g.}}$ where $\{f_1, \ldots, f_n\}$ is a set of elements of A which is not contained in any proper ideal of A.

Sheaves on a site

Definition

- A presheaf on a (small) category $\mathscr{C}$ is a functor $P : \mathscr{C}^{\text{op}} \to \mathbf{Set}$.
- Let $P : \mathscr{C}^{\text{op}} \to \mathbf{Set}$ be a presheaf on $\mathscr{C}$ and S be a sieve on an object c of $\mathscr{C}$.

  A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

  $$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d\,.$$

  An amalgamation for such a family is a single element $x \in P(c)$ such that

  $$P(f)(x) = x_f \quad \text{for all } f \text{ in } S\,.$$

Remark

For any covering family $F = \{U_i \subseteq U \mid i \in I\}$ in a topological space X and any presheaf $\mathscr{F}$ on X, giving a family of elements $s_i \in \mathscr{F}(U_i)$ such that for any $i, j \in I$ $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ is equivalent to giving a family of elements $\{s_W \in \mathscr{F}(W) \mid W \in S_F\}$ such that for any open set $W' \subseteq W$, $s_W|_{W'} = s_{W'}$, where S_F is the sieve generated by F. In other words, it is the same as giving a matching family for S_F of elements of $\mathscr{F}$.

Sheaves on a site

- Given a site $(\mathscr{C}, J)$, a presheaf on $\mathscr{C}$ is a J-sheaf if every matching family for any J-covering sieve on any object of $\mathscr{C}$ has a unique amalgamation.
- The category $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on the site $(\mathscr{C}, J)$ is the full subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.
- A Grothendieck topos is any category equivalent to the category of sheaves on a site.

Examples of toposes

The following examples show that toposes can be naturally attached to mathematical notions as different as (small) categories, topological spaces, or groups.

Examples

- For any (small) category $\mathscr{C}$, $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ is the category of sheaves $\mathbf{Sh}(\mathscr{C}, T)$ where T is the trivial topology on $\mathscr{C}$.
- For any topological space X, $\mathbf{Sh}(\mathscr{O}(X), J_{\mathscr{O}(X)})$ is equivalent to the usual category $\mathbf{Sh}(X)$ of sheaves on X.
- For any (topological) group G, the category $BG = \mathbf{Cont}(G)$ of continuous actions of G on discrete sets is a Grothendieck topos (equivalent, as we shall see, to the category $\mathbf{Sh}(\mathbf{Cont}_t(G), J_{at})$ of sheaves on the full subcategory $\mathbf{Cont}_t(G)$ on the non-empty transitive actions with respect to the atomic topology).

Basic properties of Grothendieck toposes

Grothendieck toposes satisfy all the categorical properties that one might hope for:

Theorem

Let $(\mathscr{C}, J)$ be a site. Then

- *the inclusion* $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ *has a left adjoint* $a : [\mathscr{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ *(called the associated sheaf functor), which preserves finite limits.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has all (small) limits, which are preserved by the inclusion functor* $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\text{op}}, \mathbf{Set}]$*; in particular, limits are computed pointwise and the terminal object* $1_{\mathbf{Sh}(\mathscr{C}, J)}$ *of* $\mathbf{Sh}(\mathscr{C}, J)$ *is the functor* $T : \mathscr{C}^{\text{op}} \to \mathbf{Set}$ *sending each object* $c \in Ob(\mathscr{C})$ *to the singleton* $\{*\}$*.*
- *The associated sheaf functor* $a : [\mathscr{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ *preserves colimits; in particular,* $\mathbf{Sh}(\mathscr{C}, J)$ *has all (small) colimits.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has exponentials, which are constructed as in the topos* $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$*.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has a subobject classifier.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has a separating set of objects.*

Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism. The natural notion of morphism of geometric morphisms if that of geometric transformation.

Definition

(i) Let $\mathscr{E}$ and $\mathscr{F}$ be toposes. A geometric morphism $f : \mathscr{E} \to \mathscr{F}$ consists of a pair of functors $f_* : \mathscr{E} \to \mathscr{F}$ (the direct image of f) and $f^* : \mathscr{F} \to \mathscr{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathscr{E} \to \mathscr{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

(iii) A point of a topos $\mathscr{E}$ is a geometric morphism $\mathbf{Set} \to \mathscr{E}$.

- Grothendieck toposes and geometric morphisms between them form a 2-category.
- Given two toposes $\mathscr{E}$ and $\mathscr{F}$, geometric morphisms from $\mathscr{E}$ to $\mathscr{F}$ and geometric transformations between them form a category, denoted by $\mathbf{Geom}(\mathscr{E}, \mathscr{F})$.

Examples of geometric morphisms

- A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The direct image $\mathbf{Sh}(f)_*$ sends a sheaf $F \in Ob(\mathbf{Sh}(X))$ to the sheaf $\mathbf{Sh}(f)_*(F)$ defined by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for any open subset V of Y. The inverse image $\mathbf{Sh}(f)^*$ acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.
- Every Grothendieck topos $\mathscr{E}$ has a unique geometric morphism $\mathscr{E} \to \mathbf{Set}$. The direct image is the global sections functor $\Gamma : \mathscr{E} \to \mathbf{Set}$, sending an object $e \in \mathscr{E}$ to the set $\mathrm{Hom}_{\mathscr{E}}(1_{\mathscr{E}}, e)$, while the inverse image functor $\Delta : \mathbf{Set} \to \mathscr{E}$ sends a set S to the coproduct $\coprod_{s \in S} 1_{\mathscr{E}}$.
- For any site $(\mathscr{C}, J)$, the pair of functors formed by the inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ and the associated sheaf functor $a : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ yields a geometric morphism $i : \mathbf{Sh}(\mathscr{C}, J) \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.

Geometric morphisms as flat functors I

Theorem

Let $\mathscr{C}$ be a small category and $\mathscr{E}$ be a locally small cocomplete category. Then, for any functor $A : \mathscr{C} \to \mathscr{E}$ the functor $R_A : \mathscr{E} \to [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ defined for each $e \in Ob(\mathscr{E})$ and $c \in Ob(\mathscr{C})$ by:

$$R_A(e)(c) = \text{Hom}_{\mathscr{E}}(A(c), e)$$

has a left adjoint $- \otimes_{\mathscr{C}} A : [\mathscr{C}^{\text{op}}, \mathbf{Set}] \to \mathscr{E}$.

Definition

- A functor $A : \mathscr{C} \to \mathscr{E}$ from a small category $\mathscr{C}$ to a Grothendieck topos $\mathscr{E}$ is said to be flat if the functor $- \otimes_{\mathscr{C}} A : [\mathscr{C}^{\text{op}}, \mathbf{Set}] \to \mathscr{E}$ preserves finite limits.
- The full subcategory of $[\mathscr{C}, \mathscr{E}]$ on the flat functors will be denoted by $\mathbf{Flat}(\mathscr{C}, \mathscr{E})$.

Geometric morphisms as flat functors II

Theorem

Let $\mathscr{C}$ be a small category and $\mathscr{E}$ be a Grothendieck topos. Then we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]) \simeq \mathbf{Flat}(\mathscr{C}, \mathscr{E})$$

(natural in $\mathscr{E}$), which sends

- *a flat functor $A : \mathscr{C} \to \mathscr{E}$ to the geometric morphism $\mathscr{E} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ determined by the functors R_A and $- \otimes_{\mathscr{C}} A$, and*
- *a geometric morphism $f : \mathscr{E} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ to the flat functor given by the composite $f^* \circ y$ of $f^* : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathscr{E}$ with the Yoneda embedding $y : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.*

Geometric morphisms to $\mathbf{Sh}(\mathscr{C}, J)$ I

Definition

Let $\mathscr{E}$ be a Grothendieck topos.

- A family $\{f_i : a_i \to a \mid i \in I\}$ of arrows in $\mathscr{E}$ with common codomain is said to be epimorphic if for any pair of arrows $g, h : a \to b$ with domain a, $g = h$ if and only if $g \circ f_i = h \circ f_i$ for all $i \in I$.
- If $(\mathscr{C}, J)$ is a site, a functor $F : \mathscr{C} \to \mathscr{E}$ is said to be J-continuous if it sends J-covering sieves to epimorphic families.

The full subcategory of $\mathbf{Flat}(\mathscr{C}, \mathscr{E})$ on the J-continuous flat functors will be denoted by $\mathbf{Flat}_J(\mathscr{C}, \mathscr{E})$.

Geometric morphisms to $\mathbf{Sh}(\mathscr{C}, J)$ II

Theorem

For any site $(\mathscr{C}, J)$ and Grothendieck topos $\mathscr{E}$, the above-mentioned equivalence between geometric morphisms and flat functors restricts to an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathbf{Sh}(\mathscr{C}, J)) \simeq \mathbf{Flat}_J(\mathscr{C}, \mathscr{E})$$

natural in $\mathscr{E}$.

Sketch of proof.

Appeal to the previous theorem

- identifying the geometric morphisms $\mathscr{E} \to \mathbf{Sh}(\mathscr{C}, J)$ with the geometric morphisms $\mathscr{E} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ which factor through the canonical geometric inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, and
- using the characterization of such morphisms as the geometric morphisms $f : \mathscr{E} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ such that the composite $f^* \circ y$ of the inverse image functor f^* of f with the Yoneda embedding $y : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ sends J-covering sieves to colimits in $\mathscr{E}$ (equivalently, to epimorphic families in $\mathscr{E}$).

□

Flat = filtering

Definition

A functor $F : \mathscr{C} \to \mathscr{E}$ from a small category $\mathscr{C}$ to a Grothendieck topos $\mathscr{E}$ is said to be filtering if it satisfies the following conditions:

(a) For any object E of $\mathscr{E}$ there exist an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathscr{E}$ and for each $i \in I$ an object b_i of $\mathscr{C}$ and a generalized element $E_i \to F(b_i)$ in $\mathscr{E}$.

(b) For any two objects c and d in $\mathscr{C}$ and any generalized element $\langle x, y\rangle : E \to F(c) \times F(d)$ in $\mathscr{E}$ there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathscr{E}$ and for each $i \in I$ an object b_i of $\mathscr{C}$ with arrows $u_i : b_i \to c$ and $v_i : b_i \to d$ in $\mathscr{C}$ and a generalized element $z_i : E_i \to F(b_i)$ in $\mathscr{E}$ such that $\langle F(u_i), F(v_i)\rangle \circ z_i = \langle x, y\rangle \circ e_i$ for all $i \in I$.

(c) For any two parallel arrows $u, v : d \to c$ in $\mathscr{C}$ and any generalized element $x : E \to F(d)$ in $\mathscr{E}$ for which $F(u) \circ x = F(v) \circ x$, there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathscr{E}$ and for each $i \in I$ an arrow $w_i : b_i \to d$ and a generalized element $y_i : E_i \to F(b_i)$ such that $u \circ w_i = v \circ w_i$ and $F(w_i) \circ y_i = x \circ e_i$ for all $i \in I$.

Theorem

A functor $F : \mathscr{C} \to \mathscr{E}$ from a small category $\mathscr{C}$ to a Grothendieck topos $\mathscr{E}$ is flat if and only if it is filtering.

Remarks

- *For any small category $\mathscr{C}$, a functor $P : \mathscr{C} \to$ **Set** is filtering if and only if its category of elements $\int P$ is a filtered category (equivalently, if it is a filtered colimit of representables).*
- *For any small cartesian category $\mathscr{C}$, a functor $\mathscr{C} \to \mathscr{E}$ is flat if and only if it preserves finite limits.*

Morphisms and comorphisms of sites

Geometric morphisms can be naturally induced by functors between sites satisfying appropriate properties:

Definition

- A morphism of sites $(\mathscr{C}, J) \to (\mathscr{D}, K)$ is a functor $F : \mathscr{C} \to \mathscr{D}$ such that the composite $l' \circ F$, where l' is the canonical functor $\mathscr{D} \to \mathbf{Sh}(\mathscr{D}, K)$, is flat and J-continuous. If $\mathscr{C}$ and $\mathscr{D}$ have finite limits then F is a morphism of sites if and only if it preserves finite limits and is cover-preserving.
- A comorphism of sites $(\mathscr{D}, K) \to (\mathscr{C}, J)$ is a functor $\pi : \mathscr{D} \to \mathscr{C}$ which is cover-reflecting (in the sense that for any $d \in \mathscr{D}$ and any J-covering sieve S on $\pi(d)$ there is a K-covering sieve R on d such that $\pi(R) \subseteq S$).

Theorem

- *Every morphism of sites* $(\mathscr{C}, J) \to (\mathscr{D}, K)$ *induces a geometric morphism* $\mathbf{Sh}(\mathscr{D}, K) \to \mathbf{Sh}(\mathscr{C}, J)$.
- *Every comorphism of sites* $(\mathscr{D}, K) \to (\mathscr{C}, J)$ *induces a geometric morphism* $\mathbf{Sh}(\mathscr{D}, K) \to \mathbf{Sh}(\mathscr{C}, J)$.

Toposes as mathematical universes

W. Lawvere and M. Tierney discovered that a topos could not only be seen as a generalized space, but also as a mathematical universe in which one can do mathematics similarly to how one does it in the classical context of sets, with the only important exception that one must argue constructively. In fact, the internal logic of a topos, captured to a great extent by its subobject classifier, is in general intuitionistic.

Amongst other things, this view of toposes as mathematical universes paved the way for:

- Exploiting the inherent 'flexibility' of the notion of topos to construct 'new mathematical worlds' having particular properties.
- Considering models of any kind of (first-order) mathematical theory not just in the classical set-theoretic setting, but inside every topos, and hence 'relativise' Mathematics.

Subobjects in a Grothendieck topos

Since limits in a topos $\mathbf{Sh}(\mathscr{C}, J)$ are computed as in the presheaf topos $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, a subobject of a sheaf F in $\mathbf{Sh}(\mathscr{C}, J)$ is just a subsheaf, that is a subfunctor which is a sheaf.

Notice that a subfunctor $F' \subseteq F$ is a sheaf if and only if for every J-covering sieve S and every element $x \in F(c)$, $x \in F'(c)$ if and only if $F(f)(x) \in F'(\mathrm{dom}(f))$ for every $f \in S$.

Theorem

- *For any Grothendieck topos $\mathscr{E}$ and any object a of $\mathscr{E}$, the poset $\mathrm{Sub}_{\mathscr{E}}(a)$ of all subobjects of a in $\mathscr{E}$ is a complete Heyting algebra.*
- *For any arrow $f : a \to b$ in a Grothendieck topos $\mathscr{E}$, the pullback functor $f^* : \mathrm{Sub}_{\mathscr{E}}(b) \to \mathrm{Sub}_{\mathscr{E}}(a)$ has both a left adjoint $\exists_f : \mathrm{Sub}_{\mathscr{E}}(a) \to \mathrm{Sub}_{\mathscr{E}}(b)$ and a right adjoint $\forall_f : \mathrm{Sub}_{\mathscr{E}}(a) \to \mathrm{Sub}_{\mathscr{E}}(b)$.*

The Heyting operations on subobjects

Proposition

The collection $\mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(E)$ of subobjects of an object E in $\mathbf{Sh}(\mathscr{C},J)$ has the structure of a complete Heyting algebra with respect to the natural ordering $A \leq B$ if and only if for every $c \in \mathscr{C}$, $A(c) \subseteq B(c)$. We have that

- $(A \wedge B)(c) = A(c) \cap B(c)$ *for any* $c \in \mathscr{C}$;
- $(A \vee B)(c) = \{x \in E(c) \mid \{f : d \to c \mid E(f)(x) \in A(d) \cup B(d)\} \in J(c)\}$ *for any* $c \in \mathscr{C}$;

  (the infinitary analogue of this holds)
- $(A \Rightarrow B)(c) = \{x \in E(c) \mid \text{ for every } f : d \to c, E(f)(x) \in A(d) \text{ implies } E(f)(x) \in B(d)\}$ *for any* $c \in \mathscr{C}$.
- *the bottom subobject* $0 \rightarrowtail E$ *is given by the embedding into* E *of the initial object* 0 *of* $\mathbf{Sh}(\mathscr{C},J)$ *(given by:* $0(c) = \emptyset$ *if* $\emptyset \notin J(c)$ *and* $0(c) = \{*\}$ *if* $\emptyset \in J(c)$*);*
- *the top subobject is the identity one.*

Remark

From the Yoneda Lemma it follows that the subobject classifier Ω in $\mathbf{Sh}(\mathscr{C},J)$ has the structure of an internal Heyting algebra in $\mathbf{Sh}(\mathscr{C},J)$.

The interpretation of quantifiers

Let $\phi : E \to F$ be a morphism in $\mathbf{Sh}(\mathscr{C}, J)$.

- The pullback functor

$$\phi^* : \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(F) \to \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(E)$$

  is given by: $\phi^*(B)(c) = \phi(c)^{-1}(B(c))$ for any subobject $B \rightarrowtail F$ and any $c \in \mathscr{C}$.

- The left adjoint

$$\exists_\phi : \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(E) \to \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(F)$$

  is given by: $\exists_\phi(A)(c) = \{y \in E(c) \mid \{f : d \to c \mid (\exists a \in A(d))(\phi(d)(a) = E(f)(y))\} \in J(c)\}$
  for any subobject $A \rightarrowtail E$ and any $c \in \mathscr{C}$.

- The right adjoint

$$\forall_\phi : \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(E) \to \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(F)$$

  is given by $\forall_\phi(A)(c) = \{y \in E(c) \mid \text{ for all } f : d \to c, \phi(d)^{-1}(E(f)(y)) \subseteq A(d)\}$
  for any subobject $A \rightarrowtail E$ and any $c \in \mathscr{C}$.

Interpreting first-order logic in categories

- In Logic, first-order languages are a wide class of formal languages used for talking about mathematical structures of any kind (where the restriction 'first-order' means that quantification is allowed only over individuals rather than over collections of individuals or higher-order constructions on them).
- A first-order language contains sorts, which are meant to represent different *kinds* of individuals, terms, which denote individuals, and formulae, which make assertions about the individuals. Compound terms and formulae are formed by using various logical operators.
- It is well-known that first-order languages can always be interpreted in the context of (a given model of) set theory. In this lecture, we will show that these languages can also be meaningfully interpreted in a category, provided that the latter possesses enough categorical structure to allow the interpretation of the given fragment of logic. In fact, sorts will be interpreted as objects, terms as arrows and formulae as subobjects, in a way that respects the logical structure of compound expressions.

Signatures

Definition

A first-order signature Σ consists of the following data.

a) A set Σ-Sort of sorts.

b) A set Σ-Fun of function symbols, together with a map assigning to each $f \in \Sigma$-Fun its *type*, which consists of a finite non-empty list of sorts: we write

$$f : A_1 \cdots A_n \to B$$

to indicate that f has type $A_1, \ldots, A_n, B$ (if $n = 0$, f is called a constant of sort B).

c) A set Σ-Rel of relation symbols, together with a map assigning to each Σ-Rel its *type*, which consists of a finite list of sorts: we write

$$R \rightarrowtail A_1 \cdots A_n$$

to indicate that R has type $A_1, \ldots A_n$.

Terms

For each sort A of a signature Σ we assume given a supply of variables of sort A, used to denote individuals of kind A.
Starting from variables, terms are built-up by repeated 'applications' of function symbols to them, as follows.

Definition

Let Σ be a signature. The collection of terms over Σ is defined recursively by the clauses below; simultaneously, we define the sort of each term and write $t : A$ to denote that t is a term of sort A.

a) $x : A$, if x is a variable of sort A.
b) $f(t_1, \ldots, t_n) : B$ if $f : A_1 \cdots A_n \to B$ is a function symbol and $t_1 : A_1, \ldots, t_n : A_n$.

Formation rules for formulae I

Consider the following formation rules for recursively building classes of formulae F over Σ, together with, for each formula ϕ, the (finite) set $\mathrm{FV}(\phi)$ of free variables of ϕ.

(i) Relations: $R(t_1, \ldots, t_n)$ is in F, if $R \rightarrowtail A_1 \cdots A_n$ is a relation symbol and $t_1 : A_1, \ldots, t_n : A_n$ are terms; the free variables of this formula are all the variables occurring in some t_i.

(ii) Equality: $(s = t)$ is in F if s and t are terms of the same sort; $\mathrm{FV}(s = t)$ is the set of variables occurring in s or t (or both).

(iii) Truth: $\top$ is in F; $\mathrm{FV}(\top) = \emptyset$.

(iv) Binary conjunction: $(\phi \wedge \psi)$ is in F, if ϕ and ψ are in F; $\mathrm{FV}(\phi \wedge \psi) = \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi)$.

(v) Falsity: $\bot$ is in F; $\mathrm{FV}(\bot) = \emptyset$.

(vi) Binary disjunction: $(\phi \vee \psi)$ is in F, if ϕ and ψ are in F; $\mathrm{FV}(\phi \vee \psi) = \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi)$.

(vii) Implication: $(\phi \Rightarrow \psi)$ is in F, if ϕ and ψ are in F; $\mathrm{FV}(\phi \Rightarrow \psi) = \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi)$.

(viii) Negation: $\neg\phi$ is in F, if ϕ is in F; $\mathrm{FV}(\neg\phi) = \mathrm{FV}(\phi)$.

Formation rules for formulae II

(ix) Existential quantification: $(\exists x)\phi$ is in F, if ϕ is in F and x is a variable; $\mathrm{FV}((\exists x)\phi) = \mathrm{FV}(\phi) \setminus \{x\}$.

(x) Universal quantification: $(\forall x)\phi$ is in F, if ϕ is in F and x is a variable; $\mathrm{FV}((\forall x)\phi) = \mathrm{FV}(\phi) \setminus \{x\}$.

(xi) Infinitary disjunction: $\bigvee\limits_{i \in I} \phi_i$ is in F, if I is a set, ϕ_i is in F for each $i \in I$ and $\mathrm{FV}(\bigvee\limits_{i \in I} \phi_i) := \bigcup\limits_{i \in I} \mathrm{FV}(\phi_i)$ is finite.

(xii) Infinitary conjunction: $\bigwedge\limits_{i \in I} \phi_i$ is in F, if I is a set, ϕ_i is in F for each $i \in I$ and $\mathrm{FV}(\bigwedge\limits_{i \in I} \phi_i) := \bigcup\limits_{i \in I} \mathrm{FV}(\phi_i)$ is finite.

A context is a finite list $\vec{x} = x_1, \ldots, x_n$ of distinct variables (the empty context, for $n = 0$ is allowed and indicated by []).

Notation: We will often consider formulae-in-context, that is formulae ϕ equipped with a context $\vec{x}$ such that all the free variables of ϕ occur among $\vec{x}$; we will write either $\phi(\vec{x})$ or $\{\vec{x} . \phi\}$.

Classes of formulae

Definition

In relation to the above-mentioned forming rules:

- The set of atomic formulae over Σ is the smallest set closed under *Relations* and *Equality*).
- The set of Horn formulae over Σ is the smallest set containing the class of atomic formulae and closed under *Truth* and *Binary conjunction*.
- The set of regular formulae over Σ is the smallest set containing the class of atomic formulae and closed under *Truth*, *Binary conjunction* and *Existential quantification*.
- The set of coherent formulae over Σ is the smallest set containing the set of regular formulae and closed under *False* and *Binary disjunction*.
- The set of first-order formulae over Σ is the smallest set closed under all the forming rules except for the infinitary ones.
- The *class* of geometric formulae over Σ is the smallest class containing the class of coherent formulae and closed under *Infinitary disjunction*.
- The *class* of infinitary first-order formulae over Σ is the smallest class closed under all the above-mentioned forming rules.

Sequents

Definition

- By a sequent over a signature Σ we mean a formal expression of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are formulae over Σ and $\vec{x}$ is a context suitable for both of them. The intended interpretation of this expression is that ψ is a logical consequence of ϕ in the context $\vec{x}$, i.e. that any assignment of individual values to the variables in $\vec{x}$ which makes ϕ true will also make ψ true.
- We say a sequent $(\phi \vdash_{\vec{x}} \psi)$ is Horn (resp. regular, coherent, ...) if both ϕ and ψ are Horn (resp. regular, coherent, ...) formulae.

Notice that, in full first-order logic, the general notion of sequent is not really needed, since the sequent $(\phi \vdash_{\vec{x}} \psi)$ expresses the same idea as $(\top \vdash_{[]} (\forall \vec{x})(\phi \Rightarrow \psi))$.

First-order theories

Definition

- By a theory over a signature Σ, we mean a set $\mathbb{T}$ of sequents over Σ, whose elements are called the (non-logical) axioms of $\mathbb{T}$.
- We say that $\mathbb{T}$ is an algebraic theory if its signature Σ has a single sort and no relation symbols (apart from equality) and its axioms are all of the form $\top \vdash_{\vec{x}} \phi$ where ϕ is an atomic formula $(s = t)$ and $\vec{x}$ its canonical context.
- We say $\mathbb{T}$ is a Horn (resp. regular, coherent, ...) theory if all the sequents in $\mathbb{T}$ are Horn (resp. regular, coherent, ...).

Deduction systems for first-order logic I

- To each of the fragments of first-order logic introduced above, we can naturally associate a deduction system, in the same spirit as in classical first-order logic. Such systems will be formulated as *sequent-calculi*, that is they will consist of inference rules enabling us to derive a sequent from a collection of others; we will write

$$\frac{\Gamma}{\sigma}$$

  to mean that the sequent σ can be inferred by a collection of sequents Γ. A double line instead of the single line will mean that each of the sequents can be inferred from the other.
- Given the axioms and inference rules below, the notion of proof is the usual one, and allowing the axioms of theory $\mathbb{T}$ to be taken as premises yields the notion of proof relative to a theory $\mathbb{T}$.

Consider the following rules.

Deduction systems for first-order logic II

- The rules for finite conjunction are the axioms

$$(\phi \vdash_{\vec{x}} \top) \quad ((\phi \wedge \psi) \vdash_{\vec{x}} \phi) \quad ((\phi \wedge \psi) \vdash_{\vec{x}} \psi)$$

  and the rule

$$\frac{(\phi \vdash_{\vec{x}} \psi)(\phi \vdash_{\vec{x}} \chi)}{(\phi \vdash_{\vec{x}} (\psi \wedge \chi))}$$

- The rules for finite disjunction are the axioms

$$(\bot \vdash_{\vec{x}} \phi) \quad (\phi \vdash_{\vec{x}} (\phi \vee \psi)) \quad (\psi \vdash_{\vec{x}} \phi \vee \psi)$$

  and the rule

$$\frac{(\phi \vdash_{\vec{x}} \chi)(\psi \vdash_{\vec{x}} \chi)}{((\phi \vee \psi) \vdash_{\vec{x}} \chi)}$$

- The rules for infinitary conjunction (resp. disjunction) are the infinitary analogues of the rules for finite conjunction (resp. disjunction).

Deduction systems for first-order logic III

- The rules for implication consist of the double rule

$$\frac{(\phi \wedge \psi \vdash_{\vec{x}} \chi)}{\overline{(\psi \vdash_{\vec{x}} (\phi \Rightarrow \chi))}}$$

- The rules for existential quantification consist of the double rule

$$\frac{(\phi \vdash_{\vec{x},y} \psi)}{\overline{((\exists y)\phi \vdash_{\vec{x}} \psi)}}$$

  provided that y is not free in ψ.
- The rules for universal quantification consist of the double rule

$$\frac{(\phi \vdash_{\vec{x},y} \psi)}{\overline{(\phi \vdash_{\vec{x}} (\forall y)\psi)}}$$

- The distributive axiom is

$$((\phi \wedge (\psi \vee \chi)) \vdash_{\vec{x}} ((\phi \wedge \psi) \vee (\phi \wedge \chi)))$$

Deduction systems for first-order logic IV

- The Frobenius axiom is

$$((\phi \wedge (\exists y)\psi) \vdash_{\vec{x}} (\exists y)(\phi \wedge \psi))$$

where y is a variable not in the context $\vec{x}$.

- The Law of excluded middle is

$$(\top \vdash_{\vec{x}} \phi \vee \neg\phi)$$

Fragments of first-order logic

Definition

In addition to the usual structural rules of *sequent-calculi* (Identity axiom, Equality rules, Substitution rule, and Cut rule), our deduction systems consist of the following rules:

Horn logic	finite conjunction
Regular logic	finite conjunction, existential quantification and Frobenius axiom
Coherent logic	finite conjunction, finite disjunction, existential quantification, distributive axiom and Frobenius axiom
Geometric logic	finite conjunction, infinitary disjunction, existential quantification, 'infinitary' distributive axiom, Frobenius axiom
Intuitionistic first-order logic	all the finitary rules except for the law of excluded middle
Classical first-order logic	all the finitary rules

Provability in fragments of first-order logic

Definition

We say a sequent σ is provable in an algebraic (regular, coherent, ...) theory $\mathbb{T}$ if there exists a derivation of σ relative to $\mathbb{T}$, in the appropriate fragment of first-order logic.

In geometric logic, intuitionistic and classical provability of geometric sequents coincide.

Theorem

If a geometric sequent σ is derivable from the axioms of a geometric theory $\mathbb{T}$ using 'classical geometric logic' (i.e. the rules of geometric logic plus the Law of Excluded Middle), then there is also a constructive derivation of σ, not using the Law of Excluded Middle.

Categorical semantics

- Generalizing the classical Tarskian definition of satisfaction of first-order formulae in ordinary set-valued structures, one can obtain, given a signature Σ, a notion of Σ-structure in a category with finite products, and define, according to the categorical structure present on the category, a notion of interpretation of an appropriate fragment of first-order logic in it.
- Specifically, we will introduce various classes of 'logical' categories, each of them providing a semantics for a corresponding fragment of first-order logic:

Cartesian categories	Horn logic
Regular categories	Regular logic
Coherent categories	Coherent logic
Geometric categories	Geometric logic
Heyting categories	First-order logic

Structures in categories

Definition

Let $\mathscr{C}$ be a category with finite products and Σ be a signature. A Σ-structure M in $\mathscr{C}$ is specified by the following data:

(i) A function assigning to each sort A in Σ-Sort, an object MA of $\mathscr{C}$. For finite strings of sorts, we define $M(A_1, \dots, A_n) = MA_1 \times \cdots \times MA_n$ and set $M([])$ equal to the terminal object 1 of $\mathscr{C}$.

(ii) A function assigning to each function symbol $f : A_1 \cdots A_n \to B$ in Σ-Fun an arrow $Mf : M(A_1, \dots, A_n) \to MB$ in $\mathscr{C}$.

(iii) A function assigning to each relation symbol $R \rightarrowtail A_1 \cdots A_n$ in Σ-Rel a subobject $MR \rightarrowtail M(A_1, \dots, A_n)$ in $\mathscr{C}$.

Introduction

Toposes as generalized spaces

Toposes as mathematical universes

Categorical logic

Classifying toposes

Toposes as bridges

Topological Galois theory

Theories of presheaf type

Topos-theoretic Fraïssé theorem

Quotients of theories of presheaf type

Future directions

Homomorphisms of structures

Definition

A Σ-structure homomorphism $h : M \to N$ between two Σ-structures M and N in $\mathscr{C}$ is a collection of arrows $h_A : MA \to NA$ in $\mathscr{C}$ indexed by the sorts of Σ and satisfying the following two conditions:

(i) For each function symbol $f : A_1 \cdots A_n \to B$ in Σ-Fun, the diagram

$$\begin{tikzcd} M(A_1, \ldots, A_n) \arrow[r, "Mf"] \arrow[d, "h_{A_1} \times \cdots \times h_{A_n}"] & MB \arrow[d, "h_B"] \\ N(A_1, \ldots, A_n) \arrow[r, "Nf"'] & NB \end{tikzcd}$$

commutes.

(ii) For each relation symbol $R \rightarrowtail A_1 \cdots A_n$ in Σ-Rel, there is a commutative diagram in $\mathscr{C}$ of the form

$$\begin{tikzcd} MR \arrow[r] \arrow[d] & M(A_1, \ldots, A_n) \arrow[d, "h_{A_1} \times \cdots \times h_{A_n}"] \\ NR \arrow[r] & M(A_1, \ldots, A_n) \end{tikzcd}$$

The category of Σ-structures

Definition

Given a category $\mathscr{C}$ with finite products, Σ-structures in $\mathscr{C}$ and Σ-homomorphisms between them form a category, denoted by $\Sigma\text{-str}(\mathscr{C})$. Identities and composition in $\Sigma\text{-str}(\mathscr{C})$ are defined componentwise from those in $\mathscr{C}$.

Remark

If $\mathscr{C}$ and $\mathscr{D}$ are two categories with finite products, then any functor $T : \mathscr{C} \to \mathscr{D}$ which preserves finite products and monomorphisms induces a functor $\Sigma\text{-str}(T) : \Sigma\text{-str}(\mathscr{C}) \to \Sigma\text{-str}(\mathscr{D})$ in the obvious way.

The interpretation of terms

Definition

Let M be a Σ-structure in a category $\mathscr{C}$ with finite products. If $\{\vec{x}\ .\ t\}$ is a term-in-context over Σ (with $\vec{x} = x_1, \ldots, x_n$, $x_i : A_i$ $(i = 1, ..., n)$ and $t : B$, say), then an arrow

$$[[\vec{x}\ .\ t]]_M : M(A_1, \ldots, A_n) \to MB$$

in $\mathscr{C}$ is defined recursively by the following clauses:

a) If t is a variable, it is necessarily x_i for some unique $i \leq n$, and then $[[\vec{x}\ .\ t]]_M = \pi_i$, the ith product projection.

b) If t is $f(t_1, \ldots, t_m)$ (where $t_i : C_i$, say), then $[[\vec{x}\ .\ t]]_M$ is the composite

$$M(A_1, \ldots, A_n) \xrightarrow{([[\vec{x}.t_1]]_M, \ldots, [[\vec{x}.t_m]]_M)} M(C_1, \ldots, C_m) \xrightarrow{Mf} MB$$

Interpreting formulae in categories

- In order to interpret formulae in categories, we need to have a certain amount of categorical structure present on the category in order to give a meaning to the logical connectives which appear in the formulae.
- In fact, the larger is the fragment of logic, the larger is the amount of categorical structure required to interpret it. For example, to interpret finitary conjunctions, we need to form pullbacks, to interpret disjunctions we need to form unions of subobjects, etc.
- Formulae will be interpreted as subobjects in our category; specifically, given a category $\mathscr{C}$ and a Σ-structure M in it, a formula $\phi(\vec{x})$ over Σ where $\vec{x} = (x_1^{A_1}, \ldots, x_n^{A_n})$, will be interpreted as a subobject

$$[[\vec{x} \, . \, \phi]]_M \rightarrowtail M(A_1, \ldots, A_n)$$

defined recursively on the structure of ϕ.

Cartesian categories

Recall that by a finite limit in a category $\mathscr{C}$ we mean a limit of a functor $F : \mathscr{J} \to \mathscr{C}$ where $\mathscr{J}$ is a finite category (i.e. a category with only a finite number of objects and arrows).
In any category $\mathscr{C}$ with pullbacks, pullbacks of monomorphisms are again monomorphisms; thus, for any arrow $f : a \to b$ in $\mathscr{C}$, we have a pullback functor

$$f^* : \mathrm{Sub}_{\mathscr{C}}(b) \to \mathrm{Sub}_{\mathscr{C}}(a) .$$

Definition

A cartesian category is any category with finite limits.

As we shall see below, in cartesian categories we can interpret atomic formulae as well as finite conjunctions of them; in fact, conjunctions will be interpreted as pullbacks (i.e. intersections) of subobjects.

Regular categories

Definition

- Given two subobjects $m_1 : a_1 \rightarrowtail c$ and $m_2 : a_2 \rightarrowtail c$ of an object c in a category $\mathscr{C}$, we say that m_1 factors through m_2 if there is a (necessarily unique) arrow $r : a_1 \to a_2$ in $\mathscr{C}$ such that $m_2 \circ r = m_1$. (Note that this defines a preorder relation $\leq$ on the collection $\mathrm{Sub}_{\mathscr{C}}(c)$ of subobjects of a given object c.)
- We say that a cartesian category $\mathscr{C}$ has images if we are given an operation assigning to each morphism of $\mathscr{C}$ a subobject $Im(f)$ of its codomain, which is the least (in the sense of the preorder $\leq$) subobject of $cod(f)$ through which f factors.
- A regular category is a cartesian category $\mathscr{C}$ such that $\mathscr{C}$ has images and they are stable under pullback.

Fact

Given an arrow $f : a \to b$ in a regular category $\mathscr{C}$, the pullback functor $f^* : \mathrm{Sub}_{\mathscr{C}}(b) \to \mathrm{Sub}_{\mathscr{C}}(a)$ has a left adjoint $\exists_f : \mathrm{Sub}_{\mathscr{C}}(a) \to \mathrm{Sub}_{\mathscr{C}}(b)$, which assigns to a subobject $m : c \rightarrowtail a$ the image of the composite $f \circ m$.

As we shall see below, in regular categories we can interpret formulae built-up from atomic formulae by using finite conjunctions and existential quantifications; in fact, the existential quantifiers will be interpreted as images of certain arrows.

Coherent categories

Definition

A coherent category is a regular category $\mathscr{C}$ in which each $Sub_{\mathscr{C}}(c)$ has finite unions and each $f^* : \mathrm{Sub}_{\mathscr{C}}(b) \to \mathrm{Sub}_{\mathscr{C}}(a)$ preserves them.

As we shall see below, in coherent categories we can interpret formulae built-up from atomic formulae by using finite conjunctions, existential quantifications, and finite disjunctions; in fact, finite disjunctions will be interpreted as finite unions of subobjects.

Note in passing that, if coproducts exist, a union of subobjects of an object c may be constructed as the image of the induced arrow from the coproduct to c.

Geometric categories

Definition

- A (large) category $\mathscr{C}$ is said to be well-powered if each of the preorders $Sub_{\mathscr{C}}(a)$, $a \in Ob(\mathscr{C})$, is equivalent to a small category.
- A geometric category is a well-powered regular category whose subobject lattices have arbitrary unions which are stable under pullback.

As we shall see below, in coherent categories we can interpret formulae built-up from atomic formulae by using finite conjunctions, existential quantifications, and infinitary disjunctions; in fact, disjunctions will be interpreted as unions of subobjects.

Quantifiers as adjoints

Let X and Y be two sets. For any given subset $S \subseteq X \times Y$, we can consider the sets

$$\forall_p S := \{y \in Y \mid \text{for all } x \in X, (x,y) \in S\} \text{ and}$$

$$\exists_p S := \{y \in Y \mid \text{there exists } x \in X, (x,y) \in S\} .$$

The projection map $p : X \times Y \to Y$ induces a map (taking inverse images) at the level of powersets $p^* : \mathscr{P}(Y) \to \mathscr{P}(X \times Y)$. If we regard these powersets as poset categories (where the order-relation is given by the inclusion relation) then this map becomes a functor; also, the assignments $S \to \forall_p S$ and $S \to \exists_p S$ yield functors $\forall_p, \exists_p : \mathscr{P}(X \times Y) \to \mathscr{P}(Y)$.

Theorem

The functors $\exists_p$ and $\forall_p$ are respectively left and right adjoints to the functor $p^* : \mathscr{P}(Y) \to \mathscr{P}(X \times Y)$ which sends each subset $T \subseteq Y$ to its inverse image $p^* T$ under p.

The theorem generalizes to the case of an arbitrary function in place of the projection p.

Heyting categories

Definition

A Heyting category is a coherent category $\mathscr{C}$ such that for any arrow $f : a \to b$ in $\mathscr{C}$ the pullback functor $f^* : \mathrm{Sub}_{\mathscr{C}}(b) \to \mathrm{Sub}_{\mathscr{C}}(a)$ has a right adjoint $\forall_f : \mathrm{Sub}_{\mathscr{C}}(a) \to \mathrm{Sub}_{\mathscr{C}}(b)$ (as well as its left adjoint $\exists_f : \mathrm{Sub}_{\mathscr{C}}(a) \to \mathrm{Sub}_{\mathscr{C}}(b)$).

Theorem

Let $a_1 \rightarrowtail a$ and $a_2 \rightarrowtail a$ be subobjects in a Heyting category. Then there exists a largest subobject $(a_1 \Rightarrow a_2) \rightarrowtail a$ such that $(a_1 \Rightarrow a_2) \cap a_1 \leq a_2$. Moreover, the binary operation on subobjects thus defined is stable under pullback.

In particular, all the subobject lattices in a Heyting category are Heyting algebras.

Thus, in a Heyting category we may interpret full finitary first-order logic.

Fact

Any geometric category, in particular any Grothendieck topos, is a Heyting category.

The interpretation of first-order formulae I

Let M be a Σ-structure in a category $\mathscr{C}$ with finite limits. A formula-in-context $\{\vec{x} \,.\, \phi\}$ over Σ (where $\vec{x} = x_1, \ldots, x_n$ and $x_i : A_i$, say) will be interpreted as a subobject $[[\vec{x} \,.\, \phi]]_M \rightarrowtail M(A_1, \ldots, A_n)$ according to the following recursive clauses:

- If $\phi(\vec{x})$ is $R(t_1, \ldots, t_m)$ where R is a relation symbol (of type $B_1, \ldots, B_m$, say), then $[[\vec{x} \,.\, \phi]]_M$ is the pullback

$$\begin{tikzcd} {[[\vec{x} \,.\, \phi]]_M} \arrow[r] \arrow[d] & MR \arrow[d] \\ M(A_1, \ldots, A_n) \arrow[r, "{([[\vec{x}.t_1]]_M, \ldots, [[\vec{x}.t_m]]_M)}"] & M(B_1, \ldots, B_m) \end{tikzcd}$$

- If $\phi(\vec{x})$ is $(s = t)$, where s and t are terms of sort B, then $[[\vec{x} \,.\, \phi]]_M$ is the equalizer of $[[\vec{x} \,.\, s]]_M, [[\vec{x} \,.\, t]]_M : M(A_1, \ldots, A_n) \to MB$.
- If $\phi(\vec{x})$ is $\top$ then $[[\vec{x} \,.\, \phi]]_M$ is the top element of $\mathrm{Sub}_{\mathscr{C}}(M(A_1, \ldots, A_n))$.

The interpretation of first-order formulae II

- If ϕ is $\psi \wedge \chi$ then $[[\vec{x} \,.\, \phi]]_M$ is the intersection (= pullback)

$$
\begin{tikzcd}
{[[\vec{x} \,.\, \phi]]_M} \arrow[r] \arrow[d] & {[[\vec{x} \,.\, \chi]]_M} \arrow[d] \\
{[[\vec{x} \,.\, \psi]]_M} \arrow[r] & M(A_1, \dots, A_n)
\end{tikzcd}
$$

- If $\phi(\vec{x})$ is $\perp$ and $\mathscr{C}$ is a coherent category then $[[\vec{x} \,.\, \phi]]_M$ is the bottom element of $\mathrm{Sub}_{\mathscr{C}}(M(A_1, \dots, A_n))$.
- If ϕ is $\psi \vee \chi$ and $\mathscr{C}$ is a coherent category then $[[\vec{x} \,.\, \phi]]_M$ is the union of the subobjects $[[\vec{x} \,.\, \psi]]_M$ and $[[\vec{x} \,.\, \chi]]_M$.
- If ϕ is $\psi \Rightarrow \chi$ and $\mathscr{C}$ is a Heyting category, $[[\vec{x} \,.\, \phi]]_M$ is the implication $[[\vec{x} \,.\, \psi]]_M \Rightarrow [[\vec{x} \,.\, \chi]]_M$ in the Heyting algebra $\mathrm{Sub}_{\mathscr{C}}(M(A_1, \dots, A_n))$ (similarly, the negation $\neg\psi$ is interpreted as the pseudocomplement of $[[\vec{x} \,.\, \psi]]_M$).

The interpretation of first-order formulae III

- If ϕ is $(\exists y)\psi$ where y is of sort B, and $\mathscr{C}$ is a regular category, then $[[\vec{x} \,.\, \phi]]_M$ is the image of the composite

$$[[\vec{x}, y \,.\, \psi]]_M \longrightarrow M(A_1, \ldots, A_n, B) \xrightarrow{\pi} M(A_1, \ldots, A_n)$$

  where π is the product projection on the first n factors.
- If ϕ is $(\forall y)\psi$ where y is of sort B, and $\mathscr{C}$ is a Heyting category, then $[[\vec{x} \,.\, \phi]]_M$ is $\forall_\pi([[\vec{x}, y \,.\, \psi]]_M)$, where π is the same projection as above.
- If ϕ is $\bigvee_{i \in I} \phi_i$ and $\mathscr{C}$ is a geometric category then $[[\vec{x} \,.\, \phi]]_M$ is the union of the subobjects $[[\vec{x} \,.\, \phi_i]]_M$.
- If ϕ is $\bigwedge_{i \in I} \phi_i$ and $\mathscr{C}$ has arbitrary intersections of subobjects then $[[\vec{x} \,.\, \phi]]_M$ is the intersection of the subobjects $[[\vec{x} \,.\, \phi_i]]_M$.

Models of first-order theories in categories

Definition

Let M be a Σ-structure in a category $\mathscr{C}$.

a) If $\sigma = \phi \vdash_{\vec{x}} \psi$ is a sequent over Σ interpretable in $\mathscr{C}$, we say that σ is satisfied in M if $[[\vec{x} \,.\, \phi]]_M \leq [[\vec{x} \,.\, \psi]]_M$ in $\mathrm{Sub}_{\mathscr{C}}(M(A_1, \ldots, A_n))$.
b) If $\mathbb{T}$ is a theory over Σ interpretable in $\mathscr{C}$, we say M is a model of $\mathbb{T}$ if all the axioms of $\mathbb{T}$ are satisfied in M.
c) We write $\mathbb{T}$-mod($\mathscr{C}$) for the full subcategory of Σ-str($\mathscr{C}$) whose objects are models of T.

We say that a functor $F : \mathscr{C} \to \mathscr{D}$ between two cartesian (resp. regular, coherent, geometric, Heyting) categories is cartesian (resp. regular, coherent, geometric, Heyting) if it preserves finite limits (resp. finite limits and images, finite limits and images and finite unions of subobjects, finite limits and images and arbitrary unions of subobjects, finite limits and images and Heyting implications between subobjects).

Theorem

If $\mathbb{T}$ is a regular (resp. coherent, ...) theory over Σ, then for any regular (resp. coherent, ...) functor $T : \mathscr{C} \to \mathscr{D}$ the functor Σ-str$(T) : \Sigma$-str$(\mathscr{C}) \to \Sigma$-str$(\mathscr{D})$ defined above restricts to a functor $\mathbb{T}$-mod$(T) : \mathbb{T}$-mod$(\mathscr{C}) \to \mathbb{T}$-mod$(\mathscr{D})$. If T is moreover conservative (that is, reflects isomorphisms) then the functor Σ-str(T) reflects the property of being a $\mathbb{T}$-model.

Examples

- A topological group can be seen as a model of the theory of groups in the category of topological spaces.
- Similarly, an algebraic (resp. Lie) group is a model of the theory of groups in the category of algebraic varieties (resp. the category of smooth manifolds).
- A sheaf of rings (more generally, a sheaf of models of a Horn theory $\mathbb{T}$) on a topological space X can be seen as a model of the theory of rings (resp. of the theory $\mathbb{T}$) in the topos $\mathbf{Sh}(X)$ of sheaves on X.
- A sheaf of models of a geometric theory $\mathbb{T}$ over a signature Σ in a topos $\mathbf{Sh}(X)$ of sheaves on a topological space X is a Σ-structure in $\mathbf{Sh}(X)$ whose stalks are models of $\mathbb{T}$.
- A bunch of set-based models of a theory $\mathbb{T}$ indexed over a set I can be seen as a model of $\mathbb{T}$ in the functor category $[I, \mathbf{Set}]$. More generally, we have that $\mathbb{T}\text{-mod}([\mathscr{C}, \mathbf{Set}]) \simeq [\mathscr{C}, \mathbb{T}\text{-mod}(\mathbf{Set})]$.

Soundness and completeness

Theorem (Soundness)

Let $\mathbb{T}$ be a Horn (resp. regular, coherent, first-order, geometric) theory over a signature $\mathbb{T}$, and let M be a model of $\mathbb{T}$ in a cartesian (resp. regular, coherent, Heyting, geometric) category $\mathscr{C}$. If σ is a sequent (in the appropriate fragment of first-order logic over Σ) which is provable in $\mathbb{T}$, then σ is satisfied in M.

Theorem (Completeness)

Let $\mathbb{T}$ be a Horn (resp. regular, coherent, first-order, geometric) theory. If a Horn (resp. regular, coherent, Heyting, geometric) sequent σ is satisfied in all models of $\mathbb{T}$ in cartesian (resp. regular, coherent, Heyting, geometric) categories, then it is provable in $\mathbb{T}$.

Soundness and completeness for toposes

We say that a first-order formula $\phi(\vec{x})$ over a signature Σ is valid in a topos $\mathscr{E}$ if for every Σ-structure M in $\mathscr{E}$ the sequent $\top \vdash_{\vec{x}} \phi$ is satisfied in M.

Theorem

Let Σ be a signature and $\phi(\vec{x})$ a first-order formula over Σ. Then $\phi(\vec{x})$ is provable in intuitionistic (finitary) first-order logic if and only if it is valid in every Grothendieck topos.

Sketch of proof.

The soundness result is part of a theorem mentioned above. The completeness part follows from the existence of canonical Kripke models and the fact that, given a poset P and a Kripke model $\mathscr{U}$ on P there is a model $\mathscr{U}^*$ in the topos $[P, \mathbf{Set}]$ such that the first-order sequents valid in $\mathscr{U}$ are exactly those valid in $\mathscr{U}^*$. $\square$

Hence a topos can be considered as a mathematical universe in which one can do mathematics similarly to how one does it in the classical context of sets (with the only exception that one must in general argue constructively).

The internal language of a topos I

Given a category $\mathscr{C}$ with finite products, in particular an elementary topos, one can define a first-order signature $\Sigma_{\mathscr{C}}$, called the internal language of $\mathscr{C}$, for reasoning about $\mathscr{C}$ in a set-theoretic fashion, that is by using 'elements'.

Definition

The signature $\Sigma_{\mathscr{C}}$ has one sort $\ulcorner A \urcorner$ for each object A of $\mathscr{C}$, one function symbol $\ulcorner f \urcorner : \ulcorner A_1 \urcorner, \cdots, \ulcorner A_n \urcorner \to \ulcorner B \urcorner$ for each arrow $f : A_1 \times \cdots \times A_n \to B$ in $\mathscr{C}$, and one relation symbol $\ulcorner R \urcorner \rightarrowtail \ulcorner A_1 \urcorner \cdots \ulcorner A_n \urcorner$ for each subobject $R \rightarrowtail A_1 \times \cdots \times A_n$.

Note that there is a canonical $\Sigma_{\mathscr{C}}$-structure in $\mathscr{C}$, which assigns A to $\ulcorner A \urcorner$, f to $\ulcorner f \urcorner$ and R to $\ulcorner R \urcorner$.

The usefulness of this definition lies in the fact that properties of $\mathscr{C}$ or constructions in it can often be formulated in terms of satisfaction of certain formulae over $\Sigma_{\mathscr{C}}$ in the canonical structure; the internal language can thus be used for proving things about $\mathscr{C}$.

The internal language of a topos II

If $\mathscr{C}$ is a topos, we can extend the internal language by allowing the formation of formulae of the kind $\tau \in \Gamma$, where τ is a term of sort A and Γ is a term of sort Ω^A. Indeed, we may interpret this formula as the subobject whose classifying arrow is the composite

$$W \xrightarrow{\langle \tau, \Gamma \rangle} A \times \Omega^A \xrightarrow{\in_A} \Omega$$

where W denotes the product of (the objects representing the) sorts of the variables occurring either in τ or in Γ (considered without repetitions) and $\langle \tau, \Gamma \rangle$ denotes the induced map to the product.

Note that an object A of $\mathscr{C}$ gives rise to a constant term of type Ω^A.

Thus in a topos we can also interpret all the common formulas that we use in Set Theory.

Kripke-Joyal semantics I

Kripke-Joyal semantics represents the analogue for toposes of the usual Tarskian semantics for classical first-order logic.
In the context of toposes, it makes no sense to speak of elements of a structure in a topos, but we can replace the classical notion of element of a set with that of generalized element of an object: a generalized element of an object c of a topos $\mathscr{E}$ is simply an arrow $\alpha : u \to c$ with codomain c.

Definition

Let $\mathscr{E}$ be a topos and M be a Σ-structure in $\mathscr{E}$. Given a first-order formula $\phi(x)$ over Σ in a variable x of sort A and a generalized element $\alpha : U \to MA$ of MA, we define

$$U \models_M \phi(\alpha) \quad \text{iff} \quad \alpha \text{ factors through } [[x \,.\, \phi]]_M \rightarrowtail MA$$

Of course, the definition can be extended to formulae with an arbitrary (finite) number of free variables.
In the following proposition, the notation $+$ denotes binary coproduct.

Kripke-Joyal semantics II

Proposition

If $\alpha : U \to MA$ is a generalized element of MA while $\phi(x)$ and $\psi(x)$ are formulas with a free variable x of sort A, then

- $U \models (\phi \wedge \psi)(\alpha)$ *if and only if* $U \models \phi(\alpha)$ *and* $U \models \psi(\alpha)$.
- $U \models (\phi \vee \psi)(\alpha)$ *if and only if there are arrows* $p : V \to U$ *and* $q : W \to U$ *such that* $p + q : V + W \to U$ *is epic, while both* $V \models \phi(\alpha \circ p)$ *and* $W \models \psi(\alpha \circ q)$.
- $U \models (\phi \Rightarrow \psi)(\alpha)$ *if and only if for any arrow* $p : V \to U$ *such that* $V \models \phi(\alpha \circ p)$, *then* $V \models \psi(\alpha \circ p)$.
- $U \models (\neg\phi)(\alpha)$ *if and only if whenever* $p : V \to U$ *is such that* $V \models \phi(\alpha \circ p)$, *then* $V \cong 0_{\mathscr{E}}$.

If $\phi(x, y)$ has an additional free variable y of sort B then

- $U \models (\exists y)\phi(\alpha, y)$ *if and only if there exist an epi* $p : V \to U$ *and a generalized element* $\beta : V \to B$ *such that* $V \models \phi(\alpha \circ p, \beta)$.
- $U \models (\forall y)\phi(\alpha, y)$ *if and only if for every object* V, *for every arrow* $p : V \to U$ *and every generalized element* $c : V \to B$ *one has* $V \models \phi(\alpha \circ p, \beta)$.

Geometric theories

Definition

A geometric theory $\mathbb{T}$ is a theory over a first-order signature Σ whose axioms can be presented in the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are *geometric formulae*, that is formulae in the context $\vec{x}$ built up from atomic formulae over Σ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.

Remark

Inverse image functors of geometric morphisms of toposes always preserve models of a geometric theory (but in general not those of an arbitrary first-order theory).

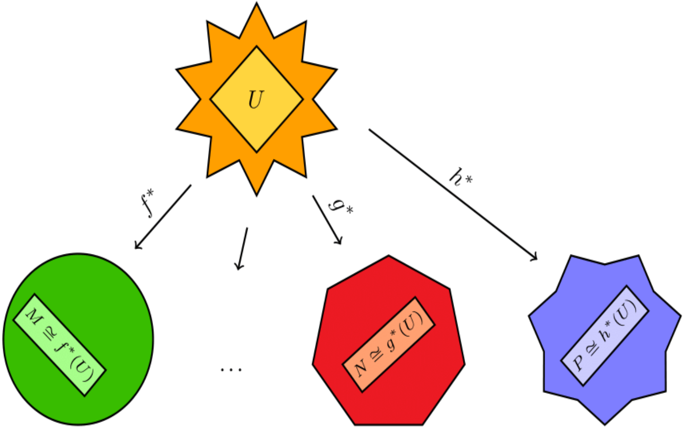
Most of the first-order theories naturally arising in Mathematics are geometric; anyway, if a finitary first-order theory is not geometric, one can always canonically associate with it a geometric theory, called its *Morleyization*, having the same set-based models.

Classifying toposes

It was realized in the seventies (thanks to the work of several people, notably including W. Lawvere, A. Joyal, G. Reyes and M. Makkai) that:

- Every geometric theory $\mathbb{T}$ has a classifying topos $\mathscr{E}_{\mathbb{T}}$ which is characterized by the following representability property: for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories
$$\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$
natural in $\mathscr{E}$, where
  - $\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}})$ is the category of geometric morphisms $\mathscr{E} \to \mathscr{E}_{\mathbb{T}}$ and
  - $\mathbb{T}\text{-mod}(\mathscr{E})$ is the category of $\mathbb{T}$-models in $\mathscr{E}$.
- The classifying topos of a geometric theory $\mathbb{T}$ can be canonically built as the category $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of sheaves on the syntactic site $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$.

Classifying toposes



Classifying topos

The syntactic category of a geometric theory

Definition (Makkai and Reyes 1977)

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. The syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the 'renaming'-equivalence classes of geometric formulae-in-context $\{\vec{x} \,.\, \phi\}$ over Σ and as arrows $\{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, \psi\}$ (where the contexts $\vec{x}$ and $\vec{y}$ are supposed to be disjoint without loss of generality) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\vec{x}, \vec{y})$ which are $\mathbb{T}$-provably functional i.e. such that the sequents

$$(\phi \vdash_{\vec{x}} (\exists y)\theta),$$
$$(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi), \text{ and}$$
$$((\theta \wedge \theta[\vec{z}/\vec{y}]) \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{y} = \vec{z}))$$

  are provable in $\mathbb{T}$.
- The composite of two arrows

$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\theta]} \{\vec{y} \,.\, \psi\} \xrightarrow{[\gamma]} \{\vec{z} \,.\, \chi\}$$

  is defined as the $\mathbb{T}$-provable-equivalence class of the formula $(\exists \vec{y})\theta \wedge \gamma$.
- The identity arrow on an object $\{\vec{x} \,.\, \phi\}$ is the arrow

$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\phi \wedge \vec{x'} = \vec{x}]} \{\vec{x'} \,.\, \phi[\vec{x'}/\vec{x}]\}$$

The syntactic site

On the syntactic category of a geometric theory it is natural to put the Grothendieck topology defined as follows:

Definition

The syntactic topology $J_{\mathbb{T}}$ on the syntactic category $\mathscr{C}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ is the geometric topology on it; in particular,

a small family $\{[\theta_i] : \{\vec{x}_i \,.\, \phi_i\} \to \{\vec{y} \,.\, \psi\}\}$ in $\mathscr{C}_{\mathbb{T}}$ is $J_{\mathbb{T}}$-covering

if and only if

the sequent $(\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x}_i)\theta_i)$ is provable in $\mathbb{T}$.

This notion is instrumental for identifying the models of the theory $\mathbb{T}$ in any geometric category $\mathscr{C}$ (and in particular in any Grothendieck topos) as suitable functors defined on the syntactic category $\mathscr{C}_{\mathbb{T}}$ with values in $\mathscr{C}$; indeed, these are precisely the $J_{\mathbb{T}}$-continuous cartesian functors $\mathscr{C}_{\mathbb{T}} \to \mathscr{C}$. So if $\mathscr{C}$ is a Grothendieck topos they correspond precisely to the geometric morphisms from $\mathscr{C}$ to $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$. This topos therefore classifies $\mathbb{T}$.

Morita equivalence

- Two mathematical theories are said to be Morita-equivalent if have the same classifying topos (up to equivalence): this means that they have equivalent categories of models in every Grothendieck topos $\mathscr{E}$, naturally in $\mathscr{E}$.
- Every Grothendieck topos is the classifying topos of *some* geometric theory (and in fact, of infinitely many theories).
- So a Grothendieck topos can be seen as a canonical representative of equivalence classes of theories modulo Morita-equivalence.

Toposes as *bridges*

- The notion of Morita-equivalence is ubiquitous in Mathematics; indeed, it formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods'.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted in terms of Morita-equivalences.
- On the other hand, Topos Theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.
- Any two theories which are bi-interpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.
- We can expect most of the categorical equivalences between categories of set-based models of geometric theories to lift to Morita equivalences.

Toposes as *bridges*

- In the topos-theoretic study of theories, the latter are represented by sites (of definition of their classifying topos or of some other topos naturally attached to them) or by other objects presenting toposes.
- The existence of theories which are Morita-equivalent to each other translates into the existence of different sites of definition (or, more generally, presentations) for the same Grothendieck topos.
- Grothendieck toposes can be effectively used as 'bridges' for transferring notions, properties and results across different Morita-equivalent theories:

$$\begin{tikzcd} & \mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'} \arrow[dr, dashed] & \\ \mathbb{T} \arrow[ur, dashed] & & \mathbb{T}' \end{tikzcd}$$

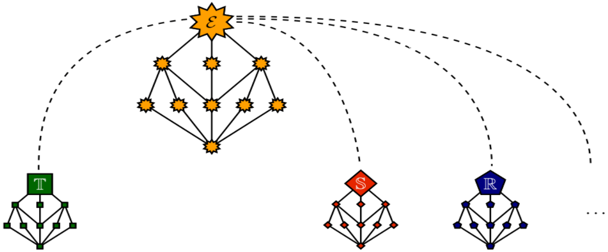
- The transfer of information takes place by expressing topos-theoretic invariants in terms of the different sites of definition (or, more generally, presentations) for the given topos.
- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

Toposes as *bridges*

- This methodology is technically effective because the relationship between a topos and its representations is often very natural, enabling us to easily transfer invariants across different representations (and hence, between different theories).
- On the other hand, the 'bridge' technique is highly non-trivial, in the sense that it often yields deep and surprising results. This is due to the fact that a given invariant can manifest itself in significanly different ways in the context of different presentations.
- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories and constructions. Indeed, many important invariants of mathematical structures are actually invariants of toposes (think for instance of cohomology or homotopy groups) and topos-theoretic invariants considered on the classifying topos $\mathscr{E}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ often translate into interesting logical (i.e. syntactic or semantic) properties of $\mathbb{T}$.

Toposes as *bridges*

- The fact that topos-theoretic invariants specialize to important properties or constructions of natural mathematical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications in Mathematics as a whole: for instance



Lattices of theories

This picture represents the lattice structure on the collection of the subtoposes of a topos $\mathscr{E}$ inducing lattice structures on the collection of 'quotients' of geometric theories $\mathbb{T}$, $\mathbb{S}$, $\mathbb{R}$ classified by it.

Introduction

Toposes as generalized spaces

Toposes as mathematical universes

Categorical logic

Classifying toposes

Toposes as bridges

Topological Galois theory

Theories of presheaf type

Topos-theoretic Fraïssé theorem

Quotients of theories of presheaf type

Future directions

The ‘bridge-building’ technique

- Decks of ‘bridges’: Morita-equivalences (or more generally morphisms or other kinds of relations between toposes)
- Arches of ‘bridges’: Site characterizations for topos-theoretic invariants (or more generally ‘unravelings’ of topos-theoretic invariants in terms of concrete representations of the relevant topos)

Invariant I across the Morita-equivalence

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{D}, K)$$

site characterization for I *site characterization for I*

$(\mathcal{C}, J)$ $(\mathcal{D}, K)$

Property $P_{(\mathcal{C},J)}$ *Property $Q_{(\mathcal{D},K)}$*

The ‘bridge’ yields a logical equivalence (or an implication) between the ‘concrete’ properties $P_{(\mathscr{C},J)}$ and $Q_{(\mathscr{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

A few selected applications

Since the theory of topos-theoretic 'bridges' was introduced in 2010, several applications of it have been obtained in different fields of Mathematics, such as:

- Model theory (topos-theoretic Fraïssé theorem)
- Proof theory (various results for first-order theories)
- Algebra (topos-theoretic generalization of topological Galois theory)
- Topology (topos-theoretic interpretation/generation of Stone-type and Priestley-type dualities)
- Functional analysis (various results on Gelfand spectra and Wallman compactifications)
- Many-valued logics and lattice-ordered groups (two joint papers with A. C. Russo)
- Cyclic homology, as reinterpreted by A. Connes (work on *"cyclic theories"*, jointly with N. Wentzlaff)
- Algebraic geometry (logical analysis of (co)homological motives, cf. the paper *"Syntactic categories for Nori motives"* joint with L. Barbieri-Viale and L. Lafforgue)

Topological Galois theory

Recall that classical topological Galois theory provides, given a Galois extension $F \subseteq L$, a bijective correspondence between the intermediate field extensions (resp. finite field extensions) $F \subseteq K \subseteq L$ and the closed (resp. open) subgroups of the Galois group $Aut_F(L)$.

This admits the following categorical reformulation: the functor $K \to \mathrm{Hom}(K, L)$ defines an equivalence of categories

$$(\mathscr{L}_F^L)^{\mathrm{op}} \simeq \mathbf{Cont}_t(Aut_F(L)),$$

where $\mathscr{L}_F^L$ is the category of finite intermediate field extensions and $\mathbf{Cont}_t(Aut_F(L))$ is the category of continuous non-empty transitive actions of $Aut_F(L)$ on discrete sets.

A natural question thus arises: can we characterize the categories $\mathscr{C}$ whose dual is equivalent to (or fully embeddable into) the category of (non-empty transitive) actions of a topological automorphism group?

The topos-theoretic interpretation

Key observation: the above equivalence extends to an equivalence of toposes

$$\mathbf{Sh}({\mathscr{L}_F^L}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_F(L)),$$

where J_{at} is the atomic topology on ${\mathscr{L}_F^L}^{\mathrm{op}}$ and $\mathbf{Cont}(Aut_F(L))$ is the topos of continuous actions of $Aut_F(L)$ on discrete sets.

It is therefore natural to investigate our problem by using the methods of topos theory: more specifically, we shall look for conditions on a small category $\mathscr{C}$ and on an object u of its ind-completion for the existence of an equivalence of toposes of the form

$$\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}(u)) \, .$$

We will then be able to obtain, starting from such an equivalence, an answer to our question, and hence build Galois-type theories in a great variety of different mathematical contexts.

The key notions I

- A category $\mathscr{C}$ is said to satisfy the amalgamation property (AP) if for every objects $a, b, c \in \mathscr{C}$ and morphisms $f : a \to b$, $g : a \to c$ in $\mathscr{C}$ there exists an object $d \in \mathscr{C}$ and morphisms $f' : b \to d$, $g' : c \to d$ in $\mathscr{C}$ such that $f' \circ f = g' \circ g$:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, dashed, "f'"] \\ c \arrow[r, dashed, "g'"'] & d \end{tikzcd}$$

- A category $\mathscr{C}$ is said to satisfy the joint embedding property (JEP) if for every pair of objects $a, b \in \mathscr{C}$ there exists an object $c \in \mathscr{C}$ and morphisms $f : a \to c$, $g : b \to c$ in $\mathscr{C}$:

$$\begin{tikzcd} & a \arrow[d, dashed, "f"] \\ b \arrow[r, dashed, "g"'] & c \end{tikzcd}$$

The key notions II

- An object $u \in \text{Ind-}\mathscr{C}$ is said to be $\mathscr{C}$-universal if for every $a \in \mathscr{C}$ there exists an arrow $\chi : a \to u$ in Ind-$\mathscr{C}$:

$$a \overset{\chi}{\dashrightarrow} u$$

- An object $u \in \text{Ind-}\mathscr{C}$ is said to be $\mathscr{C}$-ultrahomogeneous if for any object $a \in \mathscr{C}$ and arrows $\chi_1 : a \to u$, $\chi_2 : a \to u$ in Ind-$\mathscr{C}$ there exists an automorphism $j : u \to u$ such that $j \circ \chi_1 = \chi_2$:

$$\begin{tikzcd} a \arrow[r, "\chi_1"] \arrow[dr, "\chi_2"'] & u \arrow[d, dashed, "j"] \\ & u \end{tikzcd}$$

The main theorem

Theorem

Let $\mathscr{C}$ be a small category satisfying AP and JEP, and let u be a $\mathscr{C}$-universal et $\mathscr{C}$-ultrahomogeneous object of the ind-completion Ind-$\mathscr{C}$ *of $\mathscr{C}$. Then there is an equivalence of toposes*

$$\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}(u)),$$

where $\mathrm{Aut}(u)$ *is endowed with the topology in which a basis of open neighbourhoods of the identity is given by the subgroups of the form $I_\chi = \{\alpha \in \mathrm{Aut}(u) \mid \alpha \circ \chi = \chi\}$ for $\chi : c \to u$ an arrow in* Ind-$\mathscr{C}$ *from an object c of $\mathscr{C}$.*
This equivalence is induced by the functor

$$F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cont}(\mathrm{Aut}(u))$$

which sends any object c of $\mathscr{C}$ to the set $\mathrm{Hom}_{\text{Ind-}\mathscr{C}}(c, u)$ (endowed with the obvious action of $\mathrm{Aut}(u)$*) and any arrow $f : c \to d$ in $\mathscr{C}$ to the* $\mathrm{Aut}(u)$*-equivariant map*

$$- \circ f : \mathrm{Hom}_{\text{Ind-}\mathscr{C}}(d, u) \to \mathrm{Hom}_{\text{Ind-}\mathscr{C}}(c, u)\,.$$

Restricting to sites

The following result arises from two 'bridges', respectively obtained by considering the invariant notions of atom and of arrow between atoms.

$$\begin{tikzcd} & \mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}(u)) \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \mathscr{C}^{\mathrm{op}} & & \mathbf{Cont}_t(\mathrm{Aut}(u)) \end{tikzcd}$$

Theorem

Under the hypotheses of the last theorem, the functor F is full and faithful if and only if every arrow of $\mathscr{C}$ is a strict monomorphism, and it is an equivalence on the full subcategory $\mathbf{Cont}_t(\mathrm{Aut}(u))$ of $\mathbf{Cont}(\mathrm{Aut}(u))$ on the non-empty transitive actions if $\mathscr{C}$ is moreover atomically complete.

Applications

- A natural source of ultrahomogenenous and universal objects is provided by Fraïssé's construction in Model Theory and its categorical generalizations.

  For instance, if the category $\mathscr{C}$ is countable and all its arrows are monomorphisms then there always exists a $\mathscr{C}$-universal and $\mathscr{C}$-ultrahomogeneous object in Ind-$\mathscr{C}$.

- Our theorem generalizes Grothendieck's theory of Galois categories (which corresponds to the particular case when the fundamental group is profinite).

- It can be applied for generating Galois-type theories in different fields of Mathematics, which do *not* fit in the formalism of Galois categories.

Examples

Natural categories with monic arrows: $\mathscr{C}$ equal to the category of

- Finite sets and injections
- Finite graphs and embeddings
- Finite groups and injective homomorphisms

Natural categories with epic arrows: $\mathscr{C}^{\mathrm{op}}$ equal to the category of

- Finite sets and surjections
- Finite groups and surjective homomorphisms
- Finite graphs and homomorphisms which are surjective both at the level of vertices and at the level of edges.

Categories of 'imaginaries'

- If a category $\mathscr{C}$ satisfies the first but not the second condition of our last theorem, our topos-theoretic approach gives us a fully explicit way to complete it, by means of the addition of 'imaginaries', so that also the second condition gets satisfied.
- This is the case for instance for the categories considered above; so we get notions of 'imaginary finite set', 'imaginary finite group' etc.
- The objects of the atomic completion admit an explicit description in terms of equivalence relations in the topos $\mathbf{Sh}(\mathscr{C}^{\text{op}}, J_{at})$ on objects coming from the site $\mathscr{C}^{\text{op}}$.
- In a joint work with L. Lafforgue we give an alternative 'combinatorial' description of the atomic completion.

The logical interpretation

- It is interesting to study the toposes considered above from a logical point of view, that is from the perspective of the structures that they classify.

- This analysis will reveal a deep link between Galois theory (reinterpreted and generalized as above) and Fraïssé theory in Model Theory, and lead to an approach to the problem of the independence from ℓ of ℓ-adic cohomology.

- For this, we need to introduce the notion of an atomic and complete geometric theory, and that of special model of such a theory.

- We will also need to use the notion of theory of presheaf type (i.e., classified by a presheaf topos).

Atomic and complete theories

Definition

- Given a geometric theory $\mathbb{T}$ over a signature Σ, a geometric formula-in-context $\phi(\vec{x})$ over Σ is said to be $\mathbb{T}$-complete if the sequent $(\phi \vdash_{\vec{x}} \bot)$ is not provable in $\mathbb{T}$, but for any geometric formula $\psi(\vec{x})$ over Σ in the same context, either $(\phi \vdash_{\vec{x}} \psi)$ is provable in $\mathbb{T}$ or $(\phi \wedge \psi \vdash_{\vec{x}} \bot)$ is provable in $\mathbb{T}$.
- A geometric theory $\mathbb{T}$ is said to be atomic if every geometric formula-in-context is $\mathbb{T}$-provably equivalent to a disjunction of $\mathbb{T}$-complete formulae in the same context.
- A geometric theory $\mathbb{T}$ is said to be complete if every geometric assertion over its signature is either $\mathbb{T}$-provably true or $\mathbb{T}$-provably false, but not both. [N.B. if the theory is atomic then this notion becomes equivalent to the usual first-order one.]
- A set-based model M of an atomic and complete theory is defined to be special if each $\mathbb{T}$-complete formula $\phi(\vec{x})$ is realized in M and for any $\vec{a}, \vec{b} \in [[\vec{x} \,.\, \phi]]_M$ there exists an automorphism f of M such that $f(\vec{a}) = \vec{b}$.

Special models and their automorphism groups

The above categorical theorem admits the following logical formulation:

Theorem

Let $\mathbb{T}$ be an atomic and complete theory and M be a special model of $\mathbb{T}$. Then we have an equivalence

$$\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}}) \simeq \mathbf{Cont}(\mathrm{Aut}(M)),$$

where $\mathrm{Aut}(M)$ *is endowed with the topology of pointwise convergence.*

From this theorem, one immediately deduces that if M and N are respectively special models of two atomic and complete theories $\mathbb{T}$ and $\mathbb{T}'$ then $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are isomorphic as topological groups if and only if M and N are atomically bi-interpretable, generalizing and strenghtening the classical result by Ahlbrandt-Ziegler.

[This notion of bi-interpretation notably implies that for any sort A of the signature of $\mathbb{T}$, MA can be represented in the form $[[\vec{y} \, . \, \psi]]_N/R$, where $\{\vec{y} \, . \, \psi\}$ is a $\mathbb{T}$-complete formula and R is a geometrically definable equivalence relation on $[[\vec{y} \, . \, \psi]]_N$.]

Theories of presheaf type

Definition

A geometric theory is said to be of presheaf type if it is classified by a presheaf topos.

Theories of presheaf type are very important in that they constitute the basic 'building blocks' from which every geometric theory can be built. Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a 'quotient' of a theory of presheaf type.

These theories are the logical counterpart of small categories, in the sense that:

- For any theory of presheaf type $\mathbb{T}$, its category $\mathbb{T}$-mod(**Set**) of (set-based) models is equivalent to the ind-completion of the full subcategory f.p.$\mathbb{T}$-mod(**Set**) on the finitely presentable models.
- Any small category $\mathscr{C}$ is, up to idempotent splitting completion, equivalent to the category f.p.$\mathbb{T}$-mod(**Set**) for some theory of presheaf type $\mathbb{T}$.

Moreover, any geometric theory $\mathbb{T}$ can be expanded to a theory classified by the topos [f.p.$\mathbb{T}$-mod(**Set**), **Set**].

Theories of presheaf type

Every finitary algebraic (or, more generally, cartesian) theory is of presheaf type, but this class contains many other interesting mathematical theories including

- the theory of linear orders (classified by the simplicial topos)
- the theory of algebraic extensions of a given field
- the theory of flat modules over a ring
- the theory of lattice-ordered abelian groups with strong unit
- the 'cyclic theories' (classified by the cyclic topos, the epicyclic topos and the arithmetic topos)
- the theory of perfect MV-algebras (or more generally of local MV-algebras in a proper variety of MV-algebras)
- the geometric theory of finite sets

Any theory of presheaf type $\mathbb{T}$ gives rise to two different representations of its classifying topos, which can be used to build 'bridges' connecting its syntax and semantics:

$$
\begin{array}{ccccc}
 & & [\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}}) & & \\
 & ⋰ & & ⋱ & \\
\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} & & & & (\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})
\end{array}
$$

Irreducible formulae and finitely presentable models

Definition

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then a geometric formula $\phi(\vec{x})$ over Σ is said to be $\mathbb{T}$-irreducible if, regarded as an object of the syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$, it does not admit any non-trivial $J_{\mathbb{T}}$-covering sieves.

Theorem

Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ. Then

(i) *Any finitely presentable $\mathbb{T}$-model in* **Set** *is presented by a $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ;*

(ii) *Conversely, any $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ presents a $\mathbb{T}$-model.*

In fact, the category f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ *is equivalent to the full subcategory $\mathscr{C}_{\mathbb{T}}^{\text{irr}}$ of $\mathscr{C}_{\mathbb{T}}$ on the $\mathbb{T}$-irreducible formulae.*

Irreducible object

$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$
Every object

$(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$
$\mathbb{T}$-irreducible formula

A definability theorem

Theorem

Let $\mathbb{T}$ be a theory of preshef type and suppose that we are given, for every finitely presentable **Set***-model $\mathcal{M}$ of $\mathbb{T}$, a subset $R_{\mathcal{M}}$ of $\mathcal{M}^n$ in such a way that every $\mathbb{T}$-model homomorphism $h : \mathcal{M} \to \mathcal{N}$ maps $R_{\mathcal{M}}$ into $R_{\mathcal{N}}$. Then there exists a geometric formula-in-context $\phi(x_1, \dots, x_n)$ such that $R_{\mathcal{M}} = [[\vec{x} \,.\, \phi]]_{\mathcal{M}}$ for each finitely presentable $\mathbb{T}$-model $\mathcal{M}$.*

Subobject of $UA_1 \times \cdots \times UA_n$

$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$

Functorial assignment

$M \to R_M \subseteq MA_1 \times \cdots \times MA_n$

$(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

Geometric formula

$\phi(x_1^{A_1}, \dots, x_n^{A_n})$

Topos-theoretic Fraïssé theorem

The following result, which generalizes Fraïssé's theorem in classical model theory, arises from a triple 'bridge'.

Definition

A set-based model M of a geometric theory $\mathbb{T}$ is said to be homogeneous if for any arrow $y : c \to M$ in $\mathbb{T}$-mod(**Set**) and any arrow f in f.p.$\mathbb{T}$-mod(**Set**) there exists an arrow u in $\mathbb{T}$-mod(**Set**) such that $u \circ f = y$:

$$\begin{tikzcd} c \arrow[r, "y"] \arrow[d, "f"'] & M \\ d \arrow[ur, dashed, "u"'] & \end{tikzcd}$$

Theorem

Let $\mathbb{T}$ be a theory of presheaf type such that the category f.p.$\mathbb{T}$-mod(**Set**) *is non-empty and has AP and JEP. Then the theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models is complete and atomic.*

Topos-theoretic Fraïssé theorem

Atomic topos
$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
Atomic site i.e.
AP on f.p.$\mathbb{T}$-mod($\mathbf{Set}$)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Atomicity of $\mathbb{T}'$

Two-valued topos
$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
JEP on f.p.$\mathbb{T}$-mod($\mathbf{Set}$)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
Completeness of $\mathbb{T}'$

Point of
$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$
homogeneous $\mathbb{T}$-model in $\mathbf{Set}$

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$
$\mathbb{T}'$-model in $\mathbf{Set}$

Characterization theorems

Theorem

A geometric theory $\mathbb{T}$ over a signature Σ is of presheaf type if and only if every geometric formula $\phi(\vec{x})$ over Σ, when regarded as an object of $\mathscr{C}_{\mathbb{T}}$, is $J_{\mathbb{T}}$-covered by $\mathbb{T}$-irreducible formulae over Σ.

Theorem

A geometric theory $\mathbb{T}$ over a signature Σ is of presheaf type if and only if the following conditions are satisfied:

(i) *Every finitely presentable model is presented by a geometric formula over Σ.*

(ii) *Every property of finite tuples of elements of a finitely presentable $\mathbb{T}$-model which is preserved by $\mathbb{T}$-model homomorphisms is definable (in finitely presentable $\mathbb{T}$-models) by a geometric formula over Σ.*

(iii) *The finitely presentable $\mathbb{T}$-models are jointly conservative for $\mathbb{T}$.*

I have also established a characterization theorem providing necessary and sufficient semantic conditions for a theory to be of presheaf type.

'Bridges' between quotients and topologies

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the syntactic-equivalence classes of quotients (i.e. geometric theory extensions over the same signature) of $\mathbb{T}$ and the subtoposes of the classifying topos $\mathbf{Set}[\mathbb{T}]$ of $\mathbb{T}$.

This duality allows one in particular to establish 'bridges' of the following form:

Subtopos of
$\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Set}[\mathbb{T}]$

Grothendieck topology on $\mathscr{C}$ containing J — *Quotient of $\mathbb{T}$*

That is, if the classifying topos of a geometric theory $\mathbb{T}$ can be represented as the category $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a (small) site $(\mathscr{C}, J)$ then we have a natural, order-preserving bijection

$$\text{quotients of } \mathbb{T}$$
$$|||$$
$$\text{Grothendieck topologies on } \mathscr{C} \text{ which contain } J$$

Two notable cases

This result can be applied in particular in the following two cases:

(1) $(\mathscr{C}, J)$ is the syntactic site $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$

(2)
- $\mathbb{T}$ is a theory of presheaf type,
- $\mathscr{C}$ is the opposite of its category f.p.$\mathbb{T}$-mod(**Set**) of finitely presentable models, and
- J is the trivial topology on it.

In the first case, we obtain an order-preserving bijective correspondence between the quotients of $\mathbb{T}$ and the Grothendieck topologies on $\mathscr{C}_{\mathbb{T}}$ which contain $J_{\mathbb{T}}$.

In the second case, we obtain an order-preserving bijective correspondence between the quotients of $\mathbb{T}$ and the Grothendieck topologies on f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$.

In both cases, these correspondences can be naturally interpreted as proof-theoretic equivalences between the classical proof system of geometric logic over $\mathbb{T}$ and new proof systems for sieves whose inference rules correspond to the axioms of Grothendieck topologies.

Quotients of a theory of presheaf type I

The Grothendieck topology J on f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{\text{op}}$ associated with a quotient $\mathbb{T}'$ of a theory of presheaf type $\mathbb{T}$ can be explicitly described as follows.

- By using the fact that every geometric formula over Σ can be $J_{\mathbb{T}}$-covered in $\mathscr{C}_{\mathbb{T}}$ by $\mathbb{T}$-irreducible formulae, one can show that every geometric sequent over Σ is provably equivalent in $\mathbb{T}$ to a collection of sequents σ of the form $(\phi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{y_i})\theta_i)$ where, for each $i \in I$, $[\theta_i] : \{\vec{y_i} \,.\, \psi_i\} \to \{\vec{x} \,.\, \phi\}$ is an arrow in $\mathscr{C}_{\mathbb{T}}$ and $\phi(\vec{x})$, $\psi(\vec{y_i})$ are geometric formulae over Σ presenting respectively $\mathbb{T}$-models $M_{\{\vec{x} \,.\, \phi\}}$ and $M_{\{\vec{y_i} \,.\, \psi_i\}}$.
- To such a sequent σ, we can associate the cosieve S_σ on $M_{\{\vec{x} \,.\, \phi\}}$ in f.p.$\mathbb{T}$-mod$(\mathbf{Set})$ generated by the arrows s_i defined as follows. For each $i \in I$, $[[\theta_i]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}}$ is the graph of a morphism $[[\vec{y_i} \,.\, \psi_i]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}} \to [[\vec{x} \,.\, \phi]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}}$; then the image of the generators of $M_{\{\vec{y_i} \,.\, \psi_i\}}$ via this morphism is an element of $[[\vec{x} \,.\, \phi]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}}$ and this in turn determines, by definition of $M_{\{\vec{x} \,.\, \phi\}}$, a unique arrow $s_i : M_{\{\vec{x} \,.\, \phi\}} \to M_{\{\vec{y_i} \,.\, \psi_i\}}$ in $\mathbb{T}$-mod$(\mathbf{Set})$.
- Conversely, by the equivalence f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{\text{op}} \simeq \mathscr{C}_{\mathbb{T}}^{\text{irr}}$, every sieve in f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{\text{op}}$ is of the form S_σ for such a sequent σ.

Quotients of a theory of presheaf type II

The Grothendieck topology J on f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{\text{op}}$ associated with a quotient $\mathbb{T}'$ of $\mathbb{T}$ is generated by the sieves S_σ, where σ varies among the sequents of the required form which are equivalent to the axioms of $\mathbb{T}'$.

The equivalence

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

of classifying toposes for $\mathbb{T}$ restricts to an equivalence

$$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$$

of classifying toposes for $\mathbb{T}'$.
In particular, for any σ of the above form, σ is provable in $\mathbb{T}'$ if and only if S_σ belongs to J.

These equivalences are useful in that they allow us to study (the proof theory of) geometric theories through the associated Grothendieck topologies: the condition of provability of a sequent in a geometric theory gets transformed in the requirement for a sieve (or a family of sieves) to belong to a certain Grothendieck topology, something which is often much easier to investigate.

The Zariski topos

Let Σ be the one-sorted signature for the theory $\mathbb{T}$ of commutative rings with unit i.e. the signature consisting of two binary function symbols $+$ and $\cdot$, one unary function symbol $-$ and two constants 0 and 1.
The coherent theory of local rings is obtained from $\mathbb{T}$ by adding the sequents

$$((0 = 1) \vdash_{[]} \bot)$$

and

$$((\exists z)((x + y) \cdot z = 1) \vdash_{x,y} ((\exists z)(x \cdot z = 1) \vee (\exists z)(y \cdot z = 1))),$$

Definition

The Zariski topos is the topos $\mathbf{Sh}(\mathbf{Rng}_{f.g.}^{\mathrm{op}}, J)$ of sheaves on the opposite of the category $\mathbf{Rng}_{f.g.}$ of finitely generated rings with respect to the topology J on $\mathbf{Rng}_{f.g.}^{\mathrm{op}}$ defined by: given a cosieve S in $\mathbf{Rng}_{f.g.}$ on an object A, $S \in J(A)$ if and only if S contains a finite family $\{\xi_i : A \to A[{s_i}^{-1}] \mid 1 \leq i \leq n\}$ of canonical inclusions $\xi_i : A \to A[{s_i}^{-1}]$ in $\mathbf{Rng}_{f.g.}$ where $\{s_1, \ldots, s_n\}$ is any set of elements of A which is not contained in any proper ideal of A.

Fact

The (coherent) theory of local rings is classified by the Zariski topos.

The classifying topos for integral domains

The theory of integral domains is the theory obtained from the theory of commutative rings with unit by adding the axioms

$$((0 = 1) \vdash_{[]} \bot)$$

$$((x \cdot y = 0) \ \vdash_{x,y} ((x = 0) \vee (y = 0)))\,.$$

Fact

The theory of integral domains is classified by the topos $\mathbf{Sh}(\mathbf{Rng}_{f.g.}^{\mathrm{op}}, J)$ *of sheaves on the opposite of the category* $\mathbf{Rng}_{f.g.}$ *of finitely generated rings with respect to the topology* J *on* $\mathbf{Rng}_{f.g.}^{\mathrm{op}}$ *defined by: given a cosieve* S *in* $\mathbf{Rng}_{f.g.}$ *on an object* A*,* $S \in J_2(A)$ *if and only if*

- *either* A *is the zero ring and* S *is the empty sieve on it or*
- S *contains a non-empty finite family* $\{\pi_{a_i} : A \to A/(a_i) \mid 1 \leq i \leq n\}$ *of canonical projections* $\pi_{a_i} : A \to A/(a_i)$ *in* $\mathbf{Rng}_{f.g.}$ *where* $\{a_1, \ldots, a_n\}$ *is any set of elements of* A *such that* $a_1 \cdot \ldots \cdot a_n = 0$*.*

Future directions

The evidence provided by the results obtained so far shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories and for generating new equivalences, dualities and symmetries across different fields of Mathematics.

In fact, toposes have an authentic creative power in Mathematics, in the sense that their study naturally leads to the discovery of a great number of notions and 'concrete' results in different mathematical fields, which are pertinent but often unsuspected.

In the next years, we intend to continue pursuing the development of these general unifying methodologies both at the theoretical level and at the applied level, in order to continue developing the potential of toposes as fundamental tools in the study of mathematical theories and their relations, and as key concepts defining a new way of doing Mathematics liable to bring distinctly new insights in a great number of different subjects.

Future directions

Central themes in this programme will be:

- investigation of important dualities or correspondences in Mathematics from a topos-theoretic perspective (in particular, the theory of motives, class field theory and the Langlands programme)
- systematic study of invariants of toposes in terms of their presentations, and introduction of new invariants which capture important aspects of concrete mathematical problems
- interpretation and generalization of important parts of classical and modern model theory in terms of toposes and development of a functorial (topos-theoretic) model theory
- introduction of new methodologies for generating Morita-equivalences
- development of general techniques for building spectra by using classifying toposes
- generalization of the 'bridge' technique to the setting of higher categories and toposes through the introduction of higher geometric logic
- development of a relative theory of classifying toposes

An approach to stability theory

In the context of our investigation of classical and modern parts of model theory from a topos-theoretic perspective, we plan in particular to address stability theory. Our approach to this subject will be based on the following ingredients:

- Classification of the topological and localic groupoids representing the classifying topos of a geometric theory (extending and generalizing the works of Joyal-Tierney and Butz-Moerdijk).
- Investigation of the relationship between the classifying topos and Makkai's topos of types from multiple points of view (based in particular on D. Coumans' result expressing the latter in terms of the former as the localic part of the hyperconnected-localic factorization of the surjection to the classifying topos corresponding to a class of special models of the theory).
- Application of the 'bridge' technique in the context of these toposes, in order to obtain relationships between the geometry of types of the theory and the isomorphism classes of its set-based models (by considering, for instance, how the connected components of the groupoids representing the toposes can be captured through suitable topos-theoretic invariants and how the latter can in turn be expressed in syntactic terms).

For further reading

- O. Caramello
  Grothendieck toposes as unifying 'bridges' in Mathematics,
  Mémoire d'habilitation à diriger des recherches,
  Université de Paris 7 (2016),
  available from my website www.oliviacaramello.com.

- O. Caramello
  Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges',
  Oxford University Press (2017).

- S. Mac Lane and I. Moerdijk.
  Sheaves in geometry and logic: a first introduction to topos theory
  Springer-Verlag (1992).