

Duality and Morita-equivalence: a topos-theoretic perspective

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Duality and Morita-equivalence

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Topos à l'IHES

In this lecture I shall approach the subject of **duality** and **Morita-equivalence** from the perspective provided by the theory of topos-theoretic 'bridges' (introduced in 2010 in the paper "*The unification of Mathematics via Topos Theory*" in 2010 and further developed in my forthcoming book for Oxford University Press).

Throughout the past five years, this theory has found several **applications** in different mathematical domains (including Algebra, Geometry, Topology, Functional Analysis, Model Theory and Proof Theory), which illustrate the importance of the notion of (topos-theoretic) Morita-equivalence and demonstrate that this concept has **close ties** with that of duality.

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The term *duality* is used in Mathematics with many different meanings; for instance, it may refer to:

- a **contravariant** equivalence between a category of 'algebraic' (resp. 'syntactic') objects and a category of 'geometric or topological' (resp. 'semantical') objects;
 - a **covariant** equivalence between 'concrete' categories;
- or more loosely to
- the possibility of regarding a mathematical object from two (or more) different points of view (or of representing it in multiple ways).

Morita-equivalence

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The term *Morita-equivalence* also arises in different mathematical contexts (for instance, in Algebra, Functional Analysis, Geometry etc.) with different technical meanings according to the specific mathematical entities involved.

Broadly speaking, one can say that it consists in the possibility of relating two mathematical objects with each other by means of a *third object*, usually of a different mathematical kind, which can be built separately from each of them and hence admits two different representations (up to a certain notion of equivalence), one in terms of the first object and one in terms of the second.

Morita-equivalences can be understood and investigated in a most effective way in the context of **topos theory** (or its generalisations such as higher topos theory or sketch theory).

Classifying toposes

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It was realized in the seventies (thanks to the work of several people, notably including W. Lawvere, A. Joyal, G. Reyes and M. Makkai) that:

- To any (geometric first-order) mathematical theory \mathbb{T} one can canonically associate a topos $\mathcal{E}_{\mathbb{T}}$, called the **classifying topos** of the theory, which represents its 'semantical core'.
- The topos $\mathcal{E}_{\mathbb{T}}$ is characterized by the following universal property: for any Grothendieck topos \mathcal{E} we have an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathcal{E})$$

natural in \mathcal{E} , where $\mathbf{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}})$ is the category of geometric morphisms $\mathcal{E} \rightarrow \mathcal{E}_{\mathbb{T}}$ and $\mathbb{T}\text{-mod}(\mathcal{E})$ is the category of \mathbb{T} -models in \mathcal{E} .

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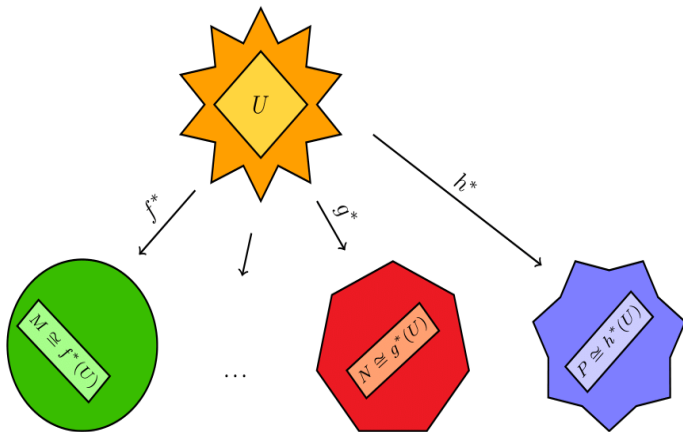
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- Two mathematical theories have the same classifying topos (up to equivalence) if and only if they have the same 'semantical core', that is if and only if they are indistinguishable from a semantic point of view; such theories are said to be **Morita-equivalent**.
- Conversely, every Grothendieck topos arises as the classifying topos of some theory.
- So a topos can be seen as a **canonical representative** of equivalence classes of theories modulo Morita-equivalence.

Morita-equivalence for geometric theories

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Two geometric theories \mathbb{T} and \mathbb{T}' are **Morita-equivalent** if they have equivalent classifying toposes, equivalently if they have equivalent categories of models in every Grothendieck topos \mathcal{E} , naturally in \mathcal{E} , that is for each Grothendieck topos \mathcal{E} there is an equivalence of categories

$$\tau_{\mathcal{E}} : \mathbb{T}\text{-mod}(\mathcal{E}) \rightarrow \mathbb{T}'\text{-mod}(\mathcal{E})$$

such that for any geometric morphism $f : \mathcal{F} \rightarrow \mathcal{E}$ the following diagram commutes (up to isomorphism):

$$\begin{array}{ccc} \mathbb{T}\text{-mod}(\mathcal{E}) & \xrightarrow{\tau_{\mathcal{E}}} & \mathbb{T}'\text{-mod}(\mathcal{E}) \\ \downarrow f^* & & \downarrow f^* \\ \mathbb{T}\text{-mod}(\mathcal{F}) & \xrightarrow{\tau_{\mathcal{F}}} & \mathbb{T}'\text{-mod}(\mathcal{F}) \end{array}$$

Toposes as *bridges*

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- The existence of **different theories** with the same classifying topos translates, at the technical level, into the existence of **different representations** (technically speaking, sites of definition) for the same topos.
- Topos-theoretic invariants can thus be used to transfer information from one theory to another:

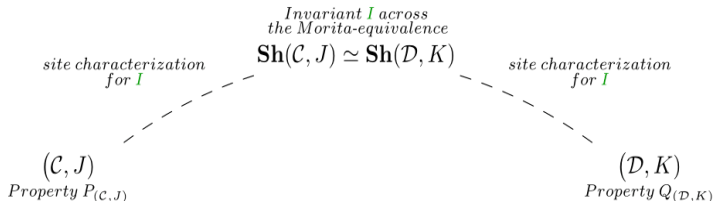
$$\mathbb{T} \overset{\mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'}}{\curvearrowright} \mathbb{T}'$$

- The **transfer of information** takes place by expressing a given invariant in terms of the different representation of the topos.

Toposes as *bridges*

More specifically, the 'bridge-building' technique works as follows:

- **Decks** of 'bridges': **Morita-equivalences** (or more generally morphisms or other kinds of relations between toposes)
- **Arches** of 'bridges': **Site characterizations** (or more generally 'unravelings' of topos-theoretic invariants in terms of concrete representations of the relevant topos)



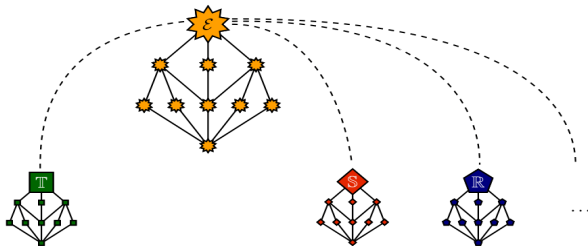
The 'bridge' yields a logical equivalence (or an implication) between the 'concrete' properties $P_{(\mathcal{C}, J)}$ and $Q_{(\mathcal{D}, K)}$, interpreted in this context as **manifestations** of a **unique** property I lying at the level of the topos.

Toposes as *bridges*

- In such a setup, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.
- Any topos-theoretic invariant behaves in this context like a 'pair of glasses' which allows to discern certain information which is 'hidden' in the given Morita-equivalence; different invariants allow to **transfer** different information.
- This methodology is technically effective because the relationship between a topos and its representations is often **very natural**, enabling us to easily **transfer invariants** across different representations (and hence, between different theories).
- The **level of generality** represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories. Indeed, as shown in my papers, important topos-theoretic invariants considered on the classifying topos $\mathcal{E}_{\mathbb{T}}$ of a geometric theory \mathbb{T} translate into interesting logical (i.e. syntactic or semantic) properties of \mathbb{T} .

Toposes as *bridges*

- The fact that topos-theoretic invariants specialize to important properties or constructions of natural mathematical interest is a clear indication of the **centrality** of these concepts in Mathematics. In fact, whatever happens at the level of toposes has '**uniform**' ramifications in Mathematics as a whole: for instance



Lattices of theories

This picture represents the lattice structure on the collection of the subtoposes of a topos \mathcal{E} inducing lattice structures on the collection of 'quotients' of geometric theories \mathbb{T} , \mathbb{S} , \mathbb{R} classified by it.

Toposes as *bridges* and the Erlangen Program

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The methodology 'toposes as bridges' is a vast extension of Felix Klein's Erlangen Program (A. Joyal)

More specifically:

- Every **group** gives rise to a **topos** (namely, the category of actions on it), but the notion of topos is much more general.
- As Klein classified geometries by means of their **automorphism groups**, so we can study first-order geometric theories by studying the associated **classifying toposes**.
- As Klein established surprising connections between very different-looking geometries through the study of the **algebraic properties** of the associated automorphism groups, so the methodology 'toposes as bridges' allows to discover non-trivial connections between properties, concepts and results pertaining to different mathematical theories through the study of the **categorical invariants** of their classifying toposes.

Dualities from topos-theoretic 'bridges'

- Categorical dualities or equivalences between 'concrete' categories can often be seen as arising from the process of 'functorializing' topos-theoretic 'bridges', whose underlying Morita-equivalences express structural relationships between each pair of objects corresponding to each other under the given duality or equivalence.
- Different sites of definition for a given topos can be interpreted logically as Morita-equivalences between different theories. The [representation theory of Grothendieck toposes](#) in terms of sites and, more generally, any technique that one may employ for obtaining a different representation for a given topos thus constitutes an effective tool for generating Morita-equivalences.
- Any mathematical object used for representing the classifying topos of a theory (resp. the topos of sheaves on a site whose underlying category can be recovered from it up to equivalence) can be seen as a [spectrum](#) for that theory (resp. for that category).
- Dualities can of course be obtained also by simply [specialising](#) Morita-equivalences (to a particular topos).

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- If two geometric theories \mathbb{T} and \mathbb{T}' have equivalent categories of models in the category **Set** then, provided that the given categorical equivalence is established by only using **constructive logic** (that is, by avoiding in particular the law of excluded middle and the axiom of choice) and **geometric constructions**, it is reasonable to expect the original equivalence to 'lift' to a Morita-equivalence between \mathbb{T} and \mathbb{T}' .
- Two **associative rings with unit** are Morita-equivalent (in the classical, ring-theoretic, sense) if and only if the algebraic theories axiomatizing the (left) modules over them are Morita-equivalent (in the topos-theoretic sense).
- Other notions of Morita-equivalence for various kinds of algebraic or geometric structures considered in the literature (for instance, **topological groups**, **localic or topological groupoids**, **small categories**, **inverse semigroups**) can be reformulated as equivalences between different representations of the same topos, and hence as Morita-equivalences between different geometric theories.

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- Given a mathematical duality, for each pair of objects related by it there should be a Morita-equivalence expressing the **structural relationships** between them. It is worth to try to identify such Morita-equivalences, since they provide new tools for transferring properties and results across the two objects by means of topos-theoretic 'bridges'.
- The notion of Morita-equivalences materializes in many situations the intuitive feeling of '**looking at the same thing in different ways**', meaning, for instance, describing the same structure(s) in different languages or constructing a given object in different ways. For instance, the different constructions of the Zariski spectrum of a ring, of the Gelfand spectrum of a C^* -algebra, and of the Stone-Cech compactification of a topological space can be interpreted as Morita-equivalences between different theories.
- Any two theories which are **biinterpretable** are Morita-equivalent but, very importantly, the converse does not hold.

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- Different ways of looking at a given mathematical theory can often be formalized as Morita-equivalences. Indeed, different ways of describing the structures axiomatized by a given theory can often give rise to a theory written in a different language whose models (in any Grothendieck topos) can be identified, in a natural way, with those of the former theory and which is therefore Morita-equivalent to it.
- A geometric theory **alone** generates an infinite number of Morita-equivalences, via its 'internal dynamics'. In fact, any way of looking at a geometric theory as an extension of a geometric theory written in its signature provides a different representation of its classifying topos, as a subtopos of the classifying topos of the latter theory.
- The usual notions of **spectra** for mathematical structures can be naturally interpreted in terms of classifying toposes, and the resulting sheaf representations as arising from Morita-equivalences between an 'algebraic' and a 'topological' representation of such toposes.

Dualities *versus* Morita-equivalences

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- If dualities arise from the functorialization of a particular kind of topos-theoretic 'bridge', their interest lies essentially in their **global, categorical** nature. Indeed, the transfer of information that they allow is realized by means of the functors defining the equivalence, that is in a **linear, dictionary-like** way. In particular, the transfer of properties and results across two particular objects related by the duality necessarily relies on the identification of some categorically invariant property that these objects satisfy inside their respective categories.
- On the other hand, a Morita-equivalence represents a concrete 'incarnation' of the structural relationships existing between the two sites of definition of the given topos, or between the two theories classified by that topos. The translations realised by means of topos-theoretic 'bridges' are no longer dictionary-oriented: they are **invariant-oriented**. As such, they are in general much **deeper** than the ones realised by dualities, in the sense that they are susceptible of transforming the shape of the properties in a very substantial way.

Galois theory as a Morita-equivalence

Theorem

Let \mathbb{T} be a theory of presheaf type such that its category $f.p.\mathbb{T}\text{-mod}(\mathbf{Set})$ of finitely presentable models satisfies AP and JEP, and let M be a $f.p.\mathbb{T}\text{-mod}(\mathbf{Set})$ -universal and $f.p.\mathbb{T}\text{-mod}(\mathbf{Set})$ -ultrahomogeneous model of \mathbb{T} . Then the theory of homogeneous \mathbb{T} -models is *complete* and we have an *equivalence of toposes*

$$\mathbf{Sh}(f.p.\mathbb{T}\text{-mod}(\mathbf{Set})^{op}, J_{at}) \simeq \mathbf{Cont}(Aut(M)),$$

where $Aut(M)$ is endowed with the topology of pointwise convergence.

This equivalence is induced by the functor

$$F : f.p.\mathbb{T}\text{-mod}(\mathbf{Set})^{op} \rightarrow \mathbf{Cont}(Aut(M))$$

sending any model c of $f.p.\mathbb{T}\text{-mod}(\mathbf{Set})$ to the set $\text{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(c, M)$ (endowed with the obvious action by $Aut(M)$) and any arrow $f : c \rightarrow d$ in $f.p.\mathbb{T}\text{-mod}(\mathbf{Set})$ to the $Aut(M)$ -equivariant map

$$- \circ f : \text{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(d, M) \rightarrow \text{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(c, M).$$

Topos-theoretic 'bridges' for Galois

The following result arises from two bridges, obtained respectively by considering the invariant notions of **atom** and of **arrow between atoms**.

Theorem

*Under the hypotheses of the last theorem, the functor F is **full and faithful** if and only if every arrow of $f.p.\mathbb{T}\text{-mod}(\mathbf{Set})$ is a **strict monomorphism** and it is an **equivalence** onto the full subcategory $\mathbf{Cont}_t(\mathbf{Aut}(M))$ of $\mathbf{Cont}(\mathbf{Aut}(M))$ on the transitive actions if moreover $f.p.\mathbb{T}\text{-mod}(\mathbf{Set})$ is **atomically complete**.*

$$\begin{array}{ccc} & \mathbf{Sh}(f.p.\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(\mathbf{Aut}(M)) & \\ \text{---} & & \text{---} \\ f.p.\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} & & \mathbf{Cont}_t(\mathbf{Aut}(M)) \end{array}$$

This theorem generalizes **Grothendieck's theory of Galois categories** and can be applied to obtain Galois-type theories in different fields of Mathematics, for instance one for **finite groups** and one for **finite graphs**.

Stone-type dualities

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All the classical Stone-type dualities/equivalences between special kinds of preorders and locales or topological spaces can be obtained by **functorializing** 'bridges' of the form

$$\mathcal{C} \text{ --- } \mathbf{Sh}(\mathcal{C}, J_{\mathcal{C}}) \simeq \mathbf{Sh}(\mathcal{D}, K_{\mathcal{D}}) \text{ --- } \mathcal{D}$$

where \mathcal{D} is a $J_{\mathcal{C}}$ -dense subcategory of a preorder category \mathcal{C} .

For instance, take \mathcal{D} equal to a Boolean algebra and \mathcal{C} equal to the lattice of open sets of its Stone space for **Stone duality**, \mathcal{C} equal to an atomic complete Boolean algebra and \mathcal{D} equal to the collection of its atoms for **Lindenbaum-Tarski duality**.

Stone-type dualities

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- This method actually allows to **recover** all the **classical Stone-type dualities** (for instance, Alexandrov duality, Moshier-Jipsen's duality, Birkhoff duality, the duality between algebraic lattices and sup-semilattices, the duality between completely distributive algebraic lattices and posets, the duality between spatial frames and sober spaces etc.).
- Moreover, it allows to generate many new dualities for different kinds of pre-ordered structures (for instance, a localic duality for **meet-semilattices**, a duality for **k -frames**, a duality for **disjunctively distributive lattices**, a duality for **preframes generated by their directedly irreducible elements** etc. It also naturally generalizes to the setting of arbitrary categories.
- By using the same methodology, I have also obtained a new duality for compact Hausdorff spaces (with a category of Alexandrov algebras), and another one for T_1 compact spaces.

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- A list of papers is available from my website www.oliviacaramello.com.

In particular, the examples mentioned above can be found in the papers

- “*A topos-theoretic approach to Stone-type dualities*”
 - “*Gelfand spectra and Wallman compactifications*”
 - “*Topological Galois theory*”.
- A **book** for Oxford University Press provisionally entitled *Lattices of Theories* will appear in a few months.

International conference on topos theory

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*Tous les chevaux du roi
y pourraient boire ensemble*
Alexander Grothendieck

Everyone is welcome!