Diagrams for meta-mathematics

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Diagrammatic reasoning

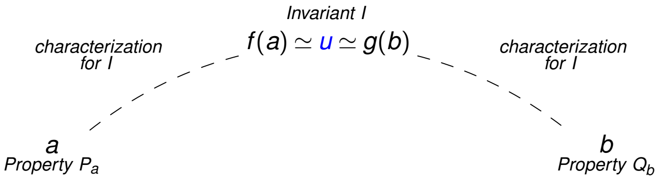
- In mathematics, a visual approach, based on a diagrammatic or schematic representation of knowledge, can be extremely useful in clarifying the key ideas or aspects of a proof, distinguishing them from more routine or technical aspects, highlighting the underlying conceptual architecture, including the different levels of abstraction involved, and thus achieving a high degree of explainability.
- More generally, experience has shown that interdisciplinary dialogue becomes increasingly feasible with the use of tools that operate at a meta or methodological level and which capture the dynamics of a problem or situation in a single picture or through a few concise gestures, offering an overarching view of the phenomenon without the need to delve into technical details.

Unification and dynamical thinking

- Knowledge transfers between different fields occurs through the identification of invariants, i.e. those abstract and essential aspects of things, which we encounter concretely through their different manifestations but which truly exist at the meta level.
- Creating conceptual architectures – both within individual disciplines and at a broader meta level – alongside abstract, schematic representations of these structures, emerges as a crucial instrument in this endeavor. The strength of a visual or artistic approach lies not merely in depicting outcomes already achieved, but more importantly in fostering a relational and dynamic understanding of reality.
- I will illustrate these principles by discussing two notable examples of diagrammatic meta-mathematical reasoning from my own research work, namely bridges and relativity.

Bridge objects

- Given two objects a and b, imagine to be able to associate with a an object $f(a)$ through a certain 'construction' f and with b an object $g(b)$ through a 'construction' g, in such a way that $f(a)$ and $g(b)$ be related by a certain equivalence relation $\simeq$. Then a and b can be seen as two distinct representations of a unique object u, equivalent on the one hand to $f(a)$ and on the other hand to $g(b)$, which can be used as a 'bridge' for transferring invariants across its different presentations:



- In the topos-theoretic implementation of the 'bridge' technique, the objects a and b are formalized contexts or theories while $f(a)$ and $g(b)$ are toposes associated with them. Note that a and b may also be seen as different 'points of view' on the bridge object u.

Examples of bridges

Geometric morphism from $\mathscr{E}$ to

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J\text{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J\text{at})$ homogeneous $\mathbb{T}$-model in $\mathscr{E}$

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$ $\mathbb{T}'$-model in $\mathscr{E}$

Atomic topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}})$ *Atomic site i.e.* AP on $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$ *Atomicity of* $\mathbb{T}'$

Two-valued topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}})$ JEP on $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$ *Completeness of* $\mathbb{T}'$

These bridges notably lead to the following wide generalization of Fraïssé's theorem in Model Theory:

Theorem (O.C.)

Let $\mathbb{T}$ be a theory of presheaf type such that the category $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ is non-empty and satisfies the amalgamation and joint embedding properties. Then the theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models is complete and atomic.

Algebra/Topology bridges

The following are other instances of bridges, arising in the context of the topos-theoretic interpretation of Stone-type dualities:

$$
\begin{array}{ccc}
\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}}) & \simeq & \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}}) \\
\downarrow & & \downarrow \\
\mathbf{Sh}(\mathscr{C}', J_{\mathscr{C}'}) & \simeq & \mathbf{Sh}(\mathscr{D}', K_{\mathscr{D}'})
\end{array}
$$

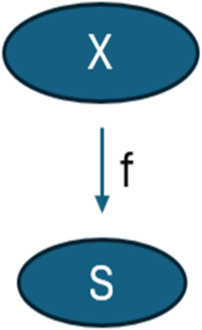
$$
\begin{array}{ccc}
\mathscr{C} & \qquad & \mathscr{D} \\
\uparrow & & \downarrow \\
\mathscr{C}' & & \mathscr{D}'
\end{array}
$$

N.B. With respect to the previous examples, the fields of mathematics are completely different, but the conceptual architecture is the *same*!

The relativity method

In Mathematics the relativity method, pioneered by Grothendieck in the context of his refoundation of Algebraic Geometry in the language of schemes, consists in trying to state notions and results in terms of morphisms, rather than objects, of a given category, so that they can be 'relativized' to an arbitrary base object.

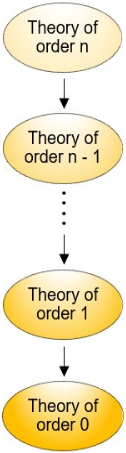
One works in the new base universe as if it were the 'classical' one, and then interprets the obtained results from the point of view of the original universe. This process is called externalization. Conversely, interpreting a concrete result - formulated in the setting of a given universe - from the perspective of a richer base universe constitutes an instance of internalization.



Relative toposes, relative theories

The power of the relativity method lies in the very high degree of technical flexibility it provides, resulting from the possibility of 'encapsulating' part of the complexity of a situation in the base topos, so that the given notions acquire a simpler (e.g. a lower-degree) expression with respect to it.

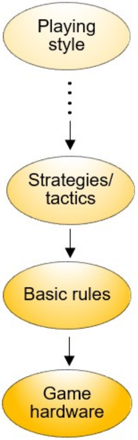
For instance, one can apply it to reduce the logical complexity of theories, yielding a logical version of Grothendieck's celebrated "dévissage" technique in scheme theory. For instance, a first-order theory can be seen as a propositional one relative to a base topos which is richer than the classical topos of sets.



The logic of games

Games can also be conveniently modelized in terms of chains of relative toposes, with the ground topos formalizing the "hardware of the game", the level 1 topos the basic, "first-order rules" of the game, and the higher-level relative toposes the "higher-level rules" (i.e. tactics or strategies).

Each morphism in the chain encodes the way in which the rules at that level are concretely realized in terms of lower-level ones; in particular, the morphism from the level 1 topos to the base topos expresses the way in which the basic rules (e.g. those governing the beheviour of the "pieces" of the game) are "arithmetically encoded" in the game's hardware.



Modelling stratified phenomena

The same architecture (chains/diagrams of relative entities) emerges in other fields, e.g. in

- AI (modelling of meta-learning processes, i.e. processes taking place at different levels of knowledge or abstraction constructed on top of each other);
- psychology (formalization of Raven matrices);
- biology (structure of organisms);
- linguistics (structure of languages).

Raven matrices	Natural language	Biology
Matrices	Texts	Living being
Rows	Words	Organs
Cells	Letters	Tissues
Cell inner structure	Alphabet	Cells

For further reading

O. Caramello
Grothendieck toposes as unifying 'bridges' in Mathematics,
Mémoire d'habilitation à diriger des recherches,
Université de Paris 7, 2016,
available from my website www.oliviacaramello.com.

O. Caramello
Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges',
Oxford University Press, 2017.

The Grothendieck Institute

- These projects are notably developed at the Grothendieck Institute, an international foundation devoted to cutting-edge research in mathematics and its interactions with other disciplines.
- Established in 2022 and based in Mondovì (Italy), the Institute is named after Alexander Grothendieck, whose work it is committed to valorize and disseminate.

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- Its main focus is interdisciplinarity, with particular reference to the development of unifying methods, both within mathematics and in relation to other areas of knowledge.

The Grothendieck Institute

- The Institute pursues its mission notably through its research centres, and by collaborating with academic institutions and scientific associations which share its interests.
- It also offers doctoral studentships and research fellowships to outstanding young scholars selected by its Scientific Council, which includes among its members three Fields Medalists.
- The Institute's Centre for Topos Theory and its Applications organized in September 2024 the fourth edition of the international school and conference on topos theory, with more than 120 participants from 30 countries and 5 continents - www.ctta.igrothendieck.org.
- The philosophical and interdisciplinary aspects of Grothendieck's work are notably investigated at the Institute's Centre for Grothendiecian Studies: www.csg.igrothendieck.org

If you wish to collaborate with us, don't hesitate to get in touch!