Deductive systems and Grothendieck topologies

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Aim of the talk

The purpose of this talk is to illustrate the proof-theoretic relevance of the notion of Grothendieck topology.

We will show that the classical proof system of geometric logic over a given geometric theory is equivalent to new proof systems based on the notion of Grothendieck topology.

These equivalences result from a proof-theoretic interpretation of a duality between the quotients (i.e. geometric theory extensions over the same signature) of a given geometric theory and the subtoposes of its classifying topos.

Interestingly, these alternative proof systems turn out to be computationally better-behaved than the classical one for many purposes, as we shall illustrate by discussing a few selected applications.

Before describing these results, we need to review the necessary background.

Geometric theories

Definition

- A geometric formula over a signature Σ is any formula (with a finite number of free variables) built from atomic formulae over Σ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.
- A geometric theory over a signature Σ is any theory whose axioms are of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are geometric formulae over Σ and $\vec{x}$ is a context suitable for both of them.

Fact

Most of the theories naturally arising in Mathematics are geometric; and if a finitary first-order theory is not geometric, we can always associate to it a finitary geometric theory over a larger signature (the so-called Morleyization of the theory) with essentially the same models in the category **Set** *of sets.*

The syntactic category of a geometric theory

Definition (Makkai and Reyes 1977)

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. The syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the 'renaming'-equivalence classes of geometric formulae-in-context $\{\vec{x} . \phi\}$ over Σ and as arrows $\{\vec{x} . \phi\} \to \{\vec{y} . \psi\}$ (where the contexts $\vec{x}$ and $\vec{y}$ are supposed to be disjoint without loss of generality) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\vec{x}, \vec{y})$ which are $\mathbb{T}$-provably functional i.e. such that the sequents

$$(\phi \vdash_{\vec{x}} (\exists y)\theta),$$
$$(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi), \text{ and}$$
$$((\theta \wedge \theta[\vec{z}/\vec{y}]) \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{y} = \vec{z}))$$

  are provable in $\mathbb{T}$.
- The composite of two arrows

$$\{\vec{x} . \phi\} \xrightarrow{[\theta]} \{\vec{y} . \psi\} \xrightarrow{[\gamma]} \{\vec{z} . \chi\}$$

  is defined as the $\mathbb{T}$-provable-equivalence class of the formula $(\exists \vec{y})\theta \wedge \gamma$.
- The identity arrow on an object $\{\vec{x} . \phi\}$ is the arrow

$$\{\vec{x} . \phi\} \xrightarrow{[\phi \wedge \vec{x'} = \vec{x}]} \{\vec{x'} . \phi[\vec{x'}/\vec{x}]\}$$

Grothendieck topologies

Definition

A Grothendieck topology on a small category $\mathscr{C}$ is a function J which assigns to each object c of $\mathscr{C}$ a collection $J(c)$ of sieves on c in such a way that

1. (maximality axiom) the maximal sieve $M_c = \{f \mid cod(f) = c\}$ is in $J(c)$;
2. (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;
3. (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

The sieves S which belong to $J(c)$ for some object c of $\mathscr{C}$ are said to be J-covering.

A site is a pair $(\mathscr{C}, J)$ consisting of a category $\mathscr{C}$ and a Grothendieck topology J on $\mathscr{C}$.

Examples of Grothendieck topologies

- For any (small) category $\mathscr{C}$, the trivial topology on $\mathscr{C}$ is the Grothendieck topology in which the only sieve covering an object c is the maximal sieve M_c.
- The dense topology D on a category $\mathscr{C}$ is defined by: for a sieve S,

$$S \in D(c) \quad \text{if and only if} \quad \begin{array}{l} \text{for any } f : d \to c \text{ there exists} \\ g : e \to d \text{ such that } f \circ g \in S\,. \end{array}$$

- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathscr{O}(X)}$ on the poset category $\mathscr{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathscr{O}(X))$,

$$S \in J_{\mathscr{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U\,.$$

- More generally, given a frame H, i.e. a Heyting algebra with arbitrary joins $\bigvee$ (and meets), we can define a Grothendieck topology J_H on H by:

$$\{a_i \leq a \mid i \in I\} \in J_H(a) \text{ if and only if } \bigvee_{i \in I} a_i = a\,.$$

Syntactic sites

- For any geometric theory $\mathbb{T}$, its syntactic category $\mathscr{C}_{\mathbb{T}}$ is a geometric category, i.e. a well-powered cartesian category in which images of morphisms and arbitrary unions of subobjects exist and are stable under pullback.
- For a geometric theory $\mathbb{T}$, the geometric topology on $\mathscr{C}_{\mathbb{T}}$ is the Grothendieck topology $J_{\mathbb{T}}$ whose covering sieves are those which contain small covering families.

Definition

The syntactic topology $J_{\mathbb{T}}$ on the syntactic category $\mathscr{C}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ is the geometric topology on it; in particular,

a small family $\{[\theta_i] : \{\vec{x}_i . \phi_i\} \to \{\vec{y} . \psi\}\}$ in $\mathscr{C}_{\mathbb{T}}$ is $J_{\mathbb{T}}$-covering

if and only if

the sequent $(\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x}_i)\theta_i)$ is provable in $\mathbb{T}$.

Grothendieck toposes

One can define sheaves on an arbitrary site in a formally analogous way to how one defines sheaves on a topological space. This leads to the following

Definition

- A Grothendieck topos is a category (equivalent to the category) $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a (small-generated) site $(\mathscr{C}, J)$.

- A geometric morphism of toposes $f : \mathscr{E} \to \mathscr{F}$ is a pair of adjoint functors whose left adjoint (called the inverse image functor) $f^* : \mathscr{F} \to \mathscr{E}$ preserves finite limits.

  For instance, the inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ of a Grothendieck topos $\mathbf{Sh}(\mathscr{C}, J)$ in the corresponding presheaf topos $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ yields a geometric morphism between these toposes (whose inverse image is the associated sheaf functor).

The notion of classifying topos

Definition

Let $\mathbb{T}$ be a geometric theory over a given signature. A classifying topos of $\mathbb{T}$ is a Grothendieck topos **Set**$[\mathbb{T}]$ such that for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathbf{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

natural in $\mathscr{E}$.

Theorem (Makkai, Reyes *et al.*)

Every geometric theory has a classifying topos. Conversely, every Grothendieck topos arises as the classifying topos of some geometric theory.

The classifying topos of a geometric theory $\mathbb{T}$ can always be constructed canonically from it as the topos of sheaves $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ on its syntactic site $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$.

Many different (not necessarily bi-interpretable) theories may have the same classifying topos (up to equivalence). This phenomenon is called Morita equivalence and corresponds to the existence of different sites presenting the same topos.

Subtoposes

Definition

A subtopos of a topos $\mathscr{E}$ is an equivalence class of geometric inclusions to $\mathscr{E}$.

Fact

- *The notion of subtopos is a topos-theoretic invariant.*
- *If $\mathscr{E}$ is the topos $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a site $(\mathscr{C}, J)$, the subtoposes of $\mathscr{E}$ are in bijective correspondence with the Grothendieck topologies J' on $\mathscr{C}$ which contain J (i.e. such that every J-covering sieve is J'-covering).*

A duality theorem

Definition

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. A quotient of $\mathbb{T}$ is a geometric theory $\mathbb{T}'$ over Σ such that every axiom of $\mathbb{T}$ is provable in $\mathbb{T}'$.
- Let $\mathbb{T}$ and $\mathbb{T}'$ be geometric theories over a signature Σ. We say that $\mathbb{T}$ and $\mathbb{T}'$ are syntactically equivalent, and we write $\mathbb{T} \equiv_s \mathbb{T}'$, if for every geometric sequent σ over Σ, σ is provable in $\mathbb{T}$ if and only if σ is provable in $\mathbb{T}'$.

Theorem

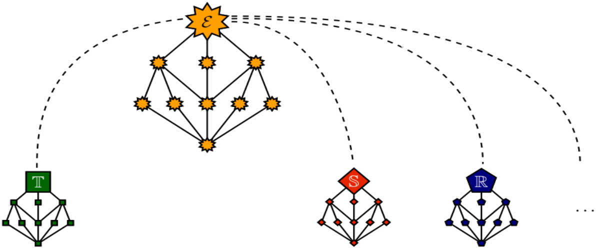
Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the $\equiv_s$-equivalence classes of quotients of $\mathbb{T}$ and the subtoposes of the classifying topos **Set**$[\mathbb{T}]$ *of $\mathbb{T}$.*

Some consequences

This duality theorem has several implications; in particular, it allows one to import many notions and results from topos theory into the realm of geometric logic. For instance, one can deduce from it that

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the collection $\mathfrak{Th}_{\Sigma}^{\mathbb{T}}$ of (syntactic-equivalence classes of) geometric theories over Σ which are quotients of $\mathbb{T}$, endowed with the order defined by $\mathbb{T}' \leq \mathbb{T}''$ if and only if all the axioms of $\mathbb{T}'$ are provable in $\mathbb{T}''$', is a Heyting algebra.



Lattices of theories

‘Bridges’ between quotients and topologies

This duality also allows one to establish ‘bridges’ of the following form:

Subtopos of
$\textbf{Sh}(\mathscr{C}, J) \simeq \textbf{Set}[\mathbb{T}]$

Grothendieck topology on $\mathscr{C}$ containing J — *Quotient of $\mathbb{T}$*

That is, if the classifying topos of a geometric theory $\mathbb{T}$ can be represented as the category $\textbf{Sh}(\mathscr{C}, J)$ of sheaves on a (small) site $(\mathscr{C}, J)$ then we have a natural, order-preserving bijection

$$\text{quotients of } \mathbb{T}$$
$$|||$$
$$\text{Grothendieck topologies on } \mathscr{C} \text{ which contain } J$$

Two notable cases

We shall focus on two particular cases of this result:

1. $(\mathscr{C}, J)$ is the syntactic site $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$
2. - $\mathbb{T}$ is a theory of presheaf type (e.g. a finitary algebraic, or more generally cartesian, theory),
   - $\mathscr{C}$ is the opposite of its category f.p.$\mathbb{T}$-mod(**Set**) of finitely presentable models, and
   - J is the trivial topology on it.

In the first case, we obtain an order-preserving bijective correspondence between the quotients of $\mathbb{T}$ and the Grothendieck topologies on $\mathscr{C}_{\mathbb{T}}$ which contain $J_{\mathbb{T}}$.

In the second case, we obtain an order-preserving bijective correspondence between the quotients of $\mathbb{T}$ and the Grothendieck topologies on f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$.

In both cases, these correspondences can be naturally interpreted as proof-theoretic equivalences between the classical proof system of geometric logic over $\mathbb{T}$ and new proof systems for sieves whose inference rules correspond to the axioms of Grothendieck topologies.

Proof systems for sieves

Given a collection $\mathscr{A}$ of sieves on a given category $\mathscr{C}$, the notion of Grothendieck topology on $\mathscr{C}$ naturally gives rise to a proof system $\mathscr{T}_{\mathscr{C}}^{\mathscr{A}}$, as follows: the axioms of $\mathscr{T}_{\mathscr{C}}^{\mathscr{A}}$ are the sieves in $\mathscr{A}$ plus all the maximal sieves, while the inference rules of $\mathscr{T}_{\mathscr{C}}^{\mathscr{A}}$ are the proof-theoretic versions of the well-known axioms for Grothendieck topologies, i.e. the following rules:

- *Stability rule:*

$$\frac{R}{f^*(R)}$$

where R is any sieve on an object c of $\mathscr{C}$ and f is any arrow in $\mathscr{C}$ with codomain c.

- *Transitivity rule:*

$$\frac{Z \quad \{f^*(R) \mid f \in Z\}}{R}$$

where R and Z are sieves in $\mathscr{C}$ on a given object of $\mathscr{C}$.

N.B. The 'closed theories' of this proof system are precisely the Grothendieck topologies on $\mathscr{C}$ which contain the sieves in $\mathscr{A}$ as covering sieves. The closure of a 'theory' in $\mathscr{T}_{\mathscr{C}}^{\mathscr{A}}$, i.e. of a collection $\mathscr{U}$ of sieves in $\mathscr{C}$, is exactly the Grothendieck topology on $\mathscr{C}$ generated by $\mathscr{A}$ and $\mathscr{U}$.

The first correspondence

Let $\mathbb{T}$ be a geometric theory over a signature Σ, $\mathscr{S}$ the collection of geometric sequents over Σ and $S(\mathscr{C}_{\mathbb{T}})$ the collection of (small-generated) sieves in the syntactic category $\mathscr{C}_{\mathbb{T}}$.

- Given a geometric sequent $\sigma \equiv (\phi \vdash_{\vec{x}} \psi)$ over Σ, we set $\mathscr{F}(\sigma)$ equal to the principal sieve in $\mathscr{C}_{\mathbb{T}}$ generated by the monomorphism

$$\{\vec{x'} \,.\, \phi \wedge \psi\} \xrightarrow{[(\phi \wedge \psi \wedge \vec{x'} = \vec{x})]} \{\vec{x} \,.\, \phi\} \,.$$

- Given a small-generated sieve $R = \{[\theta_i] : \{\vec{x}_i \,.\, \phi_i\} \to \{\vec{y} \,.\, \psi\}\}$ in $\mathscr{C}_{\mathbb{T}}$, we set $\mathscr{G}(R)$ equal to the sequent $(\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x}_i)\theta_i)$.

The first equivalence

Let $V \to \overline{V}^{\mathscr{T}}$ and $U \to \overline{U}^{\mathbb{T}}$ respectively be the operations consisting in taking the Grothendieck topology $\overline{V}^{\mathscr{T}}$ generated by $J_{\mathbb{T}}$ plus the sieves in V and in taking the collection $\overline{U}^{\mathbb{T}}$ of geometric sequents provable in $\mathbb{T} \cup U$ by using geometric logic. Let $F : \mathscr{P}(\mathscr{S}) \to \mathscr{P}(S(\mathscr{C}_{\mathbb{T}}))$ be the composite $\overline{(-)}^{\mathscr{T}} \circ \mathscr{P}(\mathscr{F})$ and $G : \mathscr{P}(S(\mathscr{C}_{\mathbb{T}})) \to \mathscr{P}(\mathscr{S})$ be the composite $\overline{(-)}^{\mathbb{T}}_{\tilde{\mathscr{F}}} \circ \mathscr{P}(\mathscr{G})$. Then

Theorem

(i) *For any $U \in \mathscr{P}(\mathscr{S})$, $\mathscr{F}(\overline{U}^{\mathbb{T}}) \subseteq \overline{\mathscr{F}(U)}^{\mathscr{T}}$.*

(ii) *For any $V \in \mathscr{P}(S(\mathscr{C}_{\mathbb{T}}))$, $\mathscr{G}(\overline{V}^{\mathscr{T}}) \subseteq \overline{\mathscr{G}(V)}^{\mathbb{T}}$.*

(iii) *For any $U \in \mathscr{P}(\mathscr{S})$, $G(F(U)) = \overline{U}^{\mathbb{T}}$.*

(iv) *For any $V \in \mathscr{P}(S(\mathscr{C}_{\mathbb{T}}))$, $F(G(V)) = \overline{V}^{\mathscr{T}}$.*

In other words, the maps F and G define a proof-theoretic equivalence between the classical deduction system for geometric logic over $\mathbb{T}$ and the proof system $\mathscr{T}_{\mathscr{C}_{\mathbb{T}}}^{J_{\mathbb{T}}}$.

Describing the second equivalence

Recall that a geometric theory is said to be of presheaf type if it is classified by a presheaf topos (equivalently, by the topos $[\text{f.p.}\mathbb{T}\text{-mod}(\textbf{Set}), \textbf{Set}]$). Theories of presheaf type are very important in that they constitute the basic 'building blocks' from which every geometric theory can be built. Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a 'quotient' of a theory of presheaf type.

Every finitary algebraic (or more generally any cartesian) theory is of presheaf type, but this class also contain many other interesting mathematical theories.

Definition

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then a geometric formula $\phi(\vec{x})$ over Σ is said to be $\mathbb{T}$-irreducible if, regarded as an object of the syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$, it does not admit any non-trivial $J_{\mathbb{T}}$-covering sieves.

Theorem

A geometric theory $\mathbb{T}$ over a signature Σ is of presheaf type if and only if every geometric formula $\phi(\vec{x})$ over Σ, when regarded as an object of $\mathscr{C}_{\mathbb{T}}$, is $J_{\mathbb{T}}$-covered by $\mathbb{T}$-irreducible formulae over Σ.

Irreducible formulae and finitely presentable models

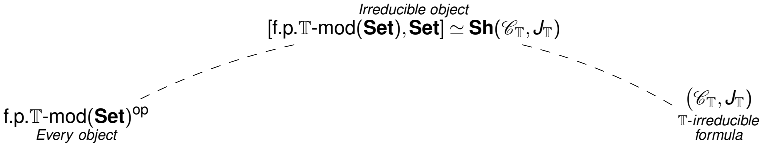
Theorem

Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ. Then

(i) *Any finitely presentable $\mathbb{T}$-model in* **Set** *is presented by a $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ;*

(ii) *Conversely, any $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ presents a $\mathbb{T}$-model.*

In fact, the category $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$ *is equivalent to the full subcategory* $\mathscr{C}_{\mathbb{T}}^{\text{irr}}$ *of* $\mathscr{C}_{\mathbb{T}}$ *on the $\mathbb{T}$-irreducible formulae.*



Sequents and sieves on f.p. models

- By using the fact that every geometric formula over Σ can be $J_{\mathbb{T}}$-covered in $\mathscr{C}_{\mathbb{T}}$ by $\mathbb{T}$-irreducible formulae, one can show that every geometric sequent over Σ is provably equivalent in $\mathbb{T}$ to a collection of sequents σ of the form $(\phi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{y_i})\theta_i)$ where, for each $i \in I$, $[\theta_i] : \{\vec{y_i} \,.\, \psi_i\} \to \{\vec{x} \,.\, \phi\}$ is an arrow in $\mathscr{C}_{\mathbb{T}}$ and $\phi(\vec{x})$, $\psi(\vec{y_i})$ are geometric formulae over Σ presenting respectively $\mathbb{T}$-models $M_{\{\vec{x} \,.\, \phi\}}$ and $M_{\{\vec{y_i} \,.\, \psi_i\}}$.
- To such a sequent σ, we can associate the cosieve S_σ on $M_{\{\vec{x} \,.\, \phi\}}$ in f.p.$\mathbb{T}$-mod(**Set**) defined as follows. For each $i \in I$, $[[\theta_i]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}}$ is the graph of a morphism $[[\vec{y_i} \,.\, \psi_i]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}} \to [[\vec{x} \,.\, \phi]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}}$; then the image of the generators of $M_{\{\vec{y_i} \,.\, \psi_i\}}$ via this morphism is an element of $[[\vec{x} \,.\, \phi]]_{M_{\{\vec{y_i} \,.\, \psi_i\}}}$ and this in turn determines, by definition of $M_{\{\vec{x} \,.\, \phi\}}$, a unique arrow $s_i : M_{\{\vec{x} \,.\, \phi\}} \to M_{\{\vec{y_i} \,.\, \psi_i\}}$ in $\mathbb{T}$-mod(**Set**). We set S_σ equal to the sieve in f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ on M_ϕ generated by the arrows s_i as i varies in I.

Sequents and sieves on f.p. models

Conversely, by the equivalence f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}} \simeq \mathscr{C}_{\mathbb{T}}^{\text{irr}}$, every sieve in f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ is clearly of the form S_σ for such a sequent σ.

These correspondences define, similarly to above, a proof-theoretic equivalence between the classical deduction system for geometric logic over $\mathbb{T}$ and the proof system $\mathscr{T}^{T}_{\text{f.p.T-mod}(\mathbf{Set})^{\text{op}}}$ (where T is the trivial Grothendieck topology).

In particular, the Grothendieck topology J on f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ associated with a quotient $\mathbb{T}'$ of $\mathbb{T}$ is generated by the sieves S_σ, where σ varies among the sequents associated with the axioms of $\mathbb{T}'$ as above.

Moreover, for any σ of the above form, σ is provable in $\mathbb{T}'$ if and only if S_σ belongs to J.

This generalizes Coste-Lombardi-Roy's correspondence between dynamical theories (viewed as coherent quotients of universal Horn theories) and the coherent Grothendieck topologies associated with them.

Why are these equivalences interesting?

These equivalences are useful in that they allow us to study (the proof theory of) geometric theories through the associated Grothendieck topologies: the condition of provability of a sequent in a geometric theory gets transformed in the requirement for a sieve (or a family of sieves) to belong to a certain Grothendieck topology, something which is often much easier to investigate.

Indeed, we have shown that Grothendieck topologies are particularly amenable to computation by deriving

- An explicit formula for the Grothendieck topology generated by a given family of sieves
- Explicit descriptions of the lattice operations on Grothendieck topologies on a given category which refine a certain topology (recall that these correspond to the lattice operations on quotients via the above duality).

Formulas for Grothendieck topologies

Meet of Grothendieck topologies

If J_1 and J_2 are Grothendieck topologies on a category $\mathscr{C}$ respectively generated by bases K_1 and K_2, the meet $J_1 \wedge J_2$ is generated by the collection of sieves which are unions of sieves in K_1 with sieves in K_2.

Grothendieck topology generated by a family of sieves

The Grothendieck topology G_D generated by a family D of sieves in $\mathscr{C}$ which is stable under pullback is given by

$$\begin{aligned} G_D(c) \quad = \quad & \{S \text{ sieve on } c \mid \text{for any arrow } d \xrightarrow{f} c \text{ and sieve } T \text{ on } d, \\ & [(\text{for any arrow } e \xrightarrow{g} d \text{ and sieve } Z \text{ on } e \\ & (Z \in D(e) \text{ and } Z \subseteq g^*(T)) \text{ implies } g \in T) \text{ and } (f^*(S) \subseteq T)] \\ & \text{implies } T = M_d\} \end{aligned}$$

for any object $c \in \mathscr{C}$.

Heyting implication of Grothendieck topologies

$$\begin{aligned} (J_1 \Rightarrow J_2)(c) \quad = \quad & \{S \text{ sieve on } c \mid \text{for any arrow } d \xrightarrow{f} c \text{ and sieve } Z \text{ on } d, \\ & [Z \text{ is } J_1\text{-covering and } J_2\text{-closed and } f^*(S) \subseteq Z] \\ & \text{implies } Z = M_d\}\ . \end{aligned}$$

Some applications

Theorem (A deduction theorem for geometric logic)

Let $\mathbb{T}$ be a geometric theory over a signature Σ and ϕ, ψ geometric sentences over Σ such that the sequent $(\top \vdash_{[]} \psi)$ is provable in the theory $\mathbb{T} \cup \{(\top \vdash_{[]} \phi)\}$. Then the sequent $(\phi \vdash_{[]} \psi)$ is provable in the theory $\mathbb{T}$.

We have proved this theorem by showing (using the above-mentioned formula for the Grothendieck topology generated by a given family of sieves) that if the principal sieve in $\mathscr{C}_{\mathbb{T}}$ generated by the arrow $\{[] \,.\, \psi\} \overset{[\psi]}{\rightarrowtail} \{[] \,.\, \top\}$ belongs to the Grothendieck topology on $\mathscr{C}_{\mathbb{T}}$ generated over $J_{\mathbb{T}}$ by the principal sieve generated by the arrow $\{[] \,.\, \phi\} \overset{[\phi]}{\rightarrowtail} \{[] \,.\, \top\}$, then $[\phi] \leq [\psi]$ in $\mathrm{Sub}_{\mathscr{C}_{\mathbb{T}}}(\{[] \,.\, \top\})$.

Some applications

Theorem

The meet of the theory of local rings and that of integral domains in the lattice of quotients of the theory of commutative rings with unit is obtained from the latter theory by adding the sequents

$$(0 = 1 \vdash_{[]} \bot)$$

and

$$(\bigwedge_{1 \leq s \leq m} P_s(\vec{x}) = 0 \vdash_{\vec{x}} \bigvee_{1 \leq i \leq k} (\exists y)(G_i(\vec{x}) \cdot y = 1) \vee \bigvee_{1 \leq j \leq l} H_j(\vec{x}) = 0)$$

where for each $1 \leq s \leq m$, $1 \leq i \leq k$ and $1 \leq j \leq l$ the P_s's, G_i's and H_j's are any polynomials in a finite string $\vec{x} = (x_1, \ldots, x_n)$ of variables with the property that $\{P_1, \ldots, P_s, G_1, \ldots, G_k\}$ is a set of elements of $\mathbb{Z}[x_1, \ldots, x_n]$ which is not contained in any proper ideal of $\mathbb{Z}[x_1, \ldots, x_n]$ and $\prod_{1 \leq j \leq l} H_j \in (P_1, \ldots, P_s)$ in $\mathbb{Z}[x_1, \ldots, x_n]$.

We have derived this result by calculating the meet of the Grothendieck topologies associated with the two quotients by using suitable bases for them.

Some applications

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ and $\mathbb{T}_1, \mathbb{T}_2$ two quotients of $\mathbb{T}$. Then the Heyting implication $\mathbb{T}_1 \Rightarrow \mathbb{T}_2$ in $\mathfrak{Th}_{\Sigma}^{\mathbb{T}}$ is the theory obtained from $\mathbb{T}$ by adding all the geometric sequents $(\psi \vdash_{\vec{y}} \psi')$ over Σ with the property that $(\psi' \vdash_{\vec{y}} \psi)$ is provable in $\mathbb{T}$ and for any $\mathbb{T}$-provably functional geometric formula $\theta(\vec{x}, \vec{y})$ from a geometric formula-in-context $\{\vec{x} \,.\, \phi\}$ to $\{\vec{y} \,.\, \psi\}$ and any geometric formula χ in the context $\vec{x}$ such that $(\chi \vdash_{\vec{x}} \phi)$ is provable in $\mathbb{T}$, the conjunction of the facts

(i) $(\phi \vdash_{\vec{x}} \chi)$ *is provable in* $\mathbb{T}_1$

(ii) $((\exists \vec{y})(\theta(\vec{x}, \vec{y}) \wedge \psi'(\vec{y})) \vdash_{\vec{x}} \chi)$ *is provable in* $\mathbb{T}$

implies that $(\phi \vdash_{\vec{x}} \chi)$ *is provable in* $\mathbb{T}_2$.

For further reading

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