An introduction to Grothendieck toposes

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The "unifying notion" of topos

"It is the topos theme which is this "bed" or "deep river" where come to be married geometry and algebra, topology and arithmetic, mathematical logic and category theory, the world of the "continuous" and that of "discontinuous" or discrete structures. It is what I have conceived of most broad to perceive with finesse, by the same language rich of geometric resonances, an "essence" which is common to situations most distant from each other coming from one region or another of the vast universe of mathematical things".

A. Grothendieck

Topos theory can be regarded as a unifying subject in Mathematics, with great relevance as a framework for systematically investigating the relationships between different mathematical theories and studying them by means of a multiplicity of different points of view. Its methods are transversal to the various fields and complementary to their own specialized techniques. In spite of their generality, the topos-theoretic techniques are liable to generate insights which would be hardly attainable otherwise and to establish deep connections that allow effective transfers of knowledge between different contexts.

The multifaceted nature of toposes

The role of toposes as unifying spaces is intimately tied to their multifaceted nature.

For instance, a topos can be seen as:

- a generalized space
- a mathematical universe
- a theory modulo 'Morita-equivalence'

In this course we shall review each of these classical points of view, and then present the more recent theory of topos-theoretic 'bridges', which combines all of them to provide tools for making toposes effective means for studying mathematical theories from multiple points of view, relating and unifying theories with each other and constructing 'bridges' across them.

A bit of history

- Toposes were originally introduced by Alexander Grothendieck in the early 1960s, in order to provide a mathematical underpinning for the 'exotic' cohomology theories needed in algebraic geometry. Every topological space gives rise to a topos and every topos in Grothendieck's sense can be considered as a 'generalized space'.
- At the end of the same decade, William Lawvere and Myles Tierney realized that the concept of Grothendieck topos also yielded an abstract notion of mathematical universe within which one could carry out most familiar set-theoretic constructions, but which also, thanks to the inherent 'flexibility' of the notion of topos, could be profitably exploited to construct 'new mathematical worlds' having particular properties.
- A few years later, the theory of classifying toposes added a further fundamental viewpoint to the above-mentioned ones: a topos can be seen not only as a generalized space or as a mathematical universe, but also as a suitable kind of first-order theory (considered up to a general notion of equivalence of theories).

Toposes as unifying 'bridges'

Since the times of my Ph.D. studies I have been developing a theory and a number of techniques which allow to effectively use toposes as unifying spaces.

The key idea is that the possibility of representing a topos in a multitude of different ways can be effectively exploited for building unifying 'bridges' between theories having an equivalent, or strictly related, mathematical content.

These 'bridges' allow effective and often deep transfers of notions, ideas and results across the theories.

In spite of the number of applications in different fields obtained throughout the last years, the potential of these methods has just started to be explored.

In fact, 'bridges' have proved useful not only for connecting different theories with each other, but also for working inside a given mathematical theory and investigating it from a multiplicity of different points of view.

Presheaves on a topological space

Definition

Let X be a topological space. A presheaf $\mathscr{F}$ on X consists of the data:

(i) for every open subset U of X, a set $\mathscr{F}(U)$ and
(ii) for every inclusion $V \subseteq U$ of open subsets of X, a function $\rho_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ subject to the conditions
  - $\rho_{U,U}$ is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$ and
  - if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

The maps $\rho_{U,V}$ are called restriction maps, and we sometimes write $s|_V$ instead of $\rho_{U,V}(s)$, if $s \in \mathscr{F}(U)$.
A morphism of presheaves $\mathscr{F} \to \mathscr{G}$ on a topological space X is a collection of maps $\mathscr{F}(U) \to \mathscr{G}(U)$ which is compatible with respect to restriction maps.

Remark

*Categorically, a presheaf $\mathscr{F}$ on X is a functor $\mathscr{F} : \mathscr{O}(X)^{op} \to$ **Set**, where $\mathscr{O}(X)$ is the poset category corresponding to the lattice of open sets of the topological space X (with respect to the inclusion relation).*
A morphism of presheaves is then just a natural transformation between the corresponding functors.
So we have a category $[\mathscr{O}(X)^{op}, \mathbf{Set}]$ of presheaves on X.

Sheaves on a topological space

Definition

A sheaf $\mathscr{F}$ on a topological space X is a presheaf on X satisfying the additional conditions

(i) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if $s, t \in \mathscr{F}(U)$ are elements such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$;

(ii) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each $i, j \in I$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathscr{F}(U)$ (necessarily unique by (i)) such that $s|_{V_i} = s_i$ for each i.

A morphism of sheaves is defined as a morphism of the underlying presheaves.

Examples

- the sheaf of continuous real-valued functions on any topological space
- the sheaf of regular functions on a variety
- the sheaf of differentiable functions on a differentiable manifold
- the sheaf of holomorphic functions on a complex manifold

In each of the above examples, the restriction maps of the sheaf are the usual set-theoretic restrictions of functions to a subset.

Sheaves from a categorical point of view

Sheaves arising in Mathematics are often equipped with more structure than the mere set-theoretic one; for example, one may wish to consider sheaves of modules (resp. rings, abelian groups, ...) on a topological space X.

The natural categorical way of looking at these notions is to consider them as models of certain (geometric) theories in a category $\mathbf{Sh}(X)$ of sheaves of sets.

Remarks

- *Categorically, a sheaf is a functor $\mathscr{O}(X)^{op} \to \mathbf{Set}$ which satisfies certain conditions expressible in categorical language entirely in terms of the poset category $\mathscr{O}(X)$ and of the usual notion of covering on it. The category $\mathbf{Sh}(X)$ of sheaves on a topological space X is thus a full subcategory of the category $[\mathscr{O}(X)^{op}, \mathbf{Set}]$ of presheaves on X.*
- *Many important properties of topological spaces X can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.*

These remarks led Grothendieck to introduce a significant categorical generalization of the notion of sheaf, and hence the notion of Grothendieck topos.

Limits and colimits in $\mathbf{Sh}(X)$

Theorem

(i) *The category* $\mathbf{Sh}(X)$ *is closed in* $[\mathscr{O}(X)^{\text{op}}, \mathbf{Set}]$ *under arbitrary (small) limits.*

(ii) *The associated sheaf functor* $a : [\mathscr{O}(X)^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(X)$ *(having a right adjoint) preserves all (small) colimits.*

- *Part (i) follows from the fact that limits commute with limits, in light of the characterization of sheaves in terms of limits.*
- *From part (ii) it follows that* $\mathbf{Sh}(X)$ *has all small colimits, which are computed by applying the associated sheaf functor to the colimit of the diagram considered with values in* $[\mathscr{O}(X)^{\text{op}}, \mathbf{Set}]$.

Adjunctions induced by points

Let x be a point of a topological space X.

Definition

Let A be a set. Then the skyscraper sheaf $\text{Sky}_x(A)$ of A at x is the sheaf on X defined as

- $\text{Sky}_x(A)(U) = A$ if $x \in U$
- $\text{Sky}_x(A)(U) = 1 = \{*\}$ if $x \notin U$

and in the obvious way on arrows.

The assignment $A \to \text{Sky}_x(A)$ is clearly functorial.

Theorem

The stalk functor $\text{Stalk}_x : \mathbf{Sh}(X) \to \mathbf{Set}$ *at* x *is left adjoint to the skyscraper functor* $\text{Sky}_x : \mathbf{Set} \to \mathbf{Sh}(X)$.

In fact, as we shall see later in the course, points in topos theory are defined as suitable kinds of functors (more precisely, colimit and finite-limit preserving ones).

Open sets as subterminal objects

Since limits in a category $\mathbf{Sh}(X)$ are computed as in the category of presheaves $[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$, a subobject of a sheaf F in $\mathbf{Sh}(X)$ is just a subsheaf, that is a subfunctor which is a sheaf.
Notice that a subfunctor $S \subseteq F$ is a sheaf if and only if for every open covering $\{U_i \subseteq U \mid i \in I\}$ and every element $x \in F(U)$, $x \in S(U)$ if and only if $x|_{U_i} \in S(U_i)$.

Definition

In a category with a terminal object, a subterminal object is an object whose unique arrow to the terminal object is a monomorphism.

Theorem

Let X be a topological space. Then we have a frame isomorphism

$$\mathrm{Sub}_{\mathbf{Sh}(X)}(1) \cong \mathscr{O}(X) .$$

between the subterminal objects of $\mathbf{Sh}(X)$ and the open sets of X.

Sieves

In order to 'categorify' the notion of sheaf of a topological space, the first step is to introduce an abstract notion of covering (of an object by a family of arrows to it) in a category.

Definition

- Given a category $\mathscr{C}$ and an object $c \in \mathrm{Ob}(\mathscr{C})$, a presieve P in $\mathscr{C}$ on c is a collection of arrows in $\mathscr{C}$ with codomain c.
- Given a category $\mathscr{C}$ and an object $c \in \mathrm{Ob}(\mathscr{C})$, a sieve S in $\mathscr{C}$ on c is a collection of arrows in $\mathscr{C}$ with codomain c such that

$$f \in S \Rightarrow f \circ g \in S$$

  whenever this composition makes sense.
- We say that a sieve S is generated by a presieve P on an object c if it is the smallest sieve containing it, that is if it is the collection of arrows to c which factor through an arrow in P.

If S is a sieve on c and $h : d \to c$ is any arrow to c, then

$$h^*(S) := \{g \mid \mathrm{cod}(g) = d, \; h \circ g \in S\}$$

is a sieve on d.

Grothendieck topologies

Definition

- A Grothendieck topology on a category $\mathscr{C}$ is a function J which assigns to each object c of $\mathscr{C}$ a collection $J(c)$ of sieves on c in such a way that
  - (i) (maximality axiom) the maximal sieve $M_c = \{f \mid \text{cod}(f) = c\}$ is in $J(c)$;
  - (ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;
  - (iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

  The sieves S which belong to $J(c)$ for some object c of $\mathscr{C}$ are said to be J-covering.
- A site is a pair $(\mathscr{C}, J)$ where $\mathscr{C}$ is a small category and J is a Grothendieck topology on $\mathscr{C}$.

Examples of Grothendieck topologies

- For any (small) category $\mathscr{C}$, the trivial topology on $\mathscr{C}$ is the Grothendieck topology in which the only sieve covering an object c is the maximal sieve M_c.
- The dense topology D on a category $\mathscr{C}$ is defined by: for a sieve S,

$$S \in D(c) \quad \text{if and only if} \quad \begin{array}{l}\text{for any } f : d \to c \text{ there exists} \\ g : e \to d \text{ such that } f \circ g \in S\,.\end{array}$$

If $\mathscr{C}$ satisfies the right Ore condition i.e. the property that any two arrows $f : d \to c$ and $g : e \to c$ with a common codomain c can be completed to a commutative square

$$\begin{array}{ccc} \bullet & \dashrightarrow & d \\ \downarrow & & \downarrow f \\ e & \xrightarrow{g} & c \end{array}$$

then the dense topology on $\mathscr{C}$ specializes to the atomic topology on $\mathscr{C}$ i.e. the topology J_{at} defined by: for a sieve S,

$$S \in J_{at}(c) \text{ if and only if } S \neq \emptyset\,.$$

Examples of Grothendieck topologies

- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathscr{O}(X)}$ on the poset category $\mathscr{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathscr{O}(X))$,

$$S \in J_{\mathscr{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U\,.$$

- More generally, given a frame (or complete Heyting algebra) H, we can define a Grothendieck topology J_H, called the *canonical topology on H*, by:

$$\{a_i \mid i \in I\} \in J_H(a) \text{ if and only if } \bigvee_{i \in I} a_i = a\,.$$

- Given a small category of topological spaces which is closed under finite limits and under taking open subspaces, one may define the open cover topology on it by specifying as basis the collection of open embeddings $\{Y_i \hookrightarrow X \mid i \in I\}$ such that $\bigcup_{i \in I} Y_i = X$.
- The Zariski topology on the opposite of the category $\mathbf{Rng}_{\text{f.g.}}$ of finitely generated commutative rings with unit is defined by: for any cosieve S in $\mathbf{Rng}_{\text{f.g.}}$ on an object A, $S \in Z(A)$ if and only if S contains a finite family $\{\xi_i : A \to A_{f_i} \mid 1 \leq i \leq n\}$ of canonical maps $\xi_i : A \to A_{f_i}$ in $\mathbf{Rng}_{\text{f.g.}}$ where $\{f_1, \ldots, f_n\}$ is a set of elements of A which is not contained in any proper ideal of A.

Sheaves on a site

Definition

- A presheaf on a (small) category $\mathscr{C}$ is a functor $P : \mathscr{C}^{\text{op}} \to \textbf{Set}$.
- Let $P : \mathscr{C}^{\text{op}} \to \textbf{Set}$ be a presheaf on $\mathscr{C}$ and S be a sieve on an object c of $\mathscr{C}$.

  A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

  $$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d\,.$$

  An amalgamation for such a family is a single element $x \in P(c)$ such that

  $$P(f)(x) = x_f \quad \text{for all } f \text{ in } S\,.$$

Remark

For any covering family $F = \{U_i \subseteq U \mid i \in I\}$ in a topological space X and any presheaf $\mathscr{F}$ on X, giving a family of elements $s_i \in \mathscr{F}(U_i)$ such that for any $i, j \in I$ $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ is equivalent to giving a family of elements $\{s_W \in \mathscr{F}(W) \mid W \in S_F\}$ such that for any open set $W' \subseteq W$, $s_W|_{W'} = s_{W'}$, where S_F is the sieve generated by F. In other words, it is the same as giving a matching family for S_F of elements of $\mathscr{F}$.

Sheaves on a site

- Given a site $(\mathscr{C}, J)$, a presheaf on $\mathscr{C}$ is a J-sheaf if every matching family for any J-covering sieve on any object of $\mathscr{C}$ has a unique amalgamation.
- The category $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on the site $(\mathscr{C}, J)$ is the full subcategory of $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.
- A Grothendieck topos is any category equivalent to the category of sheaves on a site.

Examples of toposes

The following examples show that toposes can be naturally attached to mathematical notions as different as (small) categories, topological spaces, or groups.

Examples

- For any (small) category $\mathscr{C}$, $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ is the category of sheaves $\mathbf{Sh}(\mathscr{C}, T)$ where T is the trivial topology on $\mathscr{C}$.
- For any topological space X, $\mathbf{Sh}(\mathscr{O}(X), J_{\mathscr{O}(X)})$ is equivalent to the usual category $\mathbf{Sh}(X)$ of sheaves on X.
- For any (topological) group G, the category $BG = \mathbf{Cont}(G)$ of continuous actions of G on discrete sets is a Grothendieck topos (equivalent, as we shall see, to the category $\mathbf{Sh}(\mathbf{Cont}_{\text{t}}(G), J_{\text{at}})$ of sheaves on the full subcategory $\mathbf{Cont}_{\text{t}}(G)$ on the non-empty transitive actions with respect to the atomic topology).

Basic properties of Grothendieck toposes

Grothendieck toposes satisfy all the categorical properties that one might hope for:

Theorem

Let $(\mathscr{C}, J)$ be a site. Then

- *the inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ has a left adjoint $a : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ (called the associated sheaf functor), which preserves finite limits.*
- *The category $\mathbf{Sh}(\mathscr{C}, J)$ has all (small) limits, which are preserved by the inclusion functor $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$; in particular, limits are computed pointwise and the terminal object $1_{\mathbf{Sh}(\mathscr{C}, J)}$ of $\mathbf{Sh}(\mathscr{C}, J)$ is the functor $T : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ sending each object $c \in Ob(\mathscr{C})$ to the singleton $\{*\}$.*
- *The associated sheaf functor $a : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ preserves colimits; in particular, $\mathbf{Sh}(\mathscr{C}, J)$ has all (small) colimits.*
- *The category $\mathbf{Sh}(\mathscr{C}, J)$ has exponentials, which are constructed as in the topos $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.*
- *The category $\mathbf{Sh}(\mathscr{C}, J)$ has a subobject classifier.*
- *The category $\mathbf{Sh}(\mathscr{C}, J)$ has a separating set of objects.*

Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism. The natural notion of morphism of geometric morphisms if that of geometric transformation.

Definition

(i) Let $\mathscr{E}$ and $\mathscr{F}$ be toposes. A geometric morphism $f : \mathscr{E} \to \mathscr{F}$ consists of a pair of functors $f_* : \mathscr{E} \to \mathscr{F}$ (the direct image of f) and $f^* : \mathscr{F} \to \mathscr{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathscr{E} \to \mathscr{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

(iii) A point of a topos $\mathscr{E}$ is a geometric morphism $\mathbf{Set} \to \mathscr{E}$.

- Grothendieck toposes and geometric morphisms between them form a 2-category.
- Given two toposes $\mathscr{E}$ and $\mathscr{F}$, geometric morphisms from $\mathscr{E}$ to $\mathscr{F}$ and geometric transformations between them form a category, denoted by $\mathbf{Geom}(\mathscr{E}, \mathscr{F})$.

Examples of geometric morphisms

- A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The direct image $\mathbf{Sh}(f)_*$ sends a sheaf $F \in Ob(\mathbf{Sh}(X))$ to the sheaf $\mathbf{Sh}(f)_*(F)$ defined by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for any open subset V of Y. The inverse image $\mathbf{Sh}(f)^*$ acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.
- Every Grothendieck topos $\mathscr{E}$ has a unique geometric morphism $\mathscr{E} \to \mathbf{Set}$. The direct image is the global sections functor $\Gamma : \mathscr{E} \to \mathbf{Set}$, sending an object $e \in \mathscr{E}$ to the set $\mathrm{Hom}_{\mathscr{E}}(1_{\mathscr{E}}, e)$, while the inverse image functor $\Delta : \mathbf{Set} \to \mathscr{E}$ sends a set S to the coproduct $\coprod_{s \in S} 1_{\mathscr{E}}$.
- For any site $(\mathscr{C}, J)$, the pair of functors formed by the inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ and the associated sheaf functor $a : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ yields a geometric morphism $i : \mathbf{Sh}(\mathscr{C}, J) \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.

Geometric morphisms as flat functors I

Theorem

Let $\mathscr{C}$ be a small category and $\mathscr{E}$ be a locally small cocomplete category. Then, for any functor $A : \mathscr{C} \to \mathscr{E}$ the functor $R_A : \mathscr{E} \to [\mathscr{C}^{\text{op}}, \textbf{Set}]$ defined for each $e \in Ob(\mathscr{E})$ and $c \in Ob(\mathscr{C})$ by:

$$R_A(e)(c) = \text{Hom}_{\mathscr{E}}(A(c), e)$$

has a left adjoint $- \otimes_{\mathscr{C}} A : [\mathscr{C}^{\text{op}}, \textbf{Set}] \to \mathscr{E}$.

Definition

- A functor $A : \mathscr{C} \to \mathscr{E}$ from a small category $\mathscr{C}$ to a Grothendieck topos $\mathscr{E}$ is said to be flat if the functor $- \otimes_{\mathscr{C}} A : [\mathscr{C}^{\text{op}}, \textbf{Set}] \to \mathscr{E}$ preserves finite limits.
- The full subcategory of $[\mathscr{C}, \mathscr{E}]$ on the flat functors will be denoted by $\textbf{Flat}(\mathscr{C}, \mathscr{E})$.

Geometric morphisms as flat functors II

Theorem

Let $\mathscr{C}$ be a small category and $\mathscr{E}$ be a Grothendieck topos. Then we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]) \simeq \mathbf{Flat}(\mathscr{C}, \mathscr{E})$$

(natural in $\mathscr{E}$), which sends

- *a flat functor $A : \mathscr{C} \to \mathscr{E}$ to the geometric morphism $\mathscr{E} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ determined by the functors R_A and $- \otimes_{\mathscr{C}} A$, and*
- *a geometric morphism $f : \mathscr{E} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ to the flat functor given by the composite $f^* \circ y$ of $f^* : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathscr{E}$ with the Yoneda embedding $y : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.*

Geometric morphisms to $\mathbf{Sh}(\mathscr{C}, J)$ I

Definition

Let $\mathscr{E}$ be a Grothendieck topos.

- A family $\{f_i : a_i \to a \mid i \in I\}$ of arrows in $\mathscr{E}$ with common codomain is said to be epimorphic if for any pair of arrows $g, h : a \to b$ with domain a, $g = h$ if and only if $g \circ f_i = h \circ f_i$ for all $i \in I$.
- If $(\mathscr{C}, J)$ is a site, a functor $F : \mathscr{C} \to \mathscr{E}$ is said to be J-continuous if it sends J-covering sieves to epimorphic families.

The full subcategory of $\mathbf{Flat}(\mathscr{C}, \mathscr{E})$ on the J-continuous flat functors will be denoted by $\mathbf{Flat}_J(\mathscr{C}, \mathscr{E})$.

Geometric morphisms to $\mathbf{Sh}(\mathscr{C}, J)$ II

Theorem

For any site $(\mathscr{C}, J)$ and Grothendieck topos $\mathscr{E}$, the above-mentioned equivalence between geometric morphisms and flat functors restricts to an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathbf{Sh}(\mathscr{C}, J)) \simeq \mathbf{Flat}_J(\mathscr{C}, \mathscr{E})$$

natural in $\mathscr{E}$.

Sketch of proof.

Appeal to the previous theorem

- identifying the geometric morphisms $\mathscr{E} \to \mathbf{Sh}(\mathscr{C}, J)$ with the geometric morphisms $\mathscr{E} \to [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ which factor through the canonical geometric inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\text{op}}, \mathbf{Set}]$, and
- using the characterization of such morphisms as the geometric morphisms $f : \mathscr{E} \to [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ such that the composite $f^* \circ y$ of the inverse image functor f^* of f with the Yoneda embedding $y : \mathscr{C} \to [\mathscr{C}^{\text{op}}, \mathbf{Set}]$ sends J-covering sieves to colimits in $\mathscr{E}$ (equivalently, to epimorphic families in $\mathscr{E}$).

□

Flat = filtering

Definition

A functor $F : \mathscr{C} \to \mathscr{E}$ from a small category $\mathscr{C}$ to a Grothendieck topos $\mathscr{E}$ is said to be filtering if it satisfies the following conditions:

(a) For any object E of $\mathscr{E}$ there exist an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathscr{E}$ and for each $i \in I$ an object b_i of $\mathscr{C}$ and a generalized element $E_i \to F(b_i)$ in $\mathscr{E}$.

(b) For any two objects c and d in $\mathscr{C}$ and any generalized element $\langle x, y\rangle : E \to F(c) \times F(d)$ in $\mathscr{E}$ there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathscr{E}$ and for each $i \in I$ an object b_i of $\mathscr{C}$ with arrows $u_i : b_i \to c$ and $v_i : b_i \to d$ in $\mathscr{C}$ and a generalized element $z_i : E_i \to F(b_i)$ in $\mathscr{E}$ such that $\langle F(u_i), F(v_i)\rangle \circ z_i = \langle x, y\rangle \circ e_i$ for all $i \in I$.

(c) For any two parallel arrows $u, v : d \to c$ in $\mathscr{C}$ and any generalized element $x : E \to F(d)$ in $\mathscr{E}$ for which $F(u) \circ x = F(v) \circ x$, there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathscr{E}$ and for each $i \in I$ an arrow $w_i : b_i \to d$ and a generalized element $y_i : E_i \to F(b_i)$ such that $u \circ w_i = v \circ w_i$ and $F(w_i) \circ y_i = x \circ e_i$ for all $i \in I$.

Theorem

A functor $F : \mathscr{C} \to \mathscr{E}$ from a small category $\mathscr{C}$ to a Grothendieck topos $\mathscr{E}$ is flat if and only if it is filtering.

Remarks

- *For any small category $\mathscr{C}$, a functor $P : \mathscr{C} \to$ **Set** is filtering if and only if its category of elements $\int P$ is a filtered category (equivalently, if it is a filtered colimit of representables).*
- *For any small cartesian category $\mathscr{C}$, a functor $\mathscr{C} \to \mathscr{E}$ is flat if and only if it preserves finite limits.*

Morphisms and comorphisms of sites

Geometric morphisms can be naturally induced by functors between sites satisfying appropriate properties:

Definition

- A morphism of sites $(\mathscr{C}, J) \to (\mathscr{D}, K)$ is a functor $F : \mathscr{C} \to \mathscr{D}$ such that the composite $l' \circ F$, where l' is the canonical functor $\mathscr{D} \to \mathbf{Sh}(\mathscr{D}, K)$, is flat and J-continuous. If $\mathscr{C}$ and $\mathscr{D}$ have finite limits then F is a morphism of sites if and only if it preserves finite limits and is cover-preserving.
- A comorphism of sites $(\mathscr{D}, K) \to (\mathscr{C}, J)$ is a functor $\pi : \mathscr{D} \to \mathscr{C}$ which is cover-reflecting (in the sense that for any $d \in \mathscr{D}$ and any J-covering sieve S on $\pi(d)$ there is a K-covering sieve R on d such that $\pi(R) \subseteq S$).

Theorem

- *Every morphism of sites $(\mathscr{C}, J) \to (\mathscr{D}, K)$ induces a geometric morphism $\mathbf{Sh}(\mathscr{D}, K) \to \mathbf{Sh}(\mathscr{C}, J)$.*
- *Every comorphism of sites $(\mathscr{D}, K) \to (\mathscr{C}, J)$ induces a geometric morphism $\mathbf{Sh}(\mathscr{D}, K) \to \mathbf{Sh}(\mathscr{C}, J)$.*

Toposes as mathematical universes

W. Lawvere and M. Tierney discovered that a topos could not only be seen as a generalized space, but also as a mathematical universe in which one can do mathematics similarly to how one does it in the classical context of sets, with the only important exception that one must argue constructively. In fact, the internal logic of a topos, captured to a great extent by its subobject classifier, is in general intuitionistic.

Amongst other things, this view of toposes as mathematical universes paved the way for:

- Exploiting the inherent 'flexibility' of the notion of topos to construct 'new mathematical worlds' having particular properties.
- Considering models of any kind of (first-order) mathematical theory not just in the classical set-theoretic setting, but inside every topos, and hence 'relativise' Mathematics.

Subobjects in a Grothendieck topos

Since limits in a topos $\mathbf{Sh}(\mathscr{C}, J)$ are computed as in the presheaf topos $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$, a subobject of a sheaf F in $\mathbf{Sh}(\mathscr{C}, J)$ is just a subsheaf, that is a subfunctor which is a sheaf.

Notice that a subfunctor $F' \subseteq F$ is a sheaf if and only if for every J-covering sieve S and every element $x \in F(c)$, $x \in F'(c)$ if and only if $F(f)(x) \in F'(\text{dom}(f))$ for every $f \in S$.

Theorem

- *For any Grothendieck topos $\mathscr{E}$ and any object a of $\mathscr{E}$, the poset $\mathrm{Sub}_{\mathscr{E}}(a)$ of all subobjects of a in $\mathscr{E}$ is a complete Heyting algebra.*
- *For any arrow $f : a \to b$ in a Grothendieck topos $\mathscr{E}$, the pullback functor $f^* : \mathrm{Sub}_{\mathscr{E}}(b) \to \mathrm{Sub}_{\mathscr{E}}(a)$ has both a left adjoint $\exists_f : \mathrm{Sub}_{\mathscr{E}}(a) \to \mathrm{Sub}_{\mathscr{E}}(b)$ and a right adjoint $\forall_f : \mathrm{Sub}_{\mathscr{E}}(a) \to \mathrm{Sub}_{\mathscr{E}}(b)$.*

The Heyting operations on subobjects

Proposition

The collection $\mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(E)$ of subobjects of an object E in $\mathbf{Sh}(\mathscr{C},J)$ has the structure of a complete Heyting algebra with respect to the natural ordering $A \leq B$ if and only if for every $c \in \mathscr{C}$, $A(c) \subseteq B(c)$. We have that

- $(A \wedge B)(c) = A(c) \cap B(c)$ *for any* $c \in \mathscr{C}$*;*
- $(A \vee B)(c) = \{x \in E(c) \mid \{f : d \to c \mid E(f)(x) \in A(d) \cup B(d)\} \in J(c)\}$ *for any* $c \in \mathscr{C}$*;*

  (the infinitary analogue of this holds)
- $(A \Rightarrow B)(c) = \{x \in E(c) \mid$ for every $f : d \to c, E(f)(x) \in A(d)$ implies $E(f)(x) \in B(d)\}$ *for any* $c \in \mathscr{C}$*.*
- *the bottom subobject* $0 \rightarrowtail E$ *is given by the embedding into* E *of the initial object* 0 *of* $\mathbf{Sh}(\mathscr{C},J)$ *(given by:* $0(c) = \emptyset$ *if* $\emptyset \notin J(c)$ *and* $0(c) = \{*\}$ *if* $\emptyset \in J(c)$*);*
- *the top subobject is the identity one.*

Remark

From the Yoneda Lemma it follows that the subobject classifier Ω in $\mathbf{Sh}(\mathscr{C},J)$ has the structure of an internal Heyting algebra in $\mathbf{Sh}(\mathscr{C},J)$.

The interpretation of quantifiers

Let $\phi : E \to F$ be a morphism in $\mathbf{Sh}(\mathscr{C}, J)$.

- The pullback functor

$$\phi^* : \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(F) \to \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(E)$$

  is given by: $\phi^*(B)(c) = \phi(c)^{-1}(B(c))$ for any subobject $B \rightarrowtail F$ and any $c \in \mathscr{C}$.

- The left adjoint

$$\exists_\phi : \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(E) \to \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(F)$$

  is given by: $\exists_\phi(A)(c) = \{y \in E(c) \mid \{f : d \to c \mid (\exists a \in A(d))(\phi(d)(a) = E(f)(y))\} \in J(c)\}$
  for any subobject $A \rightarrowtail E$ and any $c \in \mathscr{C}$.

- The right adjoint

$$\forall_\phi : \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(E) \to \mathrm{Sub}_{\mathbf{Sh}(\mathscr{C},J)}(F)$$

  is given by $\forall_\phi(A)(c) = \{y \in E(c) \mid \text{ for all } f : d \to c, \phi(d)^{-1}(E(f)(y)) \subseteq A(d)\}$
  for any subobject $A \rightarrowtail E$ and any $c \in \mathscr{C}$.

Model theory in toposes

We can consider models of arbitrary first-order theories in any Grothendieck topos, thanks to the rich categorical structure present on it.

The notion of model of a first-order theory in a topos is a natural generalization of the usual Taskian definition of a (set-based) model of the theory.

Let Σ be a (possibly multi-sorted) first-order signature. A *structure* M over Σ in a category $\mathscr{E}$ with finite products is specified by the following data:

- any sort A of Σ is interpreted by an *object* MA of $\mathscr{E}$
- any function symbol $f : A_1, \dots, A_n \to B$ of Σ is interpreted as an *arrow* $Mf : MA_1 \times \cdots \times MA_n \to MB$ in $\mathscr{E}$
- any relation symbol $R \rightarrowtail A_1, \dots, A_n$ of Σ is interpreted as a *subobject* $MR \rightarrowtail MA_1 \times \cdots \times MA_n$ in $\mathscr{E}$

Any formula $\{\vec{x} \,.\, \phi\}$ in a given context $\vec{x}$ over Σ is interpreted as a subobject $[[\vec{x} \,.\, \phi]]_M \rightarrowtail MA_1 \times \cdots \times MA_n$ defined recursively on the structure of the formula.
A model of a theory $\mathbb{T}$ over a first-order signature Σ is a structure over Σ in which all the axioms of $\mathbb{T}$ are satisfied.

Geometric theories

Definition

A geometric theory $\mathbb{T}$ is a theory over a first-order signature Σ whose axioms can be presented in the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are *geometric formulae*, that is formulae in the context $\vec{x}$ built up from atomic formulae over Σ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.

Remark

Inverse image functors of geometric morphisms of toposes always preserve models of a geometric theory (but in general not those of an arbitrary first-order theory).

Most of the first-order theories naturally arising in Mathematics are geometric; anyway, if a finitary first-order theory is not geometric, one can always canonically associate with it a geometric theory, called its *Morleyization*, having the same set-based models.

Classifying toposes

It was realized in the seventies (thanks to the work of several people, notably including W. Lawvere, A. Joyal, G. Reyes and M. Makkai) that:

- Every geometric theory $\mathbb{T}$ has a classifying topos $\mathscr{E}_{\mathbb{T}}$ which is characterized by the following representability property: for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

  natural in $\mathscr{E}$, where
  - $\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}})$ is the category of geometric morphisms $\mathscr{E} \to \mathscr{E}_{\mathbb{T}}$ and
  - $\mathbb{T}\text{-mod}(\mathscr{E})$ is the category of $\mathbb{T}$-models in $\mathscr{E}$.
- The classifying topos of a geometric theory $\mathbb{T}$ can be canonically built as the category $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of sheaves on the syntactic site $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$.

The syntactic category of a geometric theory

Definition (Makkai and Reyes 1977)

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. The syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the 'renaming'-equivalence classes of geometric formulae-in-context $\{\vec{x} \, . \, \phi\}$ over Σ and as arrows $\{\vec{x} \, . \, \phi\} \to \{\vec{y} \, . \, \psi\}$ (where the contexts $\vec{x}$ and $\vec{y}$ are supposed to be disjoint without loss of generality) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\vec{x}, \vec{y})$ which are $\mathbb{T}$-provably functional i.e. such that the sequents

$$(\phi \vdash_{\vec{x}} (\exists y)\theta),$$
$$(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi), \text{ and}$$
$$((\theta \wedge \theta[\vec{z}/\vec{y}]) \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{y} = \vec{z}))$$

are provable in $\mathbb{T}$.

- The composite of two arrows

$$\{\vec{x} \, . \, \phi\} \xrightarrow{[\theta]} \{\vec{y} \, . \, \psi\} \xrightarrow{[\gamma]} \{\vec{z} \, . \, \chi\}$$

is defined as the $\mathbb{T}$-provable-equivalence class of the formula $(\exists \vec{y})\theta \wedge \gamma$.

- The identity arrow on an object $\{\vec{x} \, . \, \phi\}$ is the arrow

$$\{\vec{x} \, . \, \phi\} \xrightarrow{[\phi \wedge \vec{x'} = \vec{x}]} \{\vec{x'} \, . \, \phi[\vec{x'}/\vec{x}]\}$$

The syntactic site

On the syntactic category of a geometric theory it is natural to put the Grothendieck topology defined as follows:

Definition

The syntactic topology $J_{\mathbb{T}}$ on the syntactic category $\mathscr{C}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ is the geometric topology on it; in particular,

a small family $\{[\theta_i] : \{\vec{x_i} . \phi_i\} \to \{\vec{y} . \psi\}\}$ in $\mathscr{C}_{\mathbb{T}}$ is $J_{\mathbb{T}}$-covering

if and only if

the sequent $(\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x_i})\theta_i)$ is provable in $\mathbb{T}$.

This notion is instrumental for identifying the models of the theory $\mathbb{T}$ in any geometric category $\mathscr{C}$ (and in particular in any Grothendieck topos) as suitable functors defined on the syntactic category $\mathscr{C}_{\mathbb{T}}$ with values in $\mathscr{C}$; indeed, these are precisely the $J_{\mathbb{T}}$-continuous cartesian functors $\mathscr{C}_{\mathbb{T}} \to \mathscr{C}$. So if $\mathscr{C}$ is a Grothendieck topos they correspond precisely to the geometric morphisms from $\mathscr{C}$ to $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$. This topos therefore classifies $\mathbb{T}$.

Morita equivalence

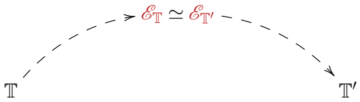
- Two mathematical theories are said to be Morita-equivalent if have the same classifying topos (up to equivalence): this means that they have equivalent categories of models in every Grothendieck topos $\mathscr{E}$, naturally in $\mathscr{E}$.
- Every Grothendieck topos is the classifying topos of *some* geometric theory (and in fact, of infinitely many theories).
- So a Grothendieck topos can be seen as a canonical representative of equivalence classes of theories modulo Morita-equivalence.

Toposes as *bridges*

- The notion of Morita-equivalence is ubiquitous in Mathematics; indeed, it formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods'.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted in terms of Morita-equivalences.
- On the other hand, Topos Theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.
- Any two theories which are bi-interpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.
- We can expect most of the categorical equivalences between categories of set-based models of geometric theories to lift to Morita equivalences.

Toposes as *bridges*

- In the topos-theoretic study of theories, the latter are represented by sites (of definition of their classifying topos or of some other topos naturally attached to them).
- The existence of theories which are Morita-equivalent to each other translates into the existence of different sites of definition (or, more generally, presentations) for the same Grothendieck topos.
- Grothendieck toposes can be effectively used as 'bridges' for transferring notions, properties and results across different Morita-equivalent theories:



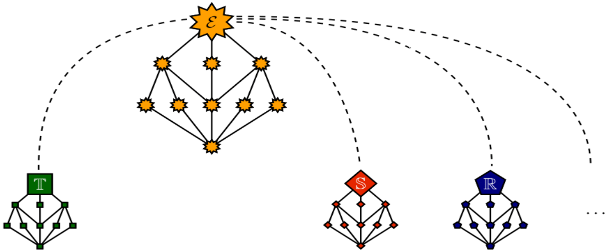
- The transfer of information takes place by expressing topos-theoretic invariants in terms of the different sites of definition (or, more generally, presentations) for the given topos.
- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

Toposes as *bridges*

- This methodology is technically effective because the relationship between a topos and its representations is often very natural, enabling us to easily transfer invariants across different representations (and hence, between different theories).
- On the other hand, the 'bridge' technique is highly non-trivial, in the sense that it often yields deep and surprising results. This is due to the fact that a given invariant can manifest itself in significanly different ways in the context of different presentations.
- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories and constructions. Indeed, many important invariants of mathematical structures are actually invariants of toposes (think for instance of cohomology or homotopy groups) and topos-theoretic invariants considered on the classifying topos $\mathscr{E}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ often translate into interesting logical (i.e. syntactic or semantic) properties of $\mathbb{T}$.

Toposes as *bridges*

- The fact that topos-theoretic invariants specialize to important properties or constructions of natural mathematical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications in Mathematics as a whole: for instance

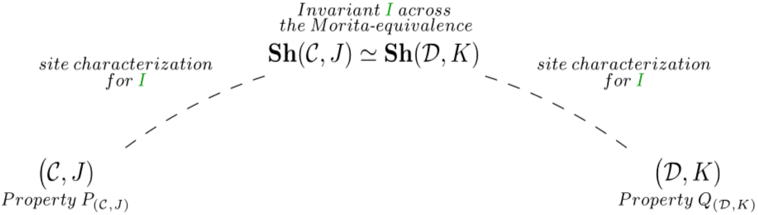


Lattices of theories

This picture represents the lattice structure on the collection of the subtoposes of a topos $\mathscr{E}$ inducing lattice structures on the collection of 'quotients' of geometric theories $\mathbb{T}$, $\mathbb{S}$, $\mathbb{R}$ classified by it.

The 'bridge-building' technique

- Decks of 'bridges': Morita-equivalences (or more generally morphisms or other kinds of relations between toposes)
- Arches of 'bridges': Site characterizations for topos-theoretic invariants (or more generally 'unravelings' of topos-theoretic invariants in terms of concrete representations of the relevant topos)



The 'bridge' yields a logical equivalence (or an implication) between the 'concrete' properties $P_{(\mathscr{C},J)}$ and $Q_{(\mathscr{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

Topos-theoretic invariants

- By a topos-theoretic invariant we mean any notion which is invariant under categorical equivalence of toposes.
- The notion of a geometric morphism of toposes has notably allowed to build general comology theories starting from the categories of internal abelian groups or modules in toposes. In particular, the topos-theoretic viewpoint has allowed Grothendieck to refine and enrich the study of cohomology, up to the so-called 'six-operation formalism'.
  The cohomological invariants have had a tremendous impact on the development of modern Algebraic Geometry and beyond.
- On the other hand, also homotopy-theoretic invariants such as the fundamental group and the higher homotopy groups can be defined as invarants of toposes.
- Still, these are by no means the only invariants that one can consider on toposes: indeed, there are infinitely many invariants of toposes (of algebraic, logical, geometric or whatever nature), the notion of identity for toposes being simply categorical equivalence.

A few selected applications

Since the theory of topos-theoretic 'bridges' was introduced in 2010, several applications of it have been obtained in different fields of Mathematics, such as:

- Model theory (topos-theoretic Fraïssé theorem)
- Proof theory (various results for first-order theories)
- Algebra (topos-theoretic generalization of topological Galois theory)
- Topology (topos-theoretic interpretation/generation of Stone-type and Priestley-type dualities)
- Functional analysis (various results on Gelfand spectra and Wallman compactifications)
- Many-valued logics and lattice-ordered groups (two joint papers with A. C. Russo)
- Cyclic homology, as reinterpreted by A. Connes (work on *"cyclic theories"*, jointly with N. Wentzlaff)
- Algebraic geometry (logical analysis of (co)homological motives, cf. the paper *"Syntactic categories for Nori motives"* joint with L. Barbieri-Viale and L. Lafforgue)

Some examples of 'bridges'

We shall now discuss a few 'bridges' established in the context of the above-mentioned applications:

- Topological Galois theory
- Theories of presheaf type
- Topos-theoretic Fraïssé theorem
- Stone-type dualities

The results are completely *different*... but the methodology is always the *same*!

Topological Galois theory

Recall that classical topological Galois theory provides, given a Galois extension $F \subseteq L$, a bijective correspondence between the intermediate field extensions (resp. finite field extensions) $F \subseteq K \subseteq L$ and the closed (resp. open) subgroups of the Galois group $\mathrm{Aut}_F(L)$.

This admits the following categorical reformulation: the functor $K \to \mathrm{Hom}(K, L)$ defines an equivalence of categories

$$(\mathscr{L}_F^L)^{\mathrm{op}} \simeq \mathbf{Cont}_t(\mathrm{Aut}_F(L)),$$

where $\mathscr{L}_F^L$ is the category of finite intermediate field extensions and $\mathbf{Cont}_t(\mathrm{Aut}_F(L))$ is the category of continuous non-empty transitive actions of $\mathrm{Aut}_F(L)$ on discrete sets.

A natural question thus arises: can we characterize the categories $\mathscr{C}$ whose dual is equivalent to (or fully embeddable into) the category of (non-empty transitive) actions of a topological automorphism group?

The topos-theoretic interpretation

Key observation: the above equivalence extends to an equivalence of toposes

$$\mathbf{Sh}({\mathscr{L}_F^L}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}_F(L)),$$

where J_{at} is the atomic topology on ${\mathscr{L}_F^L}^{\mathrm{op}}$ and $\mathbf{Cont}(\mathrm{Aut}_F(L))$ is the topos of continuous actions of $\mathrm{Aut}_F(L)$ on discrete sets.

It is therefore natural to investigate our problem by using the methods of topos theory: more specifically, we shall look for conditions on a small category $\mathscr{C}$ and on an object u of its ind-completion for the existence of an equivalence of toposes of the form

$$\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}(u)) .$$

We will then be able to obtain, starting from such an equivalence, an answer to our question, and hence build Galois-type theories in a great variety of different mathematical contexts.

The key notions I

- A category $\mathscr{C}$ is said to satisfy the amalgamation property (AP) if for every objects $a, b, c \in \mathscr{C}$ and morphisms $f : a \to b$, $g : a \to c$ in $\mathscr{C}$ there exists an object $d \in \mathscr{C}$ and morphisms $f' : b \to d$, $g' : c \to d$ in $\mathscr{C}$ such that $f' \circ f = g' \circ g$:

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ c & \xrightarrow[g']{} & d \end{array}$$

- A category $\mathscr{C}$ is said to satisfy the joint embedding property (JEP) if for every pair of objects $a, b \in \mathscr{C}$ there exists an object $c \in \mathscr{C}$ and morphisms $f : a \to c$, $g : b \to c$ in $\mathscr{C}$:

$$\begin{array}{ccc} & & a \\ & & \downarrow{\scriptstyle f} \\ b & \xrightarrow[g]{} & c \end{array}$$

The key notions II

- An object $u \in \text{Ind-}\mathscr{C}$ is said to be $\mathscr{C}$-universal if for every $a \in \mathscr{C}$ there exists an arrow $\chi : a \to u$ in $\text{Ind-}\mathscr{C}$:

$$a \overset{\chi}{\dashrightarrow} u$$

- An object $u \in \text{Ind-}\mathscr{C}$ is said to be $\mathscr{C}$-ultrahomogeneous if for any object $a \in \mathscr{C}$ and arrows $\chi_1 : a \to u$, $\chi_2 : a \to u$ in $\text{Ind-}\mathscr{C}$ there exists an automorphism $j : u \to u$ such that $j \circ \chi_1 = \chi_2$:

$$\begin{array}{ccc} a & \xrightarrow{\chi_1} & u \\ & \underset{\chi_2}{\searrow} & \downarrow{\scriptstyle j} \\ & & u \end{array}$$

Topological Galois theory as a 'bridge'

Theorem

Let $\mathscr{C}$ be a small category satisfying the amalgamation and joint embedding properties, et let u be a $\mathscr{C}$-universal et $\mathscr{C}$-ultrahomogeneous object of the ind-completion Ind-$\mathscr{C}$ *of $\mathscr{C}$. Then there is an equivalence of toposes*

$$\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}(u)),$$

where $\mathrm{Aut}(u)$ *is endowed with the topology in which a basis of open neighbourhoods of the identity is given by the subgroups of the form* $I_\chi = \{\alpha \in \mathrm{Aut}(u) \mid \alpha \circ \chi = \chi\}$ *for $\chi : c \to u$ an arrow in* Ind-$\mathscr{C}$ *from an object c of $\mathscr{C}$.*
This equivalence is induced by the functor

$$F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cont}(\mathrm{Aut}(u))$$

which sends any object c of $\mathscr{C}$ on the set $\mathrm{Hom}_{\text{Ind-}\mathscr{C}}(c, u)$ *(endowed with the obvious action of* $\mathrm{Aut}(u)$*) and any arrow $f : c \to d$ in $\mathscr{C}$ to the* $\mathrm{Aut}(u)$*-equivariant map*

$$- \circ f : \mathrm{Hom}_{\text{Ind-}\mathscr{C}}(d, u) \to \mathrm{Hom}_{\text{Ind-}\mathscr{C}}(c, u) .$$

Topological Galois theory as a 'bridge'

The following result arises from two 'bridges', respectively obtained by considering the invariant notions of atom and of arrow between atoms.

Theorem

Under the hypotheses of the last theorem, the functor F is full and faithful if and only if every arrow of $\mathscr{C}$ is a strict monomorphism, and it is an equivalence on the full subcategory $\mathbf{Cont}_t(\mathrm{Aut}(u))$ of $\mathbf{Cont}(\mathrm{Aut}(u))$ on the non-empty transitive actions if $\mathscr{C}$ is moreover atomically complete.

$$\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\mathrm{Aut}(u))$$

$$\mathscr{C}^{\mathrm{op}} \qquad\qquad\qquad \mathbf{Cont}_t(\mathrm{Aut}(u))$$

This theorem generalizes Grothendieck's theory of Galois categories and can be applied for generating Galois-type theories in different fields of Mathematics, for example that of finite groups and that of finite graphs.

Moreover, if a category $\mathscr{C}$ satisfies the first but not the second condition of the theorem, our topos-theoretic approach gives us a fully explicit way to complete it, by means of the addition of 'imaginaries', so that also the second condition gets satisfied.

Theories of presheaf type

Definition

A geometric theory is said to be of presheaf type if it is classified by a presheaf topos.

Theories of presheaf type are very important in that they constitute the basic 'building blocks' from which every geometric theory can be built. Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a 'quotient' of a theory of presheaf type.

These theories are the logical counterpart of small categories, in the sense that:

- For any theory of presheaf type $\mathbb{T}$, its category $\mathbb{T}$-mod(**Set**) of (set-based) models is equivalent to the ind-completion of the full subcategory f.p.$\mathbb{T}$-mod(**Set**) on the finitely presentable models.
- Any small category $\mathscr{C}$ is, up to idempotent splitting completion, equivalent to the category f.p.$\mathbb{T}$-mod(**Set**) for some theory of presheaf type $\mathbb{T}$.

Moreover, any geometric theory $\mathbb{T}$ can be expanded to a theory classified by the topos [f.p.$\mathbb{T}$-mod(**Set**), **Set**].

Theories of presheaf type

Every finitary algebraic (or, more generally, cartesian) theory is of presheaf type, but this class contains many other interesting mathematical theories including

- the theory of linear orders (classified by the simplicial topos)
- the theory of algebraic extensions of a given field
- the theory of flat modules over a ring
- the theory of lattice-ordered abelian groups with strong unit
- the 'cyclic theories' (classified by the cyclic topos, the epicyclic topos and the arithmetic topos)
- the theory of perfect MV-algebras (or more generally of local MV-algebras in a proper variety of MV-algebras)
- the geometric theory of finite sets

Any theory of presheaf type $\mathbb{T}$ gives rise to two different representations of its classifying topos, which can be used to build 'bridges' connecting its syntax and semantics:

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

$$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \qquad\qquad (\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

Irreducible formulae and finitely presentable models

Definition

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then a geometric formula $\phi(\vec{x})$ over Σ is said to be $\mathbb{T}$-irreducible if, regarded as an object of the syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$, it does not admit any non-trivial $J_{\mathbb{T}}$-covering sieves.

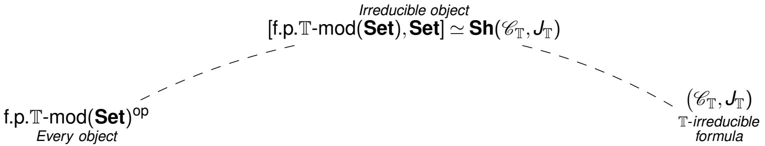
Theorem

Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ. Then

(i) *Any finitely presentable $\mathbb{T}$-model in* **Set** *is presented by a $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ;*

(ii) *Conversely, any $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ presents a $\mathbb{T}$-model.*

In fact, the category f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ *is equivalent to the full subcategory $\mathscr{C}_{\mathbb{T}}^{\text{irr}}$ of $\mathscr{C}_{\mathbb{T}}$ on the $\mathbb{T}$-irreducible formulae.*



A definability theorem

Theorem

Let $\mathbb{T}$ be a theory of preshef type and suppose that we are given, for every finitely presentable **Set***-model $\mathcal{M}$ of $\mathbb{T}$, a subset $R_{\mathcal{M}}$ of $\mathcal{M}^n$ in such a way that every $\mathbb{T}$-model homomorphism $h : \mathcal{M} \to \mathcal{N}$ maps $R_{\mathcal{M}}$ into $R_{\mathcal{N}}$. Then there exists a geometric formula-in-context $\phi(x_1, \dots, x_n)$ such that $R_{\mathcal{M}} = [[\vec{x} \,.\, \phi]]_{\mathcal{M}}$ for each finitely presentable $\mathbb{T}$-model $\mathcal{M}$.*

Subobject of $UA_1 \times \cdots \times UA_n$

$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}$

Functorial assignment

$M \to R_M \subseteq MA_1 \times \cdots \times MA_n$

$(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$

Geometric formula

$\phi(x_1^{A_1}, \dots, x_n^{A_n})$

Topos-theoretic Fraïssé theorem

The following result, which generalizes Fraïssé's theorem in classical model theory, arises from a triple 'bridge'.

Definition

A set-based model M of a geometric theory $\mathbb{T}$ is said to be homogeneous if for any arrow $y : c \to M$ in $\mathbb{T}$-mod(**Set**) and any arrow f in f.p.$\mathbb{T}$-mod(**Set**) there exists an arrow u in $\mathbb{T}$-mod(**Set**) such that $u \circ f = y$:

$$\begin{array}{ccc} c & \xrightarrow{\ y\ } & M \\ {\scriptstyle f}\downarrow & \nearrow_{u} & \\ d & & \end{array}$$

Theorem

Let $\mathbb{T}$ be a theory of presheaf type such that the category f.p.$\mathbb{T}$-mod(**Set**) *is non-empty and has AP and JEP. Then the theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models is complete and atomic.*

Topos-theoretic Fraïssé theorem

Atomic topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$

Atomic site i.e.
AP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

Atomicity of $\mathbb{T}'$

Two-valued topos

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$

JEP on f.p.$\mathbb{T}$-mod(**Set**)

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

Completeness of $\mathbb{T}'$

Point of

$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$

homogeneous $\mathbb{T}$-model in **Set**

$(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$

$\mathbb{T}'$-model in **Set**

Stone-type dualities through 'bridges'

The 'bridge-building' technique allows one to unify all the classical Stone-type dualities between special kinds of preorders and partial orders, locales or topological spaces as instances of just one topos-theoretic phenomenon, and to generate many new such dualities.

More precisely, this machinery generates Stone-type dualities/equivalences by functorializing 'bridges' of the form

$$\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}}) \simeq \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}})$$

$\mathscr{C}$ $\mathscr{D}$

where

- $\mathscr{C}$ is a preorder (regarded as a category),
- $J_{\mathscr{C}}$ is a (subcanonical) Grothendieck topology on $\mathscr{C}$,
- $\mathscr{C}$ is a $K_{\mathscr{D}}$-dense full subcategory of $\mathscr{D}$, and
- $J_{\mathscr{C}}$ is the induced Grothendieck topology $(K_{\mathscr{D}})|_{\mathscr{C}}$ on $\mathscr{C}$.

Stone-type dualities through 'bridges'

Our machinery relies on the following key points:

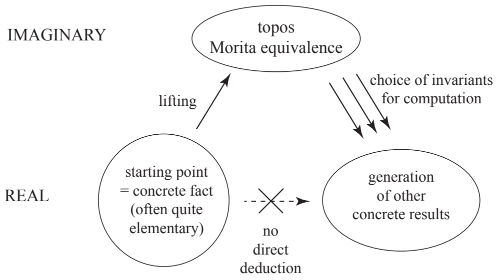
- The possibility of defining Grothendieck topologies on posets in an intrinsic way which exploits the lattice-theoretic structure present on them.
- The possibility of functorializing the assignments $\mathscr{C} \to \mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}})$ and $\mathscr{D} \to \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}})$ by means of the (covariant or contravariant) theory of morphisms of sites.
- The possibility of recovering (under suitable hypotheses which are satisfied in a great number of cases) a given preordered structure from the associated topos by means of a topos-theoretic invariant.
  More precisely, if the topologies $K_{\mathscr{D}}$ (resp. $J_{\mathscr{C}}$) can be 'uniformly described through an invariant C of families of subterminals in a topos' then the elements of $\mathscr{D}$ (resp. of $\mathscr{C}$) can be recovered as the subterminal objects of the topos $\mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}})$ (resp. $\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}})$) which satisfy a condition of C-compactness.

A mathematical morphogenesis

- The essential ambiguity given by the fact that any topos is associated in general with an infinite number of theories or different sites allows to study the relations between different theories, and hence the theories themselves, by using toposes as 'bridges' between these different presentations.
- Every topos-theoretic invariant generates a veritable mathematical morphogenesis resuting from its expression in terms of different representations of toposes, which gives rise in general to connections between properties or notions that are completely different and apparently unrelated from each other
- The mathematical exploration is therefore in a sense 'reversed' since it is guided by the Morita-equivalences and by topos-theoretic invariants, from which one proceeds to extract concrete information on the theoires that one wishes to study.

The duality between 'real' and 'imaginary'

- The passage from a site (or a theory) to the associated topos can be regarded as a sort of 'completion' by the addition of 'imaginaries' (in the model-theoretic sense), which materializes the potential contained in the site (or theory).
- The duality between the (relatively) unstructured world of presentations of theories and the maximally strucured world of toposes is of great relevance as, on the one hand, the 'simplicity' and concreteness of theories or sites makes it easy to manipulate them, while, on the other hand, computations are much easier in the 'imaginary' world of toposes thanks to their very rich internal structure and the fact that invariants live at this level.



Some key features of toposes

Here are some essential features of toposes, which account for their relevance in Mathematics:

- Generality: Unlike most of the invariants used in Mathematics, the level of generality of topos-theoretic invariants is such as to make them suitable for effectively comparing with each other theories or objects coming from different fields of Mathematics.
- Expressivity: On the other hand, many important invariants arising in Mathematics can be expressed as topos-theoretic invariants (think for instance of the cohomological and homotopy-theoretic invariants).
- Centrality: The fact that topos-theoretic invariants often manifest as important properties or constructions of natural mathematical or logical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications in Mathematics as a whole.
- Technical flexibility: Toposes are mathematical universes which are very rich in terms of internal structure; moreover, they have a very-well behaved representation theory, which makes them extremely effective computational tools, in particular when they are considered as 'bridges'.

Toposes as 'bridges' and the Erlangen Program

The methodology 'toposes as bridges' is a vast extension of Felix Klein's Erlangen Program (A. Joyal)

More specifically:

- Every group gives rise to a topos (namely, the category of actions of it), but the notion of topos is much more general.
- As Klein classified geometries by means of their automorphism groups, so we can study first-order geometric theories by studying the associated classifying toposes.
- As Klein established surprising connections between very different-looking geometries through the study of the algebraic properties of the associated automorphism groups, so the methodology 'toposes as bridges' allows to discover non-trivial connections between properties, concepts and results pertaining to different mathematical theories through the study of the categorical invariants of their classifying toposes.

Future directions

The evidence provided by the results obtained so far shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories and for generating new equivalences, dualities and symmetries across different fields of Mathematics.

In fact, toposes have an authentic creative power in Mathematics, in the sense that their study naturally leads to the discovery of a great number of notions and 'concrete' results in different mathematical fields, which are pertinent but often unsuspected.

In the next years, we intend to continue pursuing the development of these general unifying methodologies both at the theoretical level and at the applied level, in order to continue developing the potential of toposes as fundamental tools in the study of mathematical theories and their relations, and as key concepts defining a new way of doing Mathematics liable to bring distinctly new insights in a great number of different subjects.

Future directions

Central themes in this programme will be:

- investigation of important dualities or correspondences in Mathematics from a topos-theoretic perspective (in particular, the theory of motives, class field theory and the Langlands programme)
- systematic study of invariants of toposes in terms of their presentations, and introduction of new invariants which capture important aspects of concrete mathematical problems
- interpretation and generalization of important parts of classical and modern model theory in terms of toposes and development of a functorial model theory
- introduction of new methodologies for generating Morita-equivalences
- development of general techniques for building spectra by using classifying toposes
- generalization of the 'bridge' technique to the setting of higher categories and toposes through the introduction of higher geometric logic
- development of a relative theory of classifying toposes

For further reading

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Sheaves in geometry and logic: a first introduction to topos theory
Springer-Verlag (1992).