The geometry of morphisms and equivalences of toposes

Olivia Caramello

University of Insubria (Como) and IHÉS

Topics in contemporary mathematics
Universidad Nacional de Colombia, 26-30 October 2020

The “unifying notion” of topos

“It is the topos theme which is this “bed” or “deep river” where come to be married geometry and algebra, topology and arithmetic, mathematical logic and category theory, the world of the “continuous” and that of “discontinuous” or discrete structures. It is what I have conceived of most broad to perceive with finesse, by the same language rich of geometric resonances, an “essence” which is common to situations most distant from each other coming from one region or another of the vast universe of mathematical things”.

A. Grothendieck

Topos theory can be regarded as a unifying subject in Mathematics, with great relevance as a framework for systematically investigating the relationships between different mathematical theories and studying them by means of a multiplicity of different points of view. Its methods are transversal to the various fields and complementary to their own specialized techniques. In spite of their generality, the topos-theoretic techniques are liable to generate insights which would be hardly attainable otherwise and to establish deep connections that allow effective transfers of knowledge between different contexts.

Toposes as unifying 'bridges'

The unifying nature of toposes is intimately tied to the fact that Grothendieck toposes are objects that can be built from a great variety of different mathematical situations, ideally in a such a way that essential features of such situations can be captured by means of topos-theoretic invariants on the associated toposes.

Indeed, the possibility of presenting a given topos is multiple ways is at the heart of the bridge technique, introduced in 2010 in the paper "*The unification of Mathematics via Topos Theory*" as a means to effectively use toposes as unifying spaces across different mathematical contexts, as well as for studying mathematical theories in an intrinsically dynamical way.

The key idea is that the possibility of presenting a topos in a multitude of different ways can be effectively exploited for building unifying 'bridges' between theories having an equivalent, or strictly related, mathematical content.

These 'bridges' allow effective and often deep transfers of notions, ideas and results across the theories.

In fact, 'bridges' have proved useful not only for connecting different theories with each other, but also for working inside a given mathematical theory and investigating it from a multiplicity of different points of view.

Presentations of toposes

The most classical way for building toposes is through sites (indeed, a Grothendieck topos is, by definition, any category equivalent to the category of sheaves on a small-generated sites).

Still, toposes can also be canonically associated with groups (or more generally topological or localic groupoids) or with (first-order geometric) theories or with non-commutative structures such as quantales or quantaloids, etc.

In this course we shall study morphisms and equivalences of toposes from the (geometric) point of view of their site presentations.

Grothendieck toposes

- The notion of topos was introduced in the early sixties by A. Grothendieck with the aim of bringing a topological or geometric intuition also in areas where actual topological spaces do not occur.
- Grothendieck realized that many important properties of topological spaces X can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.
- He then defined toposes as more general categories of sheaves of sets, by replacing the topological space X by a (small) site, that is a pair $(\mathcal{C}, J)$ consisting of a (small) category $\mathcal{C}$ and a 'generalized notion of covering' J on it, and taking sheaves (in a generalized sense) over the site:

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, rightsquigarrow] & \mathbf{Sh}(X) \arrow[d, rightsquigarrow] \\ (\mathcal{C}, J) \arrow[r, dashed] & \mathbf{Sh}(\mathcal{C}, J) \end{tikzcd}$$

Sieves

The notion of Grothendieck topology on a category represents a 'categorification' of the classical notion of covering of an open set of a topological space by a family of open subsets. In order to define it in full generality, one needs to talk about *sieves*.

Definition

- Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a presieve P in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c.
- Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a sieve S in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c such that

$$f \in S \Rightarrow f \circ g \in S$$

  whenever this composition makes sense.
- We say that a sieve S is generated by a presieve P on an object c if it is the smallest sieve containing it, that is if it is the collection of arrows to c which factor through an arrow in P.

If S is a sieve on c and $h : d \to c$ is any arrow to c, then

$$h^*(S) := \{g \mid \mathrm{cod}(g) = d,\ h \circ g \in S\}$$

is a sieve on d.

Grothendieck topologies

Definition

- A Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each object c of $\mathcal{C}$ a collection $J(c)$ of sieves on c in such a way that
  - (i) (maximality axiom) the maximal sieve $M_c = \{f \mid \mathrm{cod}(f) = c\}$ is in $J(c)$;
  - (ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;
  - (iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

  The sieves S which belong to $J(c)$ for some object c of $\mathcal{C}$ are said to be J-covering.
- A site (resp. small site) is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a category (resp. a small category) and J is a Grothendieck topology on $\mathcal{C}$.
- A site $(\mathcal{C}, J)$ is said to be small-generated if $\mathcal{C}$ is locally small and has a small J-dense subcategory (that is, a category $\mathcal{D}$ such that every object of $\mathcal{C}$ admits a J-covering sieve generated by arrows whose domains lie in $\mathcal{D}$, and for every arrow $f : d \to c$ in $\mathcal{C}$ where d lies in $\mathcal{D}$ the family of arrows $g : \mathrm{dom}(g) \to d$ such that $f \circ g$ lies in $\mathcal{D}$ generates a J-covering sieve).

Examples of Grothendieck topologies

- For any (small) category $\mathcal{C}$, the trivial topology on $\mathcal{C}$ is the Grothendieck topology in which the only sieve covering an object c is the maximal sieve M_c.
- The dense topology D on a category $\mathcal{C}$ is defined by: for a sieve S,

$$S \in D(c) \quad \text{if and only if} \quad \text{for any } f : d \to c \text{ there exists } g : e \to d \text{ such that } f \circ g \in S\,.$$

If $\mathcal{C}$ satisfies the right Ore condition i.e. the property that any two arrows $f : d \to c$ and $g : e \to c$ with a common codomain c can be completed to a commutative square

$$\begin{tikzcd} \bullet \arrow[r, dashed] \arrow[d, dashed] & d \arrow[d, "f"] \\ e \arrow[r, "g"] & c \end{tikzcd}$$

then the dense topology on $\mathcal{C}$ specializes to the atomic topology on $\mathcal{C}$ i.e. the topology J_{at} defined by: for a sieve S,

$$S \in J_{at}(c) \text{ if and only if } S \neq \emptyset\,.$$

Examples of Grothendieck topologies

- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathcal{O}(X)}$ on the poset category $\mathcal{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathcal{O}(X))$,

$$S \in J_{\mathcal{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U \, .$$

- More generally, given a frame (or complete Heyting algebra) H, we can define a Grothendieck topology J_H, called the *canonical topology on* H, by:

$$\{a_i \mid i \in I\} \in J_H(a) \text{ if and only if } \bigvee_{i \in I} a_i = a \, .$$

- The Zariski topology on the opposite of the category $\mathbf{Rng}_{\text{f.g.}}$ of finitely generated commutative rings with unit is defined by: for any cosieve S in $\mathbf{Rng}_{\text{f.g.}}$ on an object A, $S \in Z(A)$ if and only if S contains a finite family $\{\xi_i : A \to A_{f_i} \mid 1 \leq i \leq n\}$ of canonical maps $\xi_i : A \to A_{f_i}$ in $\mathbf{Rng}_{\text{f.g.}}$ where $\{f_1, \ldots, f_n\}$ is a set of elements of A which is not contained in any proper ideal of A.
- Given a (first-order geometric) theory $\mathbb{T}$, one can naturally associate a site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ with it, called its *syntactic site*, which embodies essential aspects of the syntax and proof theory of $\mathbb{T}$.

Sheaves on a site

Definition

- A presheaf on a (small) category $\mathcal{C}$ is a functor $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$.
- Let $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ be a presheaf on $\mathcal{C}$ and S be a sieve on an object c of $\mathcal{C}$.

  A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

  $$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d\,.$$

  An amalgamation for such a family is a single element $x \in P(c)$ such that

  $$P(f)(x) = x_f \quad \text{for all } f \text{ in } S\,.$$

Sheaves on a site

- Given a site $(\mathcal{C}, J)$, a presheaf on $\mathcal{C}$ is a J-sheaf if every matching family for any J-covering sieve on any object of $\mathcal{C}$ has a unique amalgamation.
- The category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on the site $(\mathcal{C}, J)$ is the full subcategory of $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.
- A Grothendieck topos is any category equivalent to the category of sheaves on a small (or equivalently, small-generated) site.

Examples of toposes

The following basic examples show that toposes can be naturally attached to mathematical notions as different as (small) categories, topological spaces, or groups.

Examples

- For any (small) category $\mathcal{C}$, $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is the category of sheaves $\mathbf{Sh}(\mathcal{C}, T)$ where T is the trivial topology on $\mathcal{C}$.
- For any topological space X, $\mathbf{Sh}(\mathcal{O}(X), J_{\mathcal{O}(X)})$ is equivalent to the usual category $\mathbf{Sh}(X)$ of sheaves on X.
- For any (topological) group G, the category $BG = \mathbf{Cont}(G)$ of continuous actions of G on discrete sets is a Grothendieck topos (equivalent, as we shall see, to the category $\mathbf{Sh}(\mathbf{Cont}_t(G), J_{at})$ of sheaves on the full subcategory $\mathbf{Cont}_t(G)$ on the non-empty transitive actions with respect to the atomic topology).

Basic properties of Grothendieck toposes

Grothendieck toposes satisfy all the categorical properties that one might hope for:

Theorem

Let $(\mathcal{C}, J)$ be a site. Then

- *the inclusion* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ *has a left adjoint* $a : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ *(called the associated sheaf functor), which preserves finite limits.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has all (small) limits, which are preserved by the inclusion functor* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$*; in particular, limits are computed pointwise and the terminal object* $1_{\mathbf{Sh}(\mathcal{C}, J)}$ *of* $\mathbf{Sh}(\mathcal{C}, J)$ *is the functor* $T : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ *sending each object* $c \in Ob(\mathcal{C})$ *to the singleton* $\{*\}$*.*
- *The associated sheaf functor* $a : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ *preserves colimits; in particular,* $\mathbf{Sh}(\mathcal{C}, J)$ *has all (small) colimits.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has exponentials, which are constructed as in the topos* $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$*.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has a subobject classifier.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has a separating set of objects.*

Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism. The natural notion of morphism of geometric morphisms if that of geometric transformation.

Definition

(i) Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of a pair of functors $f_* : \mathcal{E} \to \mathcal{F}$ (the direct image of f) and $f^* : \mathcal{F} \to \mathcal{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathcal{E} \to \mathcal{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

(iii) A point of a topos $\mathcal{E}$ is a geometric morphism $\mathbf{Set} \to \mathcal{E}$.

- Grothendieck toposes and geometric morphisms between them form a 2-category.
- Given two toposes $\mathcal{E}$ and $\mathcal{F}$, geometric morphisms from $\mathcal{E}$ to $\mathcal{F}$ and geometric transformations between them form a category, denoted by $\mathbf{Geom}(\mathcal{E}, \mathcal{F})$.

Examples of geometric morphisms

- A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The direct image $\mathbf{Sh}(f)_*$ sends a sheaf $F \in Ob(\mathbf{Sh}(X))$ to the sheaf $\mathbf{Sh}(f)_*(F)$ defined by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for any open subset V of Y. The inverse image $\mathbf{Sh}(f)^*$ acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.
- Every Grothendieck topos $\mathcal{E}$ has a unique geometric morphism $\mathcal{E} \to \mathbf{Set}$. The direct image is the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$, sending an object $e \in \mathcal{E}$ to the set $\mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, e)$, while the inverse image functor $\Delta : \mathbf{Set} \to \mathcal{E}$ sends a set S to the coproduct $\coprod_{s \in S} 1_{\mathcal{E}}$.
- For any site $(\mathcal{C}, J)$, the pair of functors formed by the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and the associated sheaf functor $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ yields a geometric morphism $i : \mathbf{Sh}(\mathcal{C}, J) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.
- For any Grothendieck topos $\mathcal{E}$ and any morphism $f : P \to Q$ in $\mathcal{E}$, the pullback functor $f^* : \mathcal{E}/Q \to \mathcal{E}/P$ has both a left adjoint (namely, the functor Σ_f given by composition with f) and a right adjoint π_f. It is therefore the inverse image of a geometric morphism $\mathcal{E}/P \to \mathcal{E}/Q$.

A general hom-tensor adjunction I

Theorem

Let $\mathcal{C}$ be a small category, $\mathcal{E}$ be a locally small cocomplete category and $A : \mathcal{C} \to \mathcal{E}$ a functor. Then we have an adjunction

$$L_A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \rightleftarrows \mathcal{E} : R_A$$

where the right adjoint $R_A : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is defined for each $e \in Ob(\mathcal{E})$ and $c \in Ob(\mathcal{C})$ by:

$$R_A(e)(c) = \text{Hom}_{\mathcal{E}}(A(c), e)$$

and the left adjoint $L_A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}$ is defined by

$$L_A(P) = \text{colim}(A \circ \pi_P),$$

where π_P is the canonical projection functor $\int P \to \mathcal{C}$ from the category of elements $\int P$ of P to $\mathcal{C}$.

A general hom-tensor adjunction II

Remarks

- *The functor L_A can be considered as a generalized tensor product, since, by the construction of colimits in terms of coproducts and coequalizers, we have the following coequalizer diagram:*

$$\coprod_{\substack{c\in\mathcal{C}, p\in P(c)\\ u:c'\to c}} A(c') \underset{\tau}{\overset{\theta}{\rightrightarrows}} \coprod_{c\in\mathcal{C},\, p\in P(c)} A(c) \xrightarrow{\phi} L_A(P),$$

where

$$\theta(c, p, u, x) = (c', P(u)(p), x)$$

and

$$\tau(c, p, u, x) = (c, p, A(u)(x))\ .$$

For this reason, we shall also denote L_A by

$$-\otimes_{\mathcal{C}} A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}\ .$$

- *We can rewrite the above coequalizer as follows:*

$$\coprod_{c,c'\in\mathcal{C}} P(c)\times \mathrm{Hom}_{\mathcal{C}}(c', c)\times A(c') \underset{\tau}{\overset{\theta}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} P(c)\times A(c) \xrightarrow{\phi} P\otimes_{\mathcal{C}} A\ .$$

From this we see that this definition is symmetric in P and A, that is

$$P\otimes_{\mathcal{C}} A \cong A\otimes_{\mathcal{C}^{\text{op}}} P\ .$$

A couple of corollaries

Corollary

Every presheaf is a colimit of representables. More precisely, for any presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, we have

$$P \cong \mathrm{colim}(y_{\mathcal{C}} \circ \pi_P),$$

where $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is a Yoneda embedding and π_P is the canonical projection $\int P \to \mathcal{C}$.

Corollary

For any small category $\mathcal{C}$, the topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is the free cocompletion of $\mathcal{C}$ (via the Yoneda embedding $y_{\mathcal{C}}$); that is, any functor $A : \mathcal{C} \to \mathcal{E}$ to a cocomplete category $\mathcal{E}$ extends, uniquely up to isomorphism, to a colimit-preserving functor $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$ along $y_{\mathcal{C}}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "A"] \arrow[d, "y_{\mathcal{C}}"'] & \mathcal{E} \\ {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]} \arrow[ur, dashed] & \end{tikzcd}$$

Separating sets of objects

Definition

A separating set of objects for a Grothendieck topos $\mathcal{E}$ is a set C of objects of $\mathcal{E}$ such that for any object A of $\mathcal{E}$, the collection of arrows from objects in C to A is epimorphic.

Proposition

For any site $(\mathcal{C}, J)$, the collection of objects of the form $l_J(c)$ (for $c \in \mathcal{C}$), where

$$l_J : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$$

is the composite of the Yoneda embedding $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$ with the associated sheaf functor $a_J : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$, is a separating set of objects for the topos $\mathbf{Sh}(\mathcal{C}, J)$.

The following theorem provides a sort of converse to this proposition:

Theorem

For any set of objects $\mathcal{C}$ of $\mathcal{E}$ which is separating, we have an equivalence

$$\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J^{can}_{\mathcal{E}}|_{\mathcal{C}})$$

where $J^{can}_{\mathcal{E}}|_{\mathcal{C}}$ is the Grothendieck topology induced on $\mathcal{C}$ (regarded as a full subcategory of $\mathcal{E}$).

Geometric morphisms as flat functors I

Definition

- A functor $A : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a locally small topos $\mathcal{E}$ with small colimits is said to be flat if the functor $- \otimes_{\mathcal{C}} A : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$ preserves finite limits.
- The full subcategory of $[\mathcal{C}, \mathcal{E}]$ on the flat functors will be denoted by $\mathbf{Flat}(\mathcal{C}, \mathcal{E})$.

Theorem

Let $\mathcal{C}$ be a small category and $\mathcal{E}$ be a Grothendieck topos. Then we have an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]) \simeq \mathbf{Flat}(\mathcal{C}, \mathcal{E})$$

(natural in $\mathcal{E}$), which sends

- *a flat functor $A : \mathcal{C} \to \mathcal{E}$ to the geometric morphism $\mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ determined by the functors R_A and $- \otimes_{\mathcal{C}} A$, and*
- *a geometric morphism $f : \mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ to the flat functor given by the composite $f^* \circ y_{\mathcal{C}}$ of $f^* : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$ with the Yoneda embedding $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.*

Flat = filtering

Definition

A functor $F : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a Grothendieck topos $\mathcal{E}$ is said to be filtering if it satisfies the following conditions:

(a) For any object E of $\mathcal{E}$ there exist an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an object b_i of $\mathcal{C}$ and a generalized element $E_i \to F(b_i)$ in $\mathcal{E}$.

(b) For any two objects c and d in $\mathcal{C}$ and any generalized element $\langle x, y \rangle : E \to F(c) \times F(d)$ in $\mathcal{E}$ there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an object b_i of $\mathcal{C}$ with arrows $u_i : b_i \to c$ and $v_i : b_i \to d$ in $\mathcal{C}$ and a generalized element $z_i : E_i \to F(b_i)$ in $\mathcal{E}$ such that $\langle F(u_i), F(v_i) \rangle \circ z_i = \langle x, y \rangle \circ e_i$ for all $i \in I$.

(c) For any two parallel arrows $u, v : d \to c$ in $\mathcal{C}$ and any generalized element $x : E \to F(d)$ in $\mathcal{E}$ for which $F(u) \circ x = F(v) \circ x$, there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an arrow $w_i : b_i \to d$ and a generalized element $y_i : E_i \to F(b_i)$ such that $u \circ w_i = v \circ w_i$ and $F(w_i) \circ y_i = x \circ e_i$ for all $i \in I$.

Theorem (Mac Lane and Moerdijk)

A functor $F : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a Grothendieck topos $\mathcal{E}$ is flat if and only if it is filtering.

Remarks

- *For any small category $\mathcal{C}$, a functor $P : \mathcal{C} \to$ **Set** is filtering if and only if its category of elements $\int P$ is a filtered category (equivalently, if it is a filtered colimit of representables).*
- *For any small cartesian category $\mathcal{C}$, a functor $\mathcal{C} \to \mathcal{E}$ is flat if and only if it preserves finite limits.*

Geometric morphisms to $\mathbf{Sh}(\mathcal{C}, J)$

Definition

If $(\mathcal{C}, J)$ is a site, a flat functor $F : \mathcal{C} \to \mathcal{E}$ to a Grothendieck topos is said to be J-continuous if it sends J-covering sieves to epimorphic families.

The full subcategory of $\mathbf{Flat}(\mathcal{C}, \mathcal{E})$ on the J-continuous flat functors will be denoted by $\mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$.

Theorem

For any site $(\mathcal{C}, J)$ and Grothendieck topos $\mathcal{E}$, the above-mentioned equivalence between geometric morphisms and flat functors restricts to an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$$

natural in $\mathcal{E}$.

Sketch of proof.

Appeal to the previous theorem

- identifying the geometric morphisms $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ with the geometric morphisms $\mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ which factor through the canonical geometric inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, and
- using the characterization of such morphisms as the geometric morphisms $f : \mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ such that the composite $f^* \circ y$ of the inverse image functor f^* of f with the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ sends J-covering sieves to epimorphic families in $\mathcal{E}$.

Morphisms and comorphisms of sites

Geometric morphisms can be naturally induced by functors between sites satisfying appropriate properties:

Definition

- A morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that the composite $l' \circ F$, where l' is the canonical functor $\mathcal{D} \to \mathbf{Sh}(\mathcal{D}, K)$, is flat and J-continuous. If $\mathcal{C}$ and $\mathcal{D}$ have finite limits then F is a morphism of sites if and only if it preserves finite limits.
- A comorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ is a functor $\pi : \mathcal{D} \to \mathcal{C}$ which has the covering-lifting property (in the sense that for any $d \in \mathcal{D}$ and any J-covering sieve S on $\pi(d)$ there is a K-covering sieve R on d such that $\pi(R) \subseteq S$).

We have the following well-known fundamental result, which we shall discuss in detail below:

Theorem

- *Every morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ induces a geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.*
- *Every comorphism of sites $\pi : (\mathcal{D}, K) \to (\mathcal{C}, J)$ induces a geometric morphism $C_\pi : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.*

Characterizing morphisms of sites

We can explicitly characterize the functors which are morphisms of sites by using:

- the characterization of filtering functors with values in a Grothendieck topos as functors which send certain families to epimorphic families,
- the fact that the image under the associated sheaf functor of a family of natural transformations with common codomain is epimorphic if and only if the family is locally jointly surjective, and
- the following description of the arrows in a Grothendieck topos between objects coming from a site in terms of locally compatible families of arrows in the site.

Arrows in a Grothendieck topos

Given a site $(\mathcal{C}, J)$, for two arrows $h, k : c \to d$ in $\mathcal{C}$ we shall write $h \equiv_J k$ for J-local equality, that is, to mean that there exists a J-covering sieve S on c such that $h \circ f = k \circ f$ for every $f \in S$. Notice that, denoting by l the canonical functor $\mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$, $l(h) = l(k)$ if and only if $h \equiv_J k$.

Proposition

Let $(\mathcal{C}, J)$ be a small-generated site.

(i) *Then any arrow $\xi : l(c) \to l(d)$ in $\mathbf{Sh}(\mathcal{C}, J)$ admits a local representation by a family of arrows $\{f_u : c_u \to c, g_u : c_u \to d \mid u \in U\}$ such that $\{f_u : c_u \to c \mid u \in U\}$ generates a J-covering sieve, for any object e and arrows $h : e \to c_u$ and $k : e \to c_{u'}$ such that $f_u \circ h = f_{u'} \circ k$ we have $g_u \circ h \equiv_J g_{u'} \circ k$, and $\xi \circ l(f_u) = l(g_u)$ for every $u \in U$.*

(ii) *Conversely, any family $\mathcal{F} : \{f_u : c_u \to c, g_u : c_u \to d \mid u \in U\}$ such that $\{f_u : c_u \to c \mid u \in U\}$ generates a J-covering sieve and for any object e and arrows $h : e \to c_u$ and $k : e \to c_{u'}$ such that $f_u \circ h = f_{u'} \circ k$ we have $g_u \circ h \equiv_J g_{u'} \circ k$, determines a unique arrow $\xi_{\mathcal{F}} : l(c) \to l(d)$ in $\mathbf{Sh}(\mathcal{C}, J)$ such that $\xi_{\mathcal{F}} \circ l(f_u) = l(g_u)$ for every $u \in U$.*

Arrows in a Grothendieck topos

(iii) Two families $\mathcal{F} = \{f_u : c_u \to c, g_u : c_u \to d \mid u \in U\}$ and $\mathcal{F}' = \{f'_v : e_v \to c, g'_v : e_v \to d \mid v \in V\}$ as in (ii) determine the same arrow $l(c) \to l(d)$ (i.e. $\xi_{\mathcal{F}} = \xi_{\mathcal{F}'}$) if and only if they are locally equal on a common refinement, i.e. if there exist a J-covering family $\{a_k : b_k \to c \mid k \in K\}$ and factorizations of it through both of them by arrows $x_k : b_k \to c_{u(k)}$ and $y_k : b_k \to e_{v(k)}$ (i.e. $f_{u(k)} \circ x_k = a_k = f'_{v(k)} \circ y_k$ for every $k \in K$) such that $g_{u(k)} \circ x_k \equiv_J g'_{v(k)} \circ y_k$ for every $k \in K$.

(iv) Given two families $\mathcal{F} = \{f_u : c_u \to c, g_u : c_u \to d \mid u \in U\}$ and $\mathcal{G} = \{h_v : d_v \to d, k_v : d_v \to e \mid v \in V\}$, the composite arrow $\xi_{\mathcal{G}} \circ \xi_{\mathcal{F}} : l(c) \to l(e)$ is induced as in (ii) by the family $\{f_u \circ x : \mathrm{dom}(x) \to c, k_v \circ y : \mathrm{dom}(y) \to e \mid (u, v, x, y) \in Z\}$, where $Z = \{(u, v, x, y) \mid u \in U, v \in V, \mathrm{dom}(x) = \mathrm{dom}(y), \mathrm{cod}(x) = c_u, \mathrm{cod}(y) = d_v, h_v \circ y = g_u \circ x\}$.

Arrows in a Grothendieck topos

Proposition

Let $(\mathcal{C}, J)$ be a small-generated site and a_J the associated sheaf functor $[\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$. Then

(i) *An arrow $\xi : l(c) \to a_J(P)$ in $\mathbf{Sh}(\mathcal{C}, J)$ (equivalently, an element of $a_J(P)(c)$) can be identified with an equivalence class of families $\{x_f \in P(\text{dom}(f)) \mid f \in S\}$ of elements of P indexed by the arrows f of a J-covering sieve S on c which are locally matching in the sense that for any arrow g composable with an arrow $f \in S$, $x_{f \circ g} \equiv_J P(g)(x_f)$, modulo the equivalence which identifies two such families when they are locally equal on a common refinement.*

(ii) *Any such family yields a local representation of ξ in the sense that $\xi \circ l(f) = r_{x_f}$ for each $f \in S$, where r_{x_f} is the image under a_J of the arrow $y_{\mathcal{C}}(\text{dom}(f)) \to P$ corresponding to the element $x_f \in P(\text{dom}(f))$ via the Yoneda lemma.*

Remark

The proposition gives an explicit description of the associated sheaf functor $a_J(P)$ of a presheaf P, different from the usual construction of it by means of the double plus construction. This alternative construction of the associated sheaf functor seems to have been first discovered (albeit not published) by Eduardo Dubuc in the eighties.

J-functional relations

More generally, for any presheaves $P, Q \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, the arrows $a_J(P) \to a_J(Q)$ in $\mathbf{Sh}(\mathcal{C}, J)$ are in natural bijection with the J-functional relations from P to Q in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, in the sense of the following

Definition

In a presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, a relation $R \rightarrowtail P \times Q$ (that is, an assignment $c \to R(c)$ to each object c of $\mathcal{C}$ of a subset $R(c)$ of $P(c) \times Q(c)$ which is *functorial* in the sense that for any arrow $f : c \to c'$ in $\mathcal{C}$, $P(f) \times Q(f)$ sends $R(c')$ to $R(c)$), is said to be J-functional from P to Q if it satisfies the following properties:

(i) for any $c \in \mathcal{C}$ and any $(x, y) \in P(c) \times Q(c)$, if $\{f : d \to c \mid (P(f)(x), Q(f)(y)) \in R(d)\} \in J(c)$ then $(x, y) \in R(c)$;

(ii) for any $c \in \mathcal{C}$ and any $(x, y), (x', y') \in R(c)$, if $x = x'$ then $\{f : d \to c \mid Q(f)(y) = Q(f)(y')\} \in J(c)$;

(iii) for any $c \in \mathcal{C}$ and any $x \in P(c)$, $\{f : d \to c \mid \exists y \in Q(d)\, (P(f)(x), y) \in R(d)\} \in J(c)$.

Morphisms of sites

Theorem

Let $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ be small-generated sites, and $l : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$, $l' : \mathcal{C}' \to \mathbf{Sh}(\mathcal{C}', J')$ be the canonical functors (given by the composite of the relevant Yoneda embedding with the associated sheaf functor). Then, given a functor $F : \mathcal{C} \to \mathcal{C}'$, the following conditions are equivalent:

(i) *A induces a geometric morphism $u : \mathbf{Sh}(\mathcal{C}', J') \to \mathbf{Sh}(\mathcal{C}, J)$ making the following square commutative:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "l"'] & \mathcal{C}' \arrow[d, "l'"] \\ \mathbf{Sh}(\mathcal{C}, J) \arrow[r, "u^*"] & \mathbf{Sh}(\mathcal{C}', J'); \end{tikzcd}$$

(ii) *The functor F is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{C}', J')$ in the sense that it satisfies the following properties:*

(1) *A sends every J-covering family in $\mathcal{C}$ into a J'-covering family in $\mathcal{C}'$.*

(2) *Every object c' of $\mathcal{C}'$ admits a J'-covering family*

$$c'_i \longrightarrow c' \,, \quad i \in I \,,$$

by objects c'_i of $\mathcal{C}'$ which have morphisms

$$c'_i \longrightarrow F(c_i)$$

to the images under A of objects c_i of $\mathcal{C}$.

Morphisms of sites

(3) For any objects c_1, c_2 of $\mathcal{C}$ and any pair of morphisms of $\mathcal{C}'$

$$f'_1 : c' \longrightarrow F(c_1)\,, \quad f'_2 : c' \longrightarrow F(c_2)\,,$$

there exists a J'-covering family

$$g'_i : c'_i \longrightarrow c'\,, \quad i \in I\,,$$

and a family of pairs of morphisms of $\mathcal{C}$

$$f^i_1 : b_i \longrightarrow c_1\,, \quad f^i_2 : b_i \to c_2\,, \quad i \in I\,,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : c'_i \longrightarrow F(b_i)\,, \quad i \in I\,,$$

making the following squares commutative:

$$\begin{tikzcd} c'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & c' \arrow[d, "f'_1"] \\ F(b_i) \arrow[r, "F(f^i_1)"] & F(c_1) \end{tikzcd} \qquad \begin{tikzcd} c'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & c' \arrow[d, "f'_2"] \\ F(b_i) \arrow[r, "F(f^i_2)"] & F(c_2) \end{tikzcd}$$

Morphisms of sites

(4) For any pair of arrows $f_1, f_2 : c \rightrightarrows d$ of $\mathcal{C}$ and any arrow of $\mathcal{C}'$

$$f' : b' \longrightarrow F(c)$$

satisfying

$$F(f_1) \circ f' = F(f_2) \circ f' ,$$

there exist a J'-covering family

$$g'_i : b'_i \longrightarrow b' , \quad i \in I ,$$

and a family of morphisms of $\mathcal{C}$

$$h_i : b_i \longrightarrow c , \quad i \in I ,$$

satisfying

$$f_1 \circ h_i = f_2 \circ h_i , \quad \forall i \in I ,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : b'_i \longrightarrow F(b_i) , \quad i \in I ,$$

making commutative the following squares:

$$\begin{tikzcd} b'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & b' \arrow[d, "f'"] \\ F(b_i) \arrow[r, "F(h_i)"] & F(c) \end{tikzcd}$$

Morphisms of sites

If F is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$, we denote by $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ the geometric morphism which it induces.

Remarks

- *The above characterization of morphisms of sites is equivalent to that given by Mike Shulman in his paper "Exact completions and small sheaves", which specifies that a functor is a morphism of sites when it is cover-preserving and covering-flat (in the sense that for any finite diagram D in $\mathcal{C}$ every cone over an object of the form $F(c)$ factors locally through the F-image of a cone over D), and also proves the above theorem by using his definition.*
- *If $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ are cartesian sites (that is, $\mathcal{C}$ and $\mathcal{D}$ are cartesian categories) then a functor $\mathcal{C} \to \mathcal{D}$ which is cartesian and sends J-covering families to K-covering families is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$.*
- *If J and K are subcanonical then a geometric morphism $g : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ is of the form $\mathbf{Sh}(f)$ for some f if and only if the inverse image functor g^* sends representables to representables; if this is the case then f is isomorphic to the restriction of g^* to the full subcategories of representables.*

Comorphisms of sites

Recall that a comorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ is a functor $\pi : \mathcal{D} \to \mathcal{C}$ such that for any $d \in \mathcal{D}$ and any J-covering sieve S on $\pi(d)$ there is a K-covering sieve R on d such that $\pi(R) \subseteq S$.

Proposition

Every comorphism of sites $\pi : \mathcal{D} \to \mathcal{C}$ induces a flat and J-continuous functor $A_\pi : \mathcal{C} \to \mathbf{Sh}(\mathcal{D}, K)$ given by

$$A_\pi(c) = a_K(\mathrm{Hom}_{\mathcal{C}}(\pi(-), c))$$

and hence a geometric morphism

$$f : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

with inverse image $f^*(F) \cong a_K(F \circ \pi)$ for any J-sheaf F on $\mathcal{C}$.

Kan extensions

The direct and image functors of geometric morphisms induced by morphisms or comorphisms of sites can be naturally described in terms of Kan extensions.

Recall that, given a functor $f : \mathcal{C} \to \mathcal{D}$,

- the right Kan extension $\mathrm{Ran}_{f^{\mathrm{op}}}$ along f^{op}, which is right adjoint to the functor $f^* : [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, is given by the following formula:

$$\mathrm{Ran}_{f^{\mathrm{op}}}(F)(b) = \varprojlim_{\phi : fa \to b} F(a),$$

where the limit is taken over the opposite of the comma category $(f \downarrow b)$.

- The left adjoint to f^* is the left Kan extension $\mathrm{Lan}_{f^{\mathrm{op}}}$ along f^{op}, which is left adjoint to f^*, is given by the following formula:

$$\mathrm{Lan}_{f^{\mathrm{op}}}(F)(b) = \varinjlim_{\phi : b \to fa} F(a),$$

where the colimit is taken over the opposite of the comma category $(b \downarrow f)$.

Geometric morphisms and Kan extensions

Proposition

(i) *Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Then*
- *the direct image* $\mathbf{Sh}(F)_*$ *of the geometric morphism*
$$\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$
induced by F is given by the restriction to sheaves of F^*;
- *the inverse image* $\mathbf{Sh}(F)^*$ *of* $\mathbf{Sh}(F)$ *is given by*
$$a_K \circ \mathrm{Lan}_{F^{\mathrm{op}}} \circ i_J,$$
where $\mathrm{Lan}_{F^{\mathrm{op}}}$ *is the left Kan extension and i_J is the inclusion* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

(ii) *Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of small-generated sites. Then*
- *the direct image $(C_F)_*$ of the geometric morphism*
$$C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$
induced by F is given by the restriction to sheaves of the right Kan extension $\mathrm{Ran}_{F^{\mathrm{op}}}$*;*
- *the inverse image $(C_F)^*$ of C_F is given by*
$$a_J \circ F^* \circ i_K,$$
where i_K is the inclusion $\mathbf{Sh}(\mathcal{D}, K) \hookrightarrow [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$.

Unifying morphisms and comorphisms of sites

In order to better contextualize the role of morphisms and of comorphisms of sites, we will now briefly review the philosophy of toposes as 'bridges', which also inspires all the other results presented in this course.

In fact, we shall unify the notions of morphism and comorphisms of sites by interpreting them as two fundamentally different ways of describing morphisms of toposes which correspond to each other under a 'bridge'.

More specifically, morphisms of sites provide an 'algebraic' perspective on morphisms of toposes, while comorphisms of sites provide a 'geometric' perspective on them.

Topos-theoretic invariants

- By a topos-theoretic invariant we mean any notion which is invariant under categorical equivalence of toposes.
- The notion of a geometric morphism of toposes is a fundamental invariant, which has notably allowed to build general comology theories starting from the categories of internal abelian groups or modules in toposes. In particular, the topos-theoretic viewpoint has allowed Grothendieck to refine and enrich the study of cohomology, up to the so-called 'six-operation formalism'. The cohomological invariants have had a tremendous impact on the development of modern Algebraic Geometry and beyond.
- On the other hand, also homotopy-theoretic invariants such as the fundamental group and the higher homotopy groups can be defined as invarants of toposes.
- Still, these are by no means the only invariants that one can consider on toposes: indeed, there are infinitely many invariants of toposes (of algebraic, logical, geometric or whatever nature), the notion of identity for toposes being simply categorical equivalence.

Toposes as *bridges*

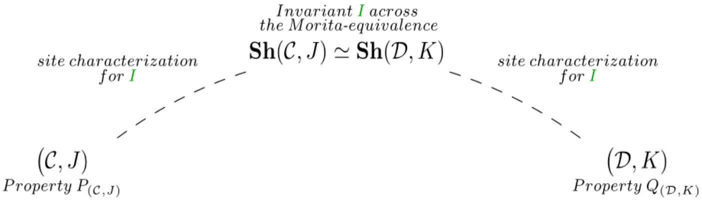
- In the topos-theoretic study of theories or 'concrete' mathematical contexts, the latter are represented by sites (of definition of their classifying topos or of some other topos naturally attached to them).
- Grothendieck toposes can be effectively used as 'bridges' for transferring notions, properties and results across them:

$$\begin{tikzcd} & \mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'} \arrow[dr, dashed, bend left] & \\ \mathbb{T} \arrow[ur, dashed, bend left] & & \mathbb{T}' \end{tikzcd}$$

- The transfer of information takes place by expressing topos-theoretic invariants in terms of the different sites of definition (or, more generally, presentations) for the given topos.
- As such, different properties (resp. constructions) arising in the context of the two presentations are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

The ‘bridge’ technique

- Decks of ‘bridges’: Morita-equivalences (that is, equivalences between different presentations of a given topos, or more generally morphisms or other kinds of relations between toposes)
- Arches of ‘bridges’: Site characterizations for topos-theoretic invariants (or more generally ‘unravelings’ of topos-theoretic invariants in terms of concrete representations of the relevant topos)



For example, this ‘bridge’ yields a logical equivalence between the ‘concrete’ properties $P_{(\mathcal{C},J)}$ and $Q_{(\mathcal{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

Toposes as *bridges*

- This methodology is technically effective because the relationship between a topos and its representations is often very natural, enabling us to transfer invariants across different representations.
- On the other hand, the ‘bridge’ technique is highly non-trivial, in the sense that it often yields deep and surprising results. This is due to the fact that a given invariant can manifest itself in significanly different ways in the context of different presentations.
- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories and constructions.

Relating morphisms and comorphisms of sites

The inspiration for our constructions is provided by the following result:

Proposition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites, and $(F : \mathcal{C} \to \mathcal{D} \dashv G : \mathcal{D} \to \mathcal{C})$ adjoint functors. Then

(i) *G is a morphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ if and only if F is a comorphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$.*

(ii) *In the situation of (i), the geometric morphism C_F induced by F coincides with the geometric morphism $\mathbf{Sh}(G)$ induced by G.*

The key idea is to replace the given sites of definition with Morita-equivalent ones in such a way that every morphism (resp. comorphism) of sites acquires a left (resp. right) adjoint, not necessarily in the classical categorial sense but in the weaker topos-theoretic sense of the associated comma categories having equivalent associated toposes.

From morphisms to comorphisms of sites

We shall turn a morphism of sites into a comorphism of sites by replacing the original codomain site with a site related to it by a morphism inducing an equivalence of toposes such that the composite of the given morphism of sites with it admits a left adjoint; this left adjoint will then be a comorphism of sites inducing the same geometric morphism (by the above proposition).

We shall denote by $(F \downarrow G)$, for two functors $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$, the comma category whose objects are the triplets (a, b, α) where $a \in \mathcal{A}$, $b \in \mathcal{B}$ and α is an arrow $F(a) \to G(b)$ in $\mathcal{C}$ (and whose arrows are defined in the obvious way).

In particular, given a functor $F : \mathcal{C} \to \mathcal{D}$, the objects of $(1_{\mathcal{D}} \downarrow F)$ are triplets of the form $(d, c, \alpha : d \to F(c))$ where $c \in \mathcal{C}$, $d \in \mathcal{D}$ and α is an arrow in $\mathcal{D}$.

From morphisms to comorphisms of sites

Theorem

Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Let i_F be the functor $\mathcal{C} \to (1_{\mathcal{D}} \downarrow F)$ sending any object c of $\mathcal{C}$ to the triplet $(F(c), c, 1_{F(c)})$ (and acting on arrows in the obvious way), and $\pi_{\mathcal{C}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{C}$ and $\pi_{\mathcal{D}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{D}$ the canonical projection functors. Let $\tilde{K}$ be the Grothendieck topology on $(1_{\mathcal{D}} \downarrow F)$ whose covering sieves are those whose image under $\pi_{\mathcal{D}}$ is K-covering. Then

(i) *$\pi_{\mathcal{C}} \dashv i_F$, $\pi_{\mathcal{D}} \circ i_F = F$, i_F is a morphism of sites $(\mathcal{C}, J) \to ((1_{\mathcal{D}} \downarrow F), \tilde{K})$ and $c_F := \pi_{\mathcal{C}}$ is a comorphism of sites $((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{C}, J)$;*

(ii) *$\pi_{\mathcal{D}} : ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{D}, K)$ is both a morphism of sites and a comorphism of sites inducing equivalences*

$$C_{\pi_{\mathcal{D}}} : \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to \mathbf{Sh}(\mathcal{D}, K)$$

and

$$\mathbf{Sh}(\pi_{\mathcal{D}}) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K})$$

which are quasi-inverse to each other and make the following triangle commute:

$$\begin{tikzcd}
\mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \arrow[rr, shift left, "C_{\pi_{\mathcal{D}}}"] \arrow[dr, "C_{\pi_{\mathcal{C}}} \cong \mathbf{Sh}(i_F)"'] & \sim & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left, "\mathbf{Sh}(\pi_{\mathcal{D}})"] \arrow[dl, "\mathbf{Sh}(F)"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

From comorphisms to morphisms of sites

Below, we shall abbreviate by $\hat{\mathcal{D}}$ the category of presheaves on a small category $\mathcal{D}$.

Theorem

Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of small-generated sites. Let $\pi'_{\mathcal{C}} : (F \downarrow 1_{\mathcal{C}}) \to \mathcal{C}$ and $\pi'_{\mathcal{D}} : (F \downarrow 1_{\mathcal{C}}) \to \mathcal{D}$ be the canonical projection functors and $j_F : \mathcal{D} \to (F \downarrow 1_{\mathcal{C}})$ the functor sending any object d of $\mathcal{D}$ to the triplet $(d, F(d), 1_{F(d)})$. Let $\overline{K}$ be the Grothendieck topology on $(F \downarrow 1_{\mathcal{C}})$ whose covering families are those which are sent by $\pi'_{\mathcal{D}}$ to K-covering families. Then

(i) *$j_F \dashv \pi'_{\mathcal{D}}$, $\pi'_{\mathcal{C}} \circ j_F = F$, $\pi'_{\mathcal{C}}$ is a comorphism of sites $(F \downarrow 1_{\mathcal{C}}, \overline{K}) \to (\mathcal{C}, J)$ and j_F is a (full and faithful) comorphism and dense morphism of sites $(\mathcal{D}, K) \to (F \downarrow 1_{\mathcal{C}}, \overline{K})$;*

(ii) *$\pi'_{\mathcal{D}}$ is both a morphism and a comorphism of sites $((F \downarrow 1_{\mathcal{C}}), \overline{K}) \to (\mathcal{D}, K)$ inducing equivalences*

$$C_{\pi'_{\mathcal{D}}} : \mathbf{Sh}((F \downarrow 1_{\mathcal{C}}), \overline{K}) \to \mathbf{Sh}(\mathcal{D}, K)$$

and

$$\mathbf{Sh}(\pi'_{\mathcal{D}}) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}((F \downarrow 1_{\mathcal{C}}), \overline{K})$$

which are quasi-inverse to each other and make the following triangle commute:

$$\begin{tikzcd}
\mathbf{Sh}((F \downarrow 1_{\mathcal{C}}), \overline{K}) \arrow[rr, shift left, "C_{\pi'_{\mathcal{D}}} \cong \mathbf{Sh}(j_F)"] \arrow[dr, "C_{\pi'_{\mathcal{C}}}"'] & \sim & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left, "\mathbf{Sh}(\pi'_{\mathcal{D}}) \cong C_{j_F}"] \arrow[dl, "C_F"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

From comorphisms to morphisms of sites

(iii) With the comorphism of sites $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ we can associate the morphism of sites

$$m_F : (\mathcal{C}, J) \to (\hat{\mathcal{D}}, \hat{K})$$

sending an object c of $\mathcal{C}$ to the presheaf $\mathrm{Hom}_{\mathcal{C}}(F(-), c)$ and $\hat{K}$ is the extension of the Grothendieck topology K along the Yoneda embedding $\mathcal{D} \to \hat{\mathcal{D}}$, which induces a geometric morphism $\mathbf{Sh}(m_F)$ making the following triangle commute:

$$\begin{tikzcd}
\mathbf{Sh}(\hat{\mathcal{D}}, \hat{K}) \arrow[rr, shift left, "\mathbf{Sh}(y_{\mathcal{D}})"] \arrow[rr, phantom, "\sim"] \arrow[dr, "\mathbf{Sh}(m_F)"'] & & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left, "C_{y_{\mathcal{D}}}"] \arrow[dl, "C_F"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

Bridging morphisms and comorphisms of sites

Theorem

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites.

(i) *Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of sites, with corresponding comorphism of sites $c_F : ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{C}, J)$ as above. Let $\pi_{\mathcal{D}} : ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{D}, K)$ be the canonical projection functor, and let*

$$w_F : (1_{\mathcal{D}} \downarrow F) \to (c_F \downarrow 1_{\mathcal{D}})$$

be the functor j_{c_F}, sending an object A of $(1_{\mathcal{D}} \downarrow F)$ to the object $(A, c_F(A), 1_{c_F(A)} : c_F(A) \to c_F(A))$. Then w_F is both a (full and faithful) comorphism and a dense morphism of sites $((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to ((c_F \downarrow 1_{\mathcal{D}}), \bar{\tilde{K}})$ satisfying the relation $\pi'''_{\mathcal{D}} \circ w_F = \pi_{\mathcal{D}}$ and inducing an equivalence relating F and c_F, which makes the following diagram commute (where $\pi'''_{\mathcal{D}}$ denotes the canonical projection functor $(c_F \downarrow 1_{\mathcal{D}}) \to \mathcal{D}$):

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{D}, K) \arrow[rr, shift left] \arrow[rr, phantom, "="] \arrow[d, shift left, "\mathbf{Sh}(\pi_{\mathcal{D}'''})"] \arrow[d, phantom, "\wr"] & & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left] \arrow[d, shift left, "\mathbf{Sh}(\pi_{\mathcal{D}})"] \arrow[d, phantom, "\wr"] \\
\mathbf{Sh}((c_F \downarrow 1_{\mathcal{D}}), \bar{\tilde{K}}) \arrow[u, shift left, "C_{\pi'''_{\mathcal{D}}}"] \arrow[dr, "C_{\pi'_{\mathcal{C}}}"'] \arrow[rr, shift left, "\mathbf{Sh}(w_F) \cong C_{\pi'_{(1_{\mathcal{D}} \downarrow F)}}"] \arrow[rr, phantom, "\sim"] & & \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \arrow[u, shift left, "C_{\pi_{\mathcal{D}}}"] \arrow[ll, shift left, "C_{w_F} \cong \mathbf{Sh}(\pi'_{(1_{\mathcal{D}} \downarrow F)})"] \arrow[dl, "C_{c_F}"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

Bridging morphisms and comorphisms of sites

(ii) Let $G : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of sites, with corresponding morphism of sites $m_G : (\mathcal{C}, J) \to (\hat{\mathcal{D}}, \hat{K})$ as above. Let

$$z_G : (G \downarrow 1_{\mathcal{C}}) \to (1_{\hat{\mathcal{D}}} \downarrow m_G)$$

be the functor sending any object $(d, c, \alpha : G(d) \to c)$ of $(G \downarrow 1_{\mathcal{C}})$ to the object $(y_{\mathcal{D}}(d), c, \overline{\alpha} : y_{\mathcal{D}}(d) \to m_G(c))$ of $(1_{\hat{\mathcal{D}}} \downarrow m_G)$, where $\overline{\alpha}$ is the arrow corresponding to the element α of m_G via the Yoneda Lemma. Then z_G is both a (full and faithful) comorphism and a dense morphism of sites $((G \downarrow 1_{\mathcal{C}}), \overline{K}) \to ((1_{\hat{\mathcal{D}}} \downarrow m_G), \widetilde{\hat{K}})$ satisfying the relation $\pi_{\hat{\mathcal{D}}} \circ z_G = y_{\mathcal{D}} \circ \pi'_{\mathcal{D}}$ and inducing an equivalence relating G and m_G, which makes the following diagram commute:

$$\begin{tikzcd}
\mathbf{Sh}(\hat{\mathcal{D}}, \hat{K}) \arrow[rr, shift left, "\mathbf{Sh}(y_{\mathcal{D}})"] \arrow[rr, phantom, "\sim"] \arrow[d, shift left, "\mathbf{Sh}(\pi_{\hat{\mathcal{D}}})"] \arrow[d, phantom, "\wr"] & & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left, "C_{y_{\mathcal{D}}}"] \arrow[d, shift left, "\mathbf{Sh}(\pi'_{\mathcal{D}}) \cong C_{j_G}"] \arrow[d, phantom, "\wr"] \\
\mathbf{Sh}((1_{\hat{\mathcal{D}}} \downarrow m_G), \widetilde{\hat{K}}) \arrow[u, shift left, "C_{\pi_{\hat{\mathcal{D}}}}"] \arrow[rr, shift left, "\mathbf{Sh}(z_G)"] \arrow[rr, phantom, "\sim"] \arrow[dr, "C_{\pi_{\mathcal{C}}} \cong \mathbf{Sh}(i_{m_G})"'] & & \mathbf{Sh}((G \downarrow 1_{\mathcal{C}}), \overline{K}) \arrow[u, shift left, "\mathbf{Sh}(j_G) \cong C_{\pi_{\mathcal{D}'}}"] \arrow[ll, shift left, "C_{z_G}"] \arrow[dl, "C_{\pi'_{\mathcal{C}}}"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

Bridging morphisms and comorphisms of sites

We shall call a functor which both a morphism and a comorphism of sites a bimorphism of sites.

The above theorem shows that the relationship between a morphism F (resp. comorphism G) of sites and the associated comorphism c_F (resp. morphism m_F) of sites is captured by the equivalence

$$\mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \simeq \mathbf{Sh}((c_F \downarrow 1_{\mathcal{D}}), \overline{\tilde{K}})$$

(resp.

$$\mathbf{Sh}((G \downarrow 1_{\mathcal{C}}), \overline{K}) \simeq \mathbf{Sh}((1_{\hat{\mathcal{D}}} \downarrow m_G), \widetilde{\hat{K}}))$$

of toposes over $\mathbf{Sh}(\mathcal{C}, J)$ induced by the bimorphism of sites w_F (resp. z_G) over $\mathcal{C}$.

Our theorem then tells us that F and c_F (resp. G and m_G) are not adjoint to each other in a concrete sense (that is, at the level of sites), since they are not defined between a pair of categories, nor the categories $(1_{\mathcal{D}} \downarrow F)$ and $(c_F \downarrow 1_{\mathcal{D}})$ (resp. the categories $(G \downarrow 1_{\mathcal{C}})$ and $(1_{\hat{\mathcal{D}}} \downarrow m_G)$) are equivalent in general; nonetheless, they become 'abstractly' adjoint in the world of toposes since toposes naturally attached to such categories are equivalent.

The dual adjunction

Definition

Let $(\mathcal{C}, J)$ be a small-generated site.

(a) The category $\mathbf{Mor}_{(\mathcal{C},J)}$ has as objects the morphisms of sites from $(\mathcal{C}, J)$ to a small generated site $(\mathcal{D}, K)$ and as arrows

$$(F : (\mathcal{C}, J) \to (\mathcal{D}, K)) \to (F' : (\mathcal{C}, J) \to (\mathcal{D}', K'))$$

between any two such morphisms the geometric morphisms

$$f : \mathbf{Sh}(\mathcal{D}', K') \to \mathbf{Sh}(\mathcal{D}, K)$$

such that $\mathbf{Sh}(F) \circ f \cong \mathbf{Sh}(F')$:

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}, J) & \mathbf{Sh}(\mathcal{D}, K) \arrow[l, "\mathbf{Sh}(F)"'] \\ & \mathbf{Sh}(\mathcal{D}', K') \arrow[u, "f"'] \arrow[ul, "\mathbf{Sh}(F')"] \end{tikzcd}$$

(b) The category $\mathbf{Com}_{(\mathcal{C},J)}$ has as objects the comorphisms of sites from a small-generated site $(\mathcal{D}, K)$ to $(\mathcal{C}, J)$ and as arrows

$$(U : (\mathcal{D}, K) \to (\mathcal{C}, J)) \to (U' : (\mathcal{D}', K') \to (\mathcal{C}, J))$$

between any two such comorphisms the geometric morphisms

$$g : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{D}', K')$$

such that $C_{U'} \circ g \cong C_U$:

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{D}, K) \arrow[r, "C_U"] \arrow[d, "g"] & \mathbf{Sh}(\mathcal{C}, J) \\ \mathbf{Sh}(\mathcal{D}', K') \arrow[ur, "C_{U'}"'] & \end{tikzcd}$$

The dual adjunction

The assignments $F \mapsto c_F$ and $G \mapsto m_G$ introduced above naturally define two functors

$$C : (\mathbf{Mor}_{(\mathcal{C},J)})^{\mathrm{op}} \to \mathbf{Com}_{(\mathcal{C},J)}$$

and

$$M : \mathbf{Com}_{(\mathcal{C},J)} \to (\mathbf{Mor}_{(\mathcal{C},J)})^{\mathrm{op}} .$$

Theorem

The functors

$$C : (\mathbf{Mor}_{(\mathcal{C},J)})^{\mathrm{op}} \to \mathbf{Com}_{(\mathcal{C},J)}$$

and

$$M : \mathbf{Com}_{(\mathcal{C},J)} \to (\mathbf{Mor}_{(\mathcal{C},J)})^{\mathrm{op}}$$

are (2-categorically) adjoint (C on the right and M on the left) and quasi-inverse to each other.

From comorphisms of sites to fibrations

The following result shows that one can naturally associate with a comorphism of sites a fibration inducing the same geometric morphism.

Definition

The fibration of generalized elements of a functor $F : \mathcal{D} \to \mathcal{C}$ is the canonical projection functor $\pi^F_{\mathcal{C}} : (1_{\mathcal{C}} \downarrow F) \to \mathcal{C}$.

Theorem

Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of small-generated sites, i'_F the canonical functor $\mathcal{D} \to (1_{\mathcal{C}} \downarrow F)$ and $K^{i'_F}$ the Grothendieck topology on $(1_{\mathcal{C}} \downarrow F)$ whose covering sieves are those whose pullback along any arrow whose domain is an object of the form $i'_F(d)$ contains the image under i'_F of a K-covering sieve on d. Let $\pi^F_{\mathcal{C}}$ and $\pi^F_{\mathcal{D}}$ be the canonical projections from $(1_{\mathcal{C}} \downarrow F)$ respectively to $\mathcal{C}$ and $\mathcal{D}$. Then

(i) *$\pi^F_{\mathcal{D}} \dashv i'_F$, $\pi^F_{\mathcal{C}} \circ i'_F = F$, $\pi^F_{\mathcal{C}}$ is a comorphism of sites $((1_{\mathcal{C}} \downarrow F), K^{i'_F}) \to (\mathcal{C}, J)$ and $\pi^F_{\mathcal{D}}$ is a comorphism of sites $((1_{\mathcal{C}} \downarrow F), K^{i'_F}) \to (\mathcal{D}, K)$;*

(ii) *i'_F is both a (full and faithful) comorphism of sites and a dense morphism of sites $(\mathcal{D}, K) \to ((1_{\mathcal{C}} \downarrow F), K^{i'_F})$ inducing equivalences*

$$C_{i'_F} : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}((1_{\mathcal{C}} \downarrow F), K^{i'_F})$$

and

$$\mathbf{Sh}(i'_F) : \mathbf{Sh}((1_{\mathcal{C}} \downarrow F), K^{i'_F}) \to \mathbf{Sh}(\mathcal{D}, K)$$

which are quasi-inverse to each other and make the following triangle commute:

$$\begin{tikzcd}
\mathbf{Sh}((1_{\mathcal{C}} \downarrow F), K^{i'_F}) \arrow[rr, shift left, "\mathbf{Sh}(i'_F) \cong C_{\pi^F_{\mathcal{D}}}"] \arrow[rr, phantom, "\sim"] \arrow[dr, "C_{\pi^F_{\mathcal{C}}}"'] & & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left, "C_{i'_F}"] \arrow[dl, "C_F"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

Fibrations as comorphisms of sites

In the converse direction, every fibration can be naturally regarded as a comorphism of sites, as follows.

Recall that, given a functor $A : \mathcal{C} \to \mathcal{D}$ and a Grothendieck topology K in $\mathcal{D}$, there is a smallest Grothendieck topology on $\mathcal{C}$ which makes A a comorphism of sites to $(\mathcal{D}, K)$. This topology, which we denote by M^A_K, is generated by the (pullback-stable) family of sieves of the form $S^A_R := \{f : \mathrm{dom}(f) \to c \mid A(f) \in R\}$ for an object c of $\mathcal{C}$ and a K-covering sieve R on $A(c)$.

Proposition

If A is a fibration, the topology M^A_K admits the following simpler description: a sieve R is M^A_K-covering if and only if the collection of cartesian arrows in R is sent by A to a K-covering family.

This point of view on fibrations was inspired by Jean Giraud's construction of the classifying topos of a stack (see the forthcoming joint work with Riccardo Zanfa).

Proposition

For any Grothendieck topology K on $\mathcal{D}$, every morphism of fibrations $(A : \mathcal{C} \to \mathcal{D}) \to (A' : \mathcal{C}' \to \mathcal{D})$ yields a comorphism of sites $(\mathcal{C}, M^A_K) \to (\mathcal{C}', M^{A'}_K)$.

Weak morphisms of toposes

Definition

A *weak morphism of toposes* $f : \mathcal{E} \to \mathcal{F}$ is a pair of adjoint functors $(f^* \dashv f_*)$.

As in the case of geometric morphism, we call f_* the *direct image* of f and f^* the *inverse image* of f.

Proposition

Let $i : \mathcal{F} \hookrightarrow \mathcal{E}$ be the geometric inclusion of a subtopos $\mathcal{F}$ of a Grothendieck topos $\mathcal{E}$ into $\mathcal{E}$, and let $f : \mathcal{G} \to \mathcal{E}$ be a weak morphism from a Grothendieck topos $\mathcal{G}$. Then the following conditions are equivalent:

(i) *The weak morphism f factors through i;*
(ii) *The direct image f_* takes values in $\mathcal{F}$ (that is, factors through i_*);*
(iii) *The inverse image f^* factors (necessarily uniquely up to isomorphism) through i^*.*

Corollary

Let $A : \mathcal{C} \to \mathcal{E}$ be a functor from an essentially small category $\mathcal{C}$ to a Grothendieck topos $\mathcal{E}$, and J be a Grothendieck topology on $\mathcal{C}$. Then the following conditions are equivalent:

(i) *The weak morphism $(L_A \dashv R_A)$ factors through the canonical geometric inclusion $i : \mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$;*
(ii) *The functor R_A takes values in $\mathbf{Sh}(\mathcal{C}, J)$;*
(iii) *The functor L_A factors (necessarily uniquely up to isomorphism) through the associated sheaf functor $a_J : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$.*

Continuous functors

The above result motivates the following definition:

Definition

(a) Given a small-generated site $(\mathcal{C}, J)$, we say that a functor $A : \mathcal{C} \to \mathcal{E}$ is J-continuous if the hom functor $R_A : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ takes values into $\mathbf{Sh}(\mathcal{C}, J)$ (equivalently, if the functor $L_A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}$ factors through $a_J : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$).

(b) Given small-generated sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, a functor $A : \mathcal{C} \to \mathcal{D}$ is said to be (J, K)-continuous if $l' \circ A$ is J-continuous, where l' is the canonical functor $\mathcal{D} \to \mathbf{Sh}(\mathcal{D}, K)$.

The following proposition shows that the above definition is equivalent to Grothendieck's notion of continuous functor:

Proposition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites and $A : \mathcal{C} \to \mathcal{D}$ a functor. Then the following conditions are equivalent:

(i) *A is (J, K)-continuous.*

(ii) *The functor*

$$D_A := (- \circ A^{\text{op}}) : [\mathcal{D}^{\text{op}}, \mathbf{Set}] \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$$

restricts to a functor $\mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.

Classifying weak morphisms of toposes

Let $[\mathcal{C}, \mathcal{E}]_J$ be the full subcategory of $[\mathcal{C}, \mathcal{E}]$ on the J-continuous functors.

Proposition

Let $\mathcal{C}$ a locally small category and $\mathcal{E}$ a Grothendieck topos.

(i) *There is an equivalence*

$$\mathbf{Wmor}(\mathcal{E}, [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]) \simeq [\mathcal{C}, \mathcal{E}]$$

sending a weak morphism $f = (f^* \dashv f_*)$ to the functor $f^* \circ y_{\mathcal{C}}$.

(ii) *For any Grothendieck topology J on $\mathcal{C}$ making $(\mathcal{C}, J)$ a small-generated site, the above equivalence restricts to an equivalence*

$$\mathbf{Wmor}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq [\mathcal{C}, \mathcal{E}]_J$$

sending a weak morphism $g = (g^* \dashv g_*)$ to the functor $g^* \circ l$.

Weak morphisms of sites

These results motivate the following

Definition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a weak morphism of sites if it is (J, K)-continuous.

Note that this notion generalizes that of morphism of sites; indeed, as morphisms of sites induce geometric morphisms of toposes, so weak morphisms of sites induce weak morphisms of toposes:

Proposition

Any weak morphism $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ of small-generated sites induces a weak geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "l"'] & \mathcal{D} \arrow[d, "l'"] \\ \mathbf{Sh}(\mathcal{C}, J) \arrow[r, "\mathbf{Sh}(F)^*"] & \mathbf{Sh}(\mathcal{D}, K) \end{tikzcd}$$

Conversely, any weak geometric morphism $f = (f^* \dashv f_*)$ such that $f^* \circ l$ factors through l' is induced by a (necessarily unique, if K is subcanonical) weak morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$.

Continuous functors

Proposition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites and $\mathcal{E}$ a Grothendieck topos. Then

(i) *A functor $A : \mathcal{C} \to \mathcal{E}$ is J-continuous if and only if for any J-covering sieve S on an object c*

$$A(c) = \varinjlim_{f:d \to c \in S} A(d)$$

for each J-covering sieve S on an object c (where the colimit is indexed by the category $\int S$ of elements of S).

(ii) *A functor $A : \mathcal{C} \to \mathcal{D}$ is (J, K)-continuous if and only if for any J-covering sieve S on an object c the canonical cocone with vertex $A(c)$ on the diagram $\{A(\mathrm{dom}(f)) \mid f \in S\}$ indexed over $\int S$ is sent by l' to a colimit in the topos $\mathbf{Sh}(\mathcal{D}, K)$.*

(iii) *Every J-continuous functor $A : \mathcal{C} \to \mathcal{E}$ is J-continuous in the sense of Mac Lane and Moerdijk (that is, sends J-covering families to epimorphic families), and the converse is true if A is flat (but not in general). More generally, every (J, K)-continuous functor $(\mathcal{C}, J) \to (\mathcal{D}, K)$ is cover-preserving, and every morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ is (J, K)-continuous.*

Continuity and cofinaliy

The above proposition suggests that the property of J-continuity could be interpreted as a sort of cofinality condition. Indeed, if A is J-continuous then in particular A sends any J-covering sieve S on an object c of $\mathcal{C}$ to an epimorphic family and hence $A(c)$ is the colimit of the cocone under the diagram whose vertices are the objects of the form $A(d)$ where d is the domain of an arrow $f : d \to c$ in S and whose arrows are all the arrows in $\mathcal{E}$ over $A(c)$ between such objects. So the condition for A to be J-continuous amounts precisely to the assertion that A be J-continuous and that this colimit be equal to the colimit $\varinjlim_{f:d\to c\in S} A(d)$.

In order to formally express continuity as a form of cofinality, we are going to introduce relative cofinality conditions.

Relative cofinality

Proposition

Let $(\mathcal{C}, J)$ be a small-generated site and $F : \mathcal{A} \to \mathcal{C}$ and $F' : \mathcal{A}' \to \mathcal{C}$ two functors to $\mathcal{C}$ related by a functor $\xi : \mathcal{A} \to \mathcal{A}'$ and a natural transformation $\alpha : F \to F' \circ \xi$. Let R_c (resp. R'_c), for any $c \in \mathcal{C}$, be the equivalence relations on the objects of the category $(c \downarrow F)$ (resp. of $(c \downarrow F')$) given by the relation of belonging to the same connected component.
Then the canonical arrow

$$\tilde{\alpha} : \mathrm{colim}_{[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}(y_{\mathcal{C}} \circ F) \to \mathrm{colim}_{[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}(y_{\mathcal{C}} \circ F')$$

is sent by a_J to an isomorphism

$$a_J(\tilde{\alpha}) : \mathrm{colim}_{\mathbf{Sh}(\mathcal{C},J)}(l \circ F) \to \mathrm{colim}_{\mathbf{Sh}(\mathcal{C},J)}(l \circ F)$$

if and only if the pair (ξ, α) satisfies the following 'cofinality' conditions:

(i) *For any object c of $\mathcal{C}$ and any arrow $y : c \to F'(a')$ in $\mathcal{C}$ there are a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ and for each $i \in I$ an object a_i of $\mathcal{A}$ and an arrow $y_i : c_i \to F(a_i)$ such that $(y \circ f_i, \alpha(a_i) \circ y_i) \in R'_{c_i}$.*

(ii) *For any object c of $\mathcal{C}$ and any arrows $x : c \to F(a)$ and $x' : c \to F(b)$ in $\mathcal{C}$ such that $(\alpha(a) \circ x, \alpha(b) \circ x') \in R'_c$ there is a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ such that $(x \circ f_i, x' \circ f_i) \in R_{c_i}$ for each $i \in I$.*

J-cofinal functors

It is interesting to apply the proposition in two notable particular cases:

(1) $F = \xi : \mathcal{A} \to \mathcal{C}$, $F' = 1_{\mathcal{C}}$, α is the identity.

(2) F' is the forgetful functor $U_{c_0} : \mathcal{C}/c_0 \to \mathcal{C}$ for an object c_0 of $\mathcal{C}$, ξ is a cocone $\{\xi_a : F(a) \to c_0 \mid a \in \mathcal{A}\}$ under the functor F with vertex c_0 and α is the identity.

Formulating the thesis of the proposition in these particular cases leads us to introduce the following

Definition

Given a small-generated site $(\mathcal{C}, J)$, a functor $F : \mathcal{A} \to \mathcal{C}$ is said to be J-cofinal if the following conditions are satisfied:

(i) For any object c of $\mathcal{C}$ there are a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ and for each $i \in I$ an object a_i of $\mathcal{A}$ and an arrow $y_i : c_i \to F(a_i)$.

(ii) For any object c of $\mathcal{C}$ and any arrows $x : c \to F(a)$ and $x' : c \to F(b)$ in $\mathcal{C}$ there is a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ such that $x \circ f_i$ and $x' \circ f_i$ belong to the same connected component of the category $(c_i \downarrow F)$ for each $i \in I$.

Two corollaries

The proposition thus yields the following two results:

Corollary

Let $(\mathcal{C}, J)$ be a small-generated site and $F : \mathcal{A} \to \mathcal{C}$ a functor. Then F is J-cofinal if and only if the canonical arrow

$$\mathrm{colim}_{\mathbf{Sh}(\mathcal{C},J)}(l \circ F) \to 1_{\mathbf{Sh}(\mathcal{C},J)}$$

is an isomorphism.

Corollary

Let $D : \mathcal{A} \to \mathcal{C}$ be a functor and ξ a cocone $\{\xi_a : D(a) \to c_0 \mid a \in \mathcal{A}\}$ under D with vertex c_0. Let U_{c_0} be the forgetful functor $\mathcal{C}/c_0 \to \mathcal{C}$, J_{c_0} the Grothendieck topology on $\mathcal{C}/c_0$ whose covering sieves are precisely those whose image under U_{c_0} is J-covering and $D_\xi : \mathcal{A} \to \mathcal{C}/c_0$ the canonical lift of D to $\mathcal{C}/c_0$ (which satisfies $U_{c_0} \circ D_\xi = D$).
Then ξ is sent by the canonical functor $l : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$ to a colimit cocone if and only if the functor D_ξ is J_{c_0}-cofinal, equivalently if and only if the following conditions are satisfied:

(i) *For any object c of $\mathcal{C}$ and any arrow $y : c \to c_0$ in $\mathcal{C}$ there are a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ and for each $i \in I$ an object a_i of $\mathcal{A}$ and an arrow $y_i : c_i \to D(a_i)$ such that $y \circ f_i = \xi_{a_i} \circ y_i$.*
(ii) *For any object c of $\mathcal{C}$ and any arrows $x : c \to D(a)$ and $x' : c \to D(b)$ in $\mathcal{C}$ such that $\xi_a \circ x = \xi_b \circ x'$ there is a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ such that $x \circ f_i$ and $x' \circ f_i$ belong to the same connected component of the category $(c_i \downarrow D)$ for each $i \in I$.*

Characterization of colimits in toposes

This notion of relative cofinality has several applications. A basic one is the characterization of colimits in Grothendieck toposes in terms of generalized elements:

Corollary

Let $D : \mathcal{A} \to \mathcal{E}$ be a functor from a small category $\mathcal{A}$ to a Grothendieck topos $\mathcal{E}$ and ξ a cocone $\{\xi_a : D(a) \to e_0 \mid a \in \mathcal{A}\}$ under D with vertex e_0. Then ξ is a colimit cocone if and only if the functor D_ξ is $(J_{\mathcal{E}}^{\text{can}})_{e_0}$-cofinal, equivalently if and only if the following conditions are satisfied:

(i) *For any object e of $\mathcal{E}$ and any arrow $y : e \to e_0$ in $\mathcal{E}$ there are an epimorphc family $\{f_i : e_i \to e \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an object a_i of $\mathcal{A}$ and an arrow $y_i : e_i \to D(a_i)$ such that $y \circ f_i = \xi_{a_i} \circ y_i$.*

(ii) *For any object e of $\mathcal{E}$ and any arrows $x : e \to D(a)$ and $x' : e \to D(b)$ in $\mathcal{E}$ such that $\xi_a \circ x = \xi_b \circ x'$ there is an epimorphic family $\{f_i : e_i \to e \mid i \in I\}$ in $\mathcal{E}$ such that $x \circ f_i$ and $x' \circ f_i$ belong to the same connected component of the category $(e_i \downarrow D)$ for each $i \in I$.*

Characterization of continuous functors

Let

$$D_S^A : \int S \to \mathcal{D}$$

be the functor sending any (d, f) of $\int S$ to $A(d)$, together with the cocone ξ_A with vertex $A(c)$ under it (whose legs are the arrows $A(f) : A(d) = D_S^A((d, f)) \to A(c)$ for any object (d, f) of $\int S$).

Applying one of the above corollaries to it, we obtain the following explicit characterization of (J, K)-continuous functors:

Proposition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites. Then a functor $A : \mathcal{C} \to \mathcal{D}$ is (J, K)-continuous if and only if it is cover-preserving (i.e., sends J-covering families to K-covering ones) and for any J-covering sieve S on an object c and any commutative square of the form

$$\begin{tikzcd} d \arrow[r] \arrow[d] & A(c') \arrow[d, "A(f)"] \\ A(c'') \arrow[r, "A(g)"] & A(c), \end{tikzcd}$$

where $f : c' \to c$ and $g : c'' \to c$ are arbitrary arrows of S, there is a K-covering family $\{d_i \to d \mid i \in I\}$ such that for each $i \in I$, the composites $d_i \to A(c')$ and $d_i \to A(c'')$ belong to the same connected component of the category $(d_i \downarrow D_S^A)$.

Indeed, the conditions of the proposition are equivalent to the requirement that that the lift

$$(D_A^S)_{\xi_A} : \int S \to \mathcal{D}/A(c)$$

of the diagram D_A^S to $\mathcal{D}/A(c)$ induced by the cocone ξ_A be $K_{A(c)}$-cofinal.

Continuity of (morphisms of) fibrations

By using the above characterization of continuous functors, one can prove

Proposition

Let $A : \mathcal{C} \to \mathcal{D}$ be a fibration. Then, for any Grothendieck topology K on $\mathcal{D}$, A is a continuous comorphism of sites $(\mathcal{C}, M_K^A) \to (\mathcal{D}, K)$.

More generally, we have the following result:

Theorem

For any Grothendieck topology K on $\mathcal{D}$, every morphism of fibrations $(A : \mathcal{C} \to \mathcal{D}) \to (A' : \mathcal{C}' \to \mathcal{D})$ is a continuous comorphism of sites $(\mathcal{C}, M_K^A) \to (\mathcal{C}', M_K^{A'})$.

Classifying essential morphisms

Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be essential if its inverse image f^* has a left adjoint, denoted by $f_!$ and called its essential image.

Theorem

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathcal{E}$ a Grothendieck topos. Let $\mathbf{Geom}_{\text{ess}}(\mathbf{Sh}(\mathcal{C}, J), \mathcal{E})$ be the category of essential geometric morphisms, and $\mathbf{Com}_{\text{cont}}((\mathcal{C}, J), (\mathcal{E}, J^{\text{can}}_{\mathcal{E}}))$ the category of J-continuous comorphisms of sites $(\mathcal{C}, J) \to (\mathcal{E}, J^{\text{can}}_{\mathcal{E}})$. Then we have an equivalence

$$\mathbf{Geom}_{\text{ess}}(\mathbf{Sh}(\mathcal{C}, J), \mathcal{E}) \simeq \mathbf{Com}_{\text{cont}}((\mathcal{C}, J), (\mathcal{E}, J^{\text{can}}_{\mathcal{E}}))$$

sending an essential geometric morphism $f = (f_! \dashv f^* \dashv f_*)$ to the comorphism of sites $f_! \circ l$ and a J-continuous comorphism of sites A to the geometric morphism C_A induced by it.

We say that two comorphisms of sites $A, A' : (\mathcal{C}, J) \to (\mathcal{D}, K)$ are K-equivalent if the geometric morphisms C_A and $C_{A'}$ that they induce are isomorphic.

Corollary

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites. Then we have an equivalence between the essential geometric morphisms $f : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ such that $f_! \circ l : \mathcal{C} \to \mathbf{Sh}(\mathcal{D}, K)$ factors through the canonical functor $l' : \mathcal{D} \to \mathbf{Sh}(\mathcal{D}, K)$ and the (J, K)-continuous comorphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$, considered up to K-equivalence.

Continuous comorphisms of sites

The following result provides alternative characterizations for the property of a comorphism of sites to be continuous:

Proposition

Let $A : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a comorphism of sites. Then the following conditions are equivalent:

(i) *A is (J, K)-continuous.*

(ii) *The left Kan extension functor $\mathrm{Lan}_{A^{\mathrm{op}}} : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ along A^{op} satisfies the property that $a_K \circ \mathrm{Lan}_{A^{\mathrm{op}}}$ factors (necessarily uniquely) through a_J.*

(iii) *The geometric morphism C_A induced by A is essential and its essential image $(C_A)_!$ makes the following diagram commute:*

$$
\begin{tikzcd}
{[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]} \arrow[r, "\mathrm{Lan}_{A^{\mathrm{op}}}"] \arrow[d, "a_J"] & {[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "a_K"] \\
\mathbf{Sh}(\mathcal{C}, J) \arrow[r, "(C_A)_!"] & \mathbf{Sh}(\mathcal{D}, K)
\end{tikzcd}
$$

If A induces an essential geometric morphism C_A then there is a canonical morphism $(C_A)_! \circ l \to l' \circ A$, and A is (J, K)-continuous if and only if this morphism is an isomorphism, equivalently if and only if the canonical morphism

$$(C_A)_! \circ a_J \to a_K \circ \mathrm{Lan}_{A^{\mathrm{op}}}$$

is an isomorphism.

Local connectedness

The notion of locally connected morphism represents a natural strengthening of the notion of essential morphism. Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be locally connected if f^* has an $\mathcal{E}$-indexed left adjoint, equivalently for any arrow $h : A \to B$ in $\mathcal{E}$, the square

$$\begin{tikzcd}
\mathcal{F}/f^*(B) \arrow[r, "(f/B)_!"] \arrow[d, "(f^*(h))^*"] & \mathcal{E}/B \arrow[d, "h^*"] \\
\mathcal{F}/f^*(A) \arrow[r, "(f/A)_!"] & \mathcal{E}/A
\end{tikzcd}$$

commutes.

The continuity of (morphisms of) fibrations implies that such comorphisms always induce essential geometric morphisms. One might thus wonder if these morphisms always induce locally connected morphisms. Interestingly, this is true for fibrations but *not* in general for morphisms of fibrations.

Characterizing locally connected morphisms

In order to characterize locally connected morphisms, we need to introduce the following

Definition

Given a functor $F : \mathcal{C} \to \mathcal{D}$, an arrow $h : d_0 \to d_1$ in $\mathcal{D}$, an object c of $\mathcal{C}$ and an arrow $x : F(c) \to d_1$ in $\mathcal{D}$:

(i) The category $\mathcal{A}^h_{(c,x)}$ has as objects the triplets (c', y, f) where c' is an object of $\mathcal{C}$, y is an arrow $F(c') \to d_0$ in $\mathcal{D}$ and $f : c' \to c$ is an arrow of $\mathcal{C}$ such that $x \circ F(f) = h \circ y$, and as arrows $(c_1, y_1, f_1) \to (c_2, y_2, f_2)$ the arrows $t : c_1 \to c_2$ in $\mathcal{C}$ such that $f_2 \circ t = f_1$ and $y_2 \circ F(t) = y_1$.

(ii) The category $\mathcal{B}^h_{(c,x)}$ has as objects the triplets (d, z, g) where d is an object of $\mathcal{D}$, z is an arrow $d \to d_0$ in $\mathcal{D}$ and $g : d \to F(c)$ is an arrow of $\mathcal{D}$ such that $x \circ g = h \circ z$, and as arrows $(d_1, z_1, g_1) \to (d_2, z_2, g_2)$ in $\mathcal{B}^h_{(c,x)}$ the arrows $s : d_1 \to d_2$ in $\mathcal{D}$ such that $g_2 \circ s = g_1$ and $z_2 \circ s = z_1$.

(iii) The categories $\mathcal{A}^h_{(c,x)}$ and $\mathcal{B}^h_{(c,x)}$ are actually fibered over $\mathcal{D}$. We shall denote by $a^h_{(c,x)} : \mathcal{A}^h_{(c,x)} \to \mathcal{D}$ the functor sending any object (c', y, f) of $\mathcal{A}^h_{(c,x)}$ to the object $F(c')$ of $\mathcal{D}$ and any arrow $s : (c_1, y_1, f_1) \to (c_2, y_2, f_2)$ in $\mathcal{A}^h_{(c,x)}$ to the arrow $F(s) : F(c_1) \to F(c_2)$ of $\mathcal{D}$, and by $b^h_{(c,x)} : \mathcal{B}^h_{(c,x)} \to \mathcal{D}$ the canonical projection functor.

(iv) The functor

$$\xi^h_{(c,x)} : a^h_{(c,x)} \to b^h_{(c,x)}$$

sends any object (c', y, f) of $\mathcal{A}^h_{(c,x)}$ to the object $(F(c'), y, F(f))$ of $\mathcal{B}^h_{(c,x)}$ and any arrow $s : (c_1, y_1, f_1) \to (c_2, y_2, f_2)$ in $\mathcal{A}^h_{(c,x)}$ to the arrow

$$F(s) : (F(c_1), y_1, F(f_1)) \to (F(c_2), y_2, F(f_2))$$

of $\mathcal{B}^h_{(c,x)}$. In fact, $\xi^h_{(c,x)}$ is a morphism of fibrations $a^h_{(c,x)} \to b^h_{(c,x)}$.

Characterizing locally connected morphisms

The following theorem provides necessary and sufficient conditions for a continuous comorphism of sites to induce a locally connected morphism:

Theorem

Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a continuous comorphism of small-generated sites. Then the following conditions are equivalent:

(i) *The geometric morphism $C_F : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ induced by F is locally connected.*

(ii) *For any arrow $h : d_0 \to d_1$ in $\mathcal{D}$, the morphism of fibrations*

$$\xi^h_{(c,x)} : a^h_{(c,x)} \to b^h_{(c,x)}$$

to $\mathcal{D}$ satisfies (together with the identical natural transformation $a^h_{(c,x)} \to b^h_{(c,x)} \circ \xi^h_{(c,x)}$) the cofinality' conditions of the above proposition.

Characterizing locally connected morphisms

Corollary

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then the geometric morphism

$$E(F) : [\mathcal{C}^{\text{op}}, \textbf{Set}] \to [\mathcal{D}^{\text{op}}, \textbf{Set}]$$

induced by F is locally connected if and only if for any arrow $h : d_0 \to d_1$ in $\mathcal{D}$, object c of $\mathcal{C}$ and arrow $x : F(c) \to d_1$ in $\mathcal{D}$, the following conditions hold:

(a) *For any object (d, z, g) of the category $\mathcal{B}^h_{(c,x)}$ there is an object (c', y, f) of the category $\mathcal{A}^h_{(c,x)}$ and an arrow*

$$s : d \to F(c') = a^h_{(c,x)}((c', y, f))$$

such that

$$1_d : d \to d = b^h_{(c,x)}((d, z, g))$$

and

$$s : d \to F(c') = b^h_{(c,x)}((F(c'), y, F(f)))$$

belong to the same connected component of the category $(d \downarrow b^h_{(c,x)})$.

(b) *For any object d of $\mathcal{D}$ and any arrows $\alpha : d \to a^h_{(c,x)}((a, y, f)) = F(a)$ and $\beta : d \to a^h_{(c,x)}((b, y', f')) = F(b)$ in $\mathcal{D}$ such that*

$$\alpha : d \to b^h_{(c,x)}((F(a), y, F(f))) = F(a)$$

and

$$\beta : d \to b^h_{(c,x)}((F(b), y', F(f'))) = F(b)$$

belong to the same connected component of the category $(d \downarrow b^h_{(c,x)})$,

$$\alpha : d \to a^h_{(c,x)}((a, y, f)) = F(a)$$

and

$$\beta : d \to b^h_{(c,x)}((b, y', f')) = F(b)$$

belong to the same connected component of the category $(d \downarrow a^h_{(c,x)})$.

This extends a partial result obtained in this connection by Johnstone.

The terminally connected factorization

We shall say that an essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is terminally connected if $f_!(1_{\mathcal{F}})$ is the terminal object of $\mathcal{E}$.
Recall that a local homeomorphism is a geometric morphism of the form $\mathcal{E}/A \to \mathcal{E}$ for an object A of $\mathcal{E}$.

Theorem

(i) *Terminally connected morphisms are orthogonal to local homeomorphisms in the 2-category of Grothendieck toposes; that is, for any commutative square*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[d, "m"] & \mathcal{E} \arrow[d, "n"] \arrow[dl, dashed, "k"'] \\ \mathcal{G} \arrow[r, "g"] & \mathcal{H}, \end{tikzcd}$$

where f is terminally connected and g is a local homeomorphism, there exists a morphism $k : \mathcal{E} \to \mathcal{G}$ (unique up to unique 2-isomorphism) making both triangles commute.

(ii) *Any essential geometric morphism can be factored, uniquely up to equivalence, as a terminally connected morphism followed by a local homeomorphism. More specifically, an essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$ factors as a terminally connected morphism $f' : \mathcal{F} \to \mathcal{E}/f_!(1)$ followed by the canonical local homeomorphism $\mathcal{E}/f_!(1) \to \mathcal{E}$.*

This factorization generalizes the well-known factorization of a locally connected morphism as a connected and locally connected morphism followed by a local homeomorphism.

The relative comprehensive factorization

By using our notion of relative cofinality, we can interpret the terminally connected factorization of an essential geometric morphism induced by a continuous comorphism of sites at the level of sites, as follows:

Given a functor $F : \mathcal{C} \to \mathcal{D}$ and a Grothendieck topology K on $\mathcal{D}$, we associate with it the K-sheaf $F_K = \mathrm{colim}_{\mathbf{Sh}(\mathcal{D},K)}(l' \circ F)$. We say that a discrete fibration to $\mathcal{D}$ is K-glueing if the presheaf corresponding to it is a K-sheaf.

Note that there is a canonical functor $\xi_K^F : \mathcal{C} \to \int F_K$ such that $\pi_{F_K} \circ \xi_K^F = F$, where π_{F_K} is the canonical projection functor $\int F_K \to \mathcal{D}$.

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and K a Grothendieck topology on $\mathcal{D}$. The K-comprehensive factorization of F is given by the composite $F_K \circ \xi_K^F$:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\xi_K^F"'] & & \mathcal{D} \\ & \int F_K \arrow[ur, "\pi_{F_K}"'] & \end{tikzcd}$$

The relative comprehensive factorization

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and K a Grothendieck topology on $\mathcal{D}$.

(i) *The K-comprehensive factorization of F is characterized by being the unique (up to equivalence) factorization of F as a M_K^p-cofinal functor $\mathcal{C} \to \mathcal{E}$ followed by a K-glueing fibration $p : \mathcal{E} \to \mathcal{D}$.*

(ii) *If F is a continuous comorphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ then $\xi_K^F : (\mathcal{C}, J) \to (\int F_K, M_K^{\pi_{F_K}})$ and $F_K : (\int F_K, M_K^{\pi_{F_K}}) \to (\mathcal{D}, K)$ are continuous comorphism of sites and $C_F \cong C_{F_K} \circ C_{\xi_K^F}$ is the terminally connected-local homeomorphism factorization of the geometric morphism $C_F : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$.*

Equivalences of toposes

We shall present several results around the theme of equivalences of toposes.

In particular, we will present a chacterization theorem providing necessary and sufficient explicit conditions for a morphism of sites to induce an equivalence of toposes. This generalizes the following classical result:

Theorem (Grothendieck's Comparison Lemma)

Let $(\mathcal{C}, J)$ be a small-generated site and $\mathcal{D}$ be a J-dense subcategory of $\mathcal{C}$. Then the sieves in $\mathcal{D}$ of the form $R \cap arr(\mathcal{D})$ for a J-covering sieve R in $\mathcal{C}$ form a Grothendieck topology $J|_{\mathcal{D}}$ on $\mathcal{D}$, and, denoting by $i : \mathcal{D} \to \mathcal{C}$ the canonical inclusion functor, the essential geometric morphism

$$E(i) : [\mathcal{D}^{\text{op}}, \mathbf{Set}] \to [\mathcal{C}^{\text{op}}, \mathbf{Set}],$$

induced by i restricts to an equivalence of categories

$$\mathbf{Sh}(\mathcal{D}, J|_{\mathcal{D}}) \simeq \mathbf{Sh}(\mathcal{C}, J)\ .$$

For this, we need to introduce general denseness conditions.

Denseness conditions

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and J (resp. K) a Grothendieck topology on $\mathcal{C}$ (resp. on $\mathcal{D}$). Then F is said to be

(a) (J, K)-faithful (resp. J-faithful) if whenever $F(h) \equiv_K F(k)$ (resp. $F(h) = F(k)$), $h \equiv_J k$;

(b) (J, K)-full (resp. J-full) if for every $x, y \in \mathcal{C}$ and any arrow $g : F(x) \to F(y)$ in $\mathcal{D}$, there exist a J-covering family of arrows $f_i : x_i \to x$ and arrows $g_i : x_i \to y$ (for each $i \in I$) such that $g \circ F(f_i) \equiv_K F(g_i)$ (resp. $g \circ F(f_i) = F(g_i)$) for all i;

(c) K-dense if for every $d \in \mathcal{D}$, there exists a K-covering family of arrows whose domains are in the image of F.

Recall that a functor $(\mathcal{C}, J) \to (\mathcal{D}, K)$ is said to be dense if it is J-faithful, J-full and K-dense, and it preserves and reflects covering families. Actually, the J-faithfulness condition is redundant for morphisms of sites since it follows from the latter condition by definition of a morphism of sites.

It was shown by Shulman (in his paper "*Exact completions and small sheaves*") that every dense morphism of sites induces an equivalence of toposes. Still, as we shall see, being dense is *not* a necessary condition for a morphism of sites to induce an equivalence of toposes. For this one needs a more refined notion of denseness, which we shall call weak denseness.

Weakly dense morphisms

Recall that a Grothendieck topology on a locally small category is said to be subcanonical if all the representables are sheaves for it.

In order to characterize the morphisms of sites which induce an equivalence of toposes, we shall proceed in two steps:

(1) We will show that if $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ is a morphism *towards a subcanonical site* $(\mathcal{D}, K)$ which induces an equivalence

$$\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

then F is dense.

(2) We will associate with $(\mathcal{D}, K)$ a Morita-equivalent subcanonical site, replace F with a morphism to this site inducing the same geometric morphism, and rephrase in terms of F the property of this latter morphism to be dense.

Weakly dense morphisms

Given a small-generated site $(\mathcal{C}, J)$ and the canonical functor $l : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$, we define $a_J(\mathcal{C})$ to be the full (dense) subcategory of $\mathbf{Sh}(\mathcal{C}, J)$ on the objects of the form $l(c)$ for $c \in \mathcal{C}$. We denote by $C_J^{\mathcal{C}}$ the Grothendieck topology $J^{\text{can}}_{\mathbf{Sh}(\mathcal{C},J)}|_{a_J(\mathcal{C})}$ induced by the canonical Grothendieck topology on the topos $\mathbf{Sh}(\mathcal{C}, J)$ on it.

- We have an equivalence

$$\mathbf{Sh}(\mathcal{D}, K) \simeq \mathbf{Sh}(a_K(\mathcal{D}), C_K^{\mathcal{D}})$$

  induced by the morphism of sites

$$l' : (\mathcal{D}, K) \to (a_K(\mathcal{D}), C_K^{\mathcal{D}}) .$$

- The site $(a_K(\mathcal{D}), C_K^{\mathcal{D}})$ is subcanonical and the morphism of sites

$$l \circ F : (\mathcal{C}, J) \to (a_K(\mathcal{D}), C_K^{\mathcal{D}})$$

  induces the same geometric morphism (up to equivalence) as F.

This motivates the following

Definition

We shall say that a morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ is weakly dense if the morphism of sites $l' \circ F : (\mathcal{C}, J) \to (a_K(\mathcal{D}), C_K^{\mathcal{D}})$ is dense.

Characterization of weakly dense morphisms

By using notably the explicit local description of arrows in a Grothendieck topos obtained above, one can prove the following

Proposition

Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of sites. Then F is a weakly dense morphism of sites if and only if it satisfies the following conditions:

(i) *P is a J-covering family in $\mathcal{C}$ if and only if $F(P)$ is a K-covering family in $\mathcal{D}$;*

(ii) *for any object d of $\mathcal{D}$ there exist a family $\{S_i \mid i \in I\}$ of K-covering sieves on objects of the form $F(c_i)$ (where c_i is an object of $\mathcal{C}$) and for each $f \in S_i$ an arrow $g_f : \mathrm{dom}(f) \to d$ such that $g_{f \circ z} \equiv_K g_f \circ z$ whenever z is composable with f, such that the family of arrows g_f (for $f \in S_i$ for some i) is K-covering;*

(iii) *for any objects x, y of $\mathcal{C}$ and any family of arrows $g_h : \mathrm{dom}(h) \to F(y)$ indexed by the arrows of a K-covering sieve U on $F(x)$ such that $g_{h \circ k} \equiv_K g_h \circ k$ for every arrow k composable with h, there exist a J-covering family of arrows $\{f_i : x_i \to x \mid i \in I\}$ and arrows $k_i : x_i \to y$ (for each $i \in I$) such that for every arrows w and z such that $F(f_i) \circ w = h \circ z$, we have $g_h \circ z \equiv_K F(k_i) \circ w$ (for every $h \in U$ and $i \in I$).*

The characterization theorem

Summarizing, we obtain the following general version of the Comparison Lemma:

Theorem

Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Then the following conditions are equivalent:

(i) *The geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ is an equivalence.*
(ii) *$l' \circ F : (\mathcal{C}, J) \to (a_K(\mathcal{D}), C_K^{\mathcal{D}})$ is a dense morphism of sites.*
(iii) *F is a weakly dense morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ (i.e. it satisfies the conditions of the above proposition).*

If K is subcanonical then any of the above conditions is equivalent to the requirement that F should be dense.

Applying this result to flat J-continuous functors $F : \mathcal{C} \to \mathcal{E}$ (regarded as morphisms of sites $(\mathcal{C}, J) \to (\mathcal{E}, J_{\mathcal{E}}^{can})$), we obtain the following criterion:

Corollary

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathcal{E}$ a Grothendieck topos and $F : \mathcal{C} \to \mathcal{E}$ a J-continuous flat functor. Then the geometric morphism $f : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ induced by F is an equivalence if and only if F satisfies the following conditions:

(i) *If the image under F of a sieve S in $\mathcal{C}$ is epimorphic in $\mathcal{E}$ then S is J-covering;*
(ii) *the family of objects of the form $F(c)$ for $c \in \mathcal{C}$ is separating for $\mathcal{E}$;*
(iii) *for every $x, y \in \mathcal{C}$ and any arrow $g : F(x) \to F(y)$ in $\mathcal{E}$, there exist a J-covering family of arrows $f_i : x_i \to x$ and a family of arrows $g_i : x_i \to y$ such that $g \circ F(f_i) = F(g_i)$ for all i.*

Denseness and weak denseness

We have seen that the notions of weak denseness and of denseness coincide if the codomain site is subcanonical.

The following countexample shows that, indeed, they do *not* coincide in general:

Example

Let $\mathbf{2}$ be the preorder category with two distinct objects 0 and 1 and just one arrow $0 \to 1$ apart from the identities. Let us equip $\mathbf{2}$ with the atomic topology J_{at}. The functor $F : \mathbf{2} \to \mathbf{2}$ sending 0 to 1, 1 to 1 and the arrow $0 \to 1$ to the identity arrow on 1 is a morphism of sites $(\mathbf{2}, J_{at}) \to (\mathbf{2}, J_{at})$ which induces an equivalence of toposes (note that $\mathbf{Sh}(\mathbf{2}, J_{at}) \simeq \mathbf{Set}$ by the Comparison Lemma). Therefore, by our characterization theorem, F is a weakly dense. However, F is not a dense morphism of sites (since it is not J_{at}-dense).

Denseness and weak denseness

Given a morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$, we have an induced functor $a_F : a_J(\mathcal{C}) \to a_K(\mathcal{D})$ given by the restriction of the inverse image of the induced geometric morphism $\mathbf{Sh}(F)$, which is a morphism of sites

$$a_F : (a_J(\mathcal{C}), C_J^{\mathcal{C}}) \to (a_K(\mathcal{D}), C_K^{\mathcal{D}})$$

Recall that a functor between sites is said to satisfy the covering-lifting property if it is a comorphism between these sites. The following proposition further illuminates the relationship between denseness and weak denseness:

Proposition

Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of sites. Then

- (i) *F is (J, K)-faithful if and only if a_F is faithful;*
- (ii) *Supposing that F is (J, K)-faithful, if a_F is full, F is (J, K)-full, and the converse holds if F satisfies the covering-lifting property;*
- (iii) *If F is K-dense then a_F is $C_K^{\mathcal{D}}$-dense, and the converse holds if F satisfies the covering-lifting property.*

Indeed, by using the proposition, one can show the following

Corollary

Let F be a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$. Then the following conditions are equivalent:

- (i) *F is a weakly dense and has the covering-lifting property;*
- (ii) *F is dense.*

Morita equivalence of sites

Theorem

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be two small-generated sites. Then the following conditions are equivalent:

(i) *The toposes* $\mathbf{Sh}(\mathcal{C}, J)$ *and* $\mathbf{Sh}(\mathcal{D}, K)$ *are equivalent.*

(ii) *There exist a category (resp. an essentially small category, if $\mathcal{C}$ and $\mathcal{D}$ are essentially small) $\mathcal{A}$, a Grothendieck topology Z on $\mathcal{A}$ (which can be supposed subcanonical) and two functors $H : \mathcal{C} \to \mathcal{A}$ and $K : \mathcal{D} \to \mathcal{A}$ satisfying the following conditions:*

   (i) *P is a J-covering family in $\mathcal{C}$ if and only if $H(P)$ is a Z-covering family in $\mathcal{A}$;*

   (ii) *Q is a K-covering family in $\mathcal{D}$ if and only if $K(Q)$ is a Z-covering family in $\mathcal{A}$;*

   (iii) *for any object a of $\mathcal{A}$ there exists a Z-covering sieve whose arrows factor both through an arrow whose domain is in the image of H and through an arrow whose domain is in the image of K;*

   (iv) *for every $x, y \in \mathcal{C}$ (resp. $x', y' \in \mathcal{D}$) and any arrow $g : H(x) \to H(y)$ (resp. $g' : K(x') \to K(y')$) in $\mathcal{A}$, there exist a J-covering family of arrows $f_i : x_i \to x$ (resp. a K-covering family of arrows $f'_j : x'_j \to x'$) and a family of arrows $g_i : x_i \to y$ (resp. a family of arrows $g'_j : x'_j \to y'$) such that $g \circ H(f_i) = H(g_i)$ for all i (resp. $g' \circ K(f'_j) = K(g'_j)$ for all j);*

   (v) *for any arrows $h, k : x \to y$ (resp. $h', k' : x' \to y'$) in $\mathcal{C}$ (resp. in $\mathcal{D}$) such that $H(h) = H(k)$ (resp. $K(h') = K'(k')$) there exists a J-covering (resp. K-covering) family of arrows $f_i : x_i \to x$ (resp. $f'_j : x'_j \to x'$) such that $h \circ f_i = k \circ f_i$ for all i (resp. $h' \circ f'_j = k' \circ f'_j$ for all j).*

Morita equivalence of sites

Sketch of proof.

The conditions of the theorem are precisely those for the functor H and K to respectively define dense morphisms of sites $(\mathcal{C}, J) \to (\mathcal{A}, Z)$ and $(\mathcal{D}, K) \to (\mathcal{A}, Z)$.

On the other hand, if $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{D}, K)$ then, by taking, for instance, $\mathcal{A}$ to be the full subcategory of this topos on the objects that are either coming from the site $(\mathcal{C}, J)$ or from the site $(\mathcal{D}, K)$ with the Grothendieck topology Z induced on it by the canonical topology on the topos, we obtain by the Comparison Lemma equivalences $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{A}, Z)$ and $\mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{A}, Z)$, whence by one of the above theorems the canonical functors $\mathcal{C} \to \mathcal{A}$ and $\mathcal{D} \to \mathcal{A}$ are respectively dense morphisms of sites $(\mathcal{C}, J) \to (\mathcal{A}, Z)$ and $(\mathcal{D}, K) \to (\mathcal{A}, Z)$. □

Surjections and inclusions

Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be

- a surjection if f^* is faithful;
- an inclusion if f_* is full and faithful.

Every geometric morphism can be factored, uniquely up to commuting equivalence, as the composite of a surjection followed by an inclusion. In fact, surjections and inclusions are orthogonal to each other. This implies that a geometric morphism is an equivalence if and only if it is both a surjection and an inclusion.

As we shall see, surjections and inclusions, as well as the surjection-inclusion factorization, can be naturally characterized in terms of sites.

Characterizations of surjections and inclusions

Theorem

Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Then:

(i) *The geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by F is a surjection if and only if F is cover-reflecting (that is, if the image of a family of arrows with a fixed codomain is K-covering then the family is J-covering).*

(ii) *The surjection-inclusion factorization of the geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by F can be identified with the factorization $\mathbf{Sh}(i_{J_F}) \circ \mathbf{Sh}(F_r)$, where J_F is the Grothendieck topology on $\mathcal{C}$ whose covering sieves are exactly those whose image under F are K-covering families, $i_{J_F} : (\mathcal{C}, J) \to (\mathcal{C}, J_F)$ is the morphism of sites given by the canonical inclusion functor and $F_r : (\mathcal{C}, J_F) \to (\mathcal{D}, K)$ is the morphism of sites given by F.*

(iii) *The geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by F is an inclusion if and only if $F_r : (\mathcal{C}, J_F) \to (\mathcal{D}, K)$ is a weakly dense morphism of sites; in particular, if K is subcanonical then $\mathbf{Sh}(F)$ is an inclusion if and only if the following conditions are satisfied:*

  (i) *for any object d of $\mathcal{D}$ there exists a K-covering family of arrows $d_i \to d$ whose domains d_i are in the image of F;*

  (ii) *for every $x, y \in \mathcal{C}$ and any arrow $g : F(x) \to F(y)$ in $\mathcal{D}$, there exist a J_F-covering family of arrows $f_i : x_i \to x$ and a family of arrows $g_i : x_i \to y$ such that $g \circ F(f_i) = F(g_i)$ for all i.*

Alternative characterization of inclusions

Corollary

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then f is an inclusion if and only if f^* satisfies the following conditions:

(i) *f^* is locally surjective, that is every object of $\mathcal{F}$ can be covered by objects in the image of f^*;*

(ii) *f^* is locally full, that is for every $x, y \in \mathcal{E}$ and any arrow $g : f^*(x) \to f^*(y)$ in $\mathcal{F}$, there exists a family of arrows $s_i : x_i \to x$ in $\mathcal{E}$ which is sent by f^* to an epimorphic family and a family of arrows $g_i : x_i \to y$ such that $g \circ f^*(s_i) = f^*(g_i)$ for all i.*

Remark

Since a Grothendieck topos has all coproducts, the two above conditions for f to be an inclusion are equivalent to the following ones:

(i) *every object of $\mathcal{F}$ is a quotient of an object in the image of f^*;*

(ii) *for every $x, y \in \mathcal{E}$ and any arrow $g : f^*(x) \to f^*(y)$ in $\mathcal{F}$, there exist an arrow $s : x' \to x$ in $\mathcal{E}$ which is sent by f^* to an epimorphism and an arrow $g' : x' \to y$ such that $g \circ f^*(s) = f^*(g')$.*

The induced topology

The following corollary of the above theorem shows that every morphism of sites naturally induces a Grothendieck topology on the domain category admitting a very simple description:

Proposition

Let $f : \mathcal{E} \to [\mathcal{C}^{op}, \mathbf{Set}]$ be a geometric morphism (equivalently, a flat functor $F : \mathcal{C} \to \mathcal{E}$). Then there exists a Grothendieck topology J_f (resp. J_F) on $\mathcal{C}$, called the Grothendieck topology induced by f (resp. F) whose covering sieves are precisely the sieves which are sent by f^* (resp. by F) to epimorphic families in $\mathcal{E}$. This applies in particular to a morphism of small-generated sites $G : (\mathcal{C}, J) \to (\mathcal{D}, K)$, yielding a Grothendieck topology J_G on $\mathcal{C}$ whose covering sieves are exactly those whose image under G are K-covering families.

In fact, J_G is the largest topology on $\mathcal{C}$ which makes F continuous as a functor from $\mathcal{C}$ to the site $(\mathcal{D}, K)$.

As shown by the following result, the notion of induced topology can be profitably applied for establishing equivalences of toposes:

Corollary

Let $\mathcal{C}$ be an essentially small category, $\mathcal{E}$ a Grothendieck topos and $F : \mathcal{C} \to \mathcal{E}$ a flat functor. Then F induces an equivalence

$$\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J_F),$$

if and only if F is J_F-full and the objects of the form $F(c)$ for $c \in \mathcal{C}$ form a separating set for the topos $\mathcal{E}$

Hyperconnected and localic morphisms

Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be

- hyperconnected if f^* is full and faithful and its image is closed under subobjects in $\mathcal{F}$;
- localic if every object of $\mathcal{F}$ is a subquotient (that is, a quotient of a subobject) of an object of the form $f^*(A)$ for $A \in \mathcal{E}$.

Recall that every geometric morphism can be factored, uniquely up to commuting equivalence, as the composite of a hyperconnected morphism followed by a localic one. In fact, hyperconnected and localic morphisms are orthogonal to each other.

Characterizations for localic morphisms

Proposition

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathcal{E}$ a Grothendieck topos and $F : \mathcal{C} \to \mathcal{E}$ a J-continuous flat functor inducing a geometric morphism $f : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$. Then f is localic if and only if the subobjects of objects of the form $F(c)$ for $c \in \mathcal{C}$ form a separating set for the topos $\mathcal{E}$.

Proposition

Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Then the geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by F is localic if and only if for any object d of $\mathcal{D}$ there exist a family $\{S_i \mid i \in I\}$ of sieves on objects of the form $F(c_i)$ (where c_i is an object of $\mathcal{C}$) and for each $f \in S_i$ an arrow $g_f : \mathrm{dom}(f) \to d$ such that $g_{f \circ z} \equiv_K g_f \circ z$ whenever z is composable with f, such that the family of arrows g_f (for $f \in S_i$ for some i) is K-covering.

Characterizations for hyperconnected morphisms

Proposition

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathcal{E}$ a Grothendieck topos and $F : \mathcal{C} \to \mathcal{E}$ a J-continuous flat functor. Then the geometric morphism $f : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ induced by F is hyperconnected if and only if F is cover-reflecting and for every subobject $A \rightarrowtail F(c)$ in $\mathcal{E}$ there exists a (J-closed) sieve R on c such that A is the union of the images of the arrows $F(f)$ for $f \in R$.

Proposition

Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Then the geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by F is hyperconnected if and only if F is cover-reflecting and closed-sieve-lifting, in the sense that for every object c of $\mathcal{C}$ and any K-closed sieve S on $F(c)$ there exists a (J-closed) sieve R on c such that S coincides with the K-closure of the sieve on $F(c)$ generated by the arrows $F(f)$ for $f \in R$.

We have also obtained a site-theoretic description of the hyperconnected-localic factorization of the geometric morphism induced by a morphism of sites. This description specializes to a particularly elegant one in the case of the geometric morphism between the classifying toposes of two geometric theories induced by an interpretation of one theory into the other.

Other criteria for equivalence

Corollary

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathcal{E}$ a Grothendieck topos and $F : \mathcal{C} \to \mathcal{E}$ a J-continuous flat functor inducing a geometric morphism $f : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$. Then f is an equivalence if and only if the following conditions are satisfied:

(i) *F is cover-reflecting;*
(ii) *for every subobject $A \rightarrowtail F(c)$ in $\mathcal{E}$ there exists a (J-closed) sieve R on c such that A is the union of the images of the arrows $F(f)$ for $f \in R$;*
(iii) *the objects of the form $F(c)$ for $c \in \mathcal{C}$ form a separating set for the topos $\mathcal{E}$.*

Corollary

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then

(i) *f is an inclusion if and only if f^* is locally surjective and its image is closed under subobjects.*
(ii) *f is an equivalence if and only if f^* is faithful, locally surjective and its image is closed under subobjects.*

By applying this latter corollary, we have obtained explicit and elegant characterizations of

- the interpetations of one geometric theory into another which identify the latter as a quotient of the former (up to Morita equivalence);
- the interpetations of one geometric theory into another which induce a Morita equivalence between them.

Characterizations in terms of comorphisms

We have obtained characterizations for the property of a comorphism of sites to induce a surjection (resp. an inclusion), as well as of the surjection-inclusion factorization of the associated geometric morphism. We have also derived characterizations for the property of a comorphism of sites to induce a hyperconnected (resp. a localic) morphism, as well as of the hyperconnected-localic factorization. Under the most general assumptions, these characterizations are normally rather technically involved.

Here are some of the simplest examples of such characterizations:

Proposition

The geometric morphism $C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by a comorphism of sites $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ is a surjection if and only if whenever a sieve S on an object $c \in \mathcal{C}$ satisfies the property that for every object d of $\mathcal{D}$ and arrow $x : F(d) \to c$ in $\mathcal{C}$, there exists a K-covering sieve T on d such that $F(T) \subseteq x^*(S)$, then S is J-covering. This condition implies that F is J-dense and is equivalent to it if F is cover-preserving.

Proposition

Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of sites which is cover-preserving. Then the surjection-inclusion factorization of the geometric morphism $C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by F can be identified with $C_i \circ C_{F'}$, where F' is the functor F regarded as a comorphism of sites from $(\mathcal{D}, K)$ to the site $(\mathcal{C}', J')$, where $\mathcal{C}'$ is the full subcategory of $\mathcal{C}$ on the objects in the image of F and J' is the smallest Grothendieck topology on $\mathcal{C}'$ making the inclusion i of $\mathcal{C}'$ into $\mathcal{C}$ a comorphism of sites to $(\mathcal{C}, J)$.

Characterizations in terms of comorphisms

Proposition

Let $F : \mathcal{D} \to \mathcal{C}$ be a functor between essentially small categories. Then the geometric morphism $C_F : [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is hyperconnected if and only if F is full and every object of $\mathcal{D}$ is a retract of an object in the image of F.

Proposition

Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of sites which is cover-preserving. Then the hyperconnected-localic factorization of the geometric morphism $C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by F can be identified with $C_{\tilde{F}} \circ C_\pi$, where $\tilde{F}$ is the functor F regarded as a comorphism of sites from the site $(\mathcal{E}, L)$ whose underlying category $\mathcal{E}$ is the quotient of the category $\mathcal{D}$ by the congruence induced by F and whose Grothendieck topology L has as covering sieves the sieves whose inverse image under the canonical projection functor $\pi : \mathcal{D} \to \mathcal{E}$ is K-covering.

Local morphisms

Recall that a (weak) geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be local if f_* has a fully faithful right adjoint.

Theorem

Let $F : \mathcal{D} \to \mathcal{C}$ be a continuous comorphism of sites (also regarded as a weak morphism of sites) $(\mathcal{D}, K) \to (\mathcal{C}, J)$. Then:

(i) *The geometric morphism $C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ is essential, and*

$$(C_F)_! \cong \mathbf{Sh}(F)^* \dashv \mathbf{Sh}(F)_* \cong (C_F)^* = D_F := (- \circ F^{\mathrm{op}}) \dashv (C_F)_*$$

(ii) *If F is a morphism of sites then the geometric morphisms $\mathbf{Sh}(F)$ and C_F form an adjoint pair in the 2-category $\mathfrak{Top}$ of Grothendieck toposes, geometric morphisms and geometric transformations.*

(iii) *The weak morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ is local if and only if C_F is an inclusion, that is, if and only if F is K-faithful and K-full.*

(iv) *The canonical geometric transformation*

$$1_{\mathbf{Sh}(\mathcal{D},K)} \to \mathbf{Sh}(F) \circ C_F$$

(given by the unit of the adjunction between $\mathbf{Sh}(F)$ and C_F) is an isomorphism if (and only if) F is K-faithful and K-full. In this case, if F is moreover a morphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$, the morphisms C_F and $\mathbf{Sh}(F)$ realize the topos $\mathbf{Sh}(\mathcal{D}, K)$ as a (coadjoint) retract of $\mathbf{Sh}(\mathcal{C}, J)$ in $\mathfrak{Top}$.

Gros and *petit* toposes

The above result can be notably applied to construct pairs of *gros* and *petit* toposes starting from a (K-)full and (K-)faithful morphism and comorphism of sites

$$(\mathcal{D}, K) \to (\mathcal{T}/T_{\mathcal{D}}, E_{T_{\mathcal{D}}}),$$

where $\mathcal{T}$ is a category endowed with a Grothendieck topology E, $T_{\mathcal{D}}$ is an object of $\mathcal{T}$ and $E_{T_{\mathcal{D}}}$ is the Grothendieck topology induced on $(\mathcal{T}/T_{\mathcal{D}})$ by E.

As an example, consider, for any small-generated site $(\mathcal{C}, J)$, the functor $L : \mathcal{C} \to \mathfrak{Top}/\mathbf{Sh}(\mathcal{C}, J)$ sending an object c of $\mathcal{C}$ to the local homeomorphism $\mathbf{Sh}(\mathcal{C}, J)/l(c) \to \mathbf{Sh}(\mathcal{C}, J)$. By equipping $\mathfrak{Top}$ with the topology E generated by the families of local homeomorphisms $\{\mathcal{E}/A_i \to \mathcal{E} \mid i \in I\}$ such that the family $\{A_i \to 1_{\mathcal{E}}\}$ is epimorphic, L becomes a J-full and J-faithful morphism and comorphism of sites $(\mathcal{C}, J) \to (\mathfrak{Top}/\mathbf{Sh}(\mathcal{C}, J), E_{\mathbf{Sh}(\mathcal{C},J)})$ (where $E_{\mathbf{Sh}(\mathcal{C},J)}$ is the Grothendieck topology whose covering sieves are those sent by the canonical projection $\mathfrak{Top}/\mathbf{Sh}(\mathcal{C}, J) \to \mathfrak{Top}$ to E-covering families), thus realizing $\mathbf{Sh}(\mathcal{C}, J)$ as a coadjoint retract of the gros topos $\mathbf{Sh}(\mathfrak{Top}/\mathbf{Sh}(\mathcal{C}, J), E_{\mathbf{Sh}(\mathcal{C},J)}) \simeq \mathbf{Sh}(\mathfrak{Top}, E)/l(\mathbf{Sh}(\mathcal{C}, J))$ (with respect to suitable Grothendieck universe), where l is the canonical functor $\mathfrak{Top} \to \mathbf{Sh}(\mathfrak{Top}, E)$.

This is part of a joint work-in-progress with my Ph.D. student Riccardo Zanfa, which constructs a whole framework for studying (relative) Grothendieck toposes from a geometric point of view, thus providing a solution to the questions posed by Grothendieck in his lecture course at Buffalo of 1973 and recently brought to the public attention by Colin McLarty.

For further reading

- O. Caramello,
  Denseness conditions, morphisms and equivalences of toposes,
  monograph draft available as arxiv:math.CT/1906.08737v3, (2020).

- O. Caramello
  Grothendieck toposes as unifying 'bridges' in Mathematics,
  Mémoire d'habilitation à diriger des recherches,
  Université de Paris 7 (2016),
  available from my website www.oliviacaramello.com.

- O. Caramello
  Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges',
  Oxford University Press (2017).

- S. Mac Lane and I. Moerdijk.
  Sheaves in geometry and logic: a first introduction to topos theory
  Springer-Verlag (1992).