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# Toposes and Their Place in Mathematics: an Interview with Olivia Caramello

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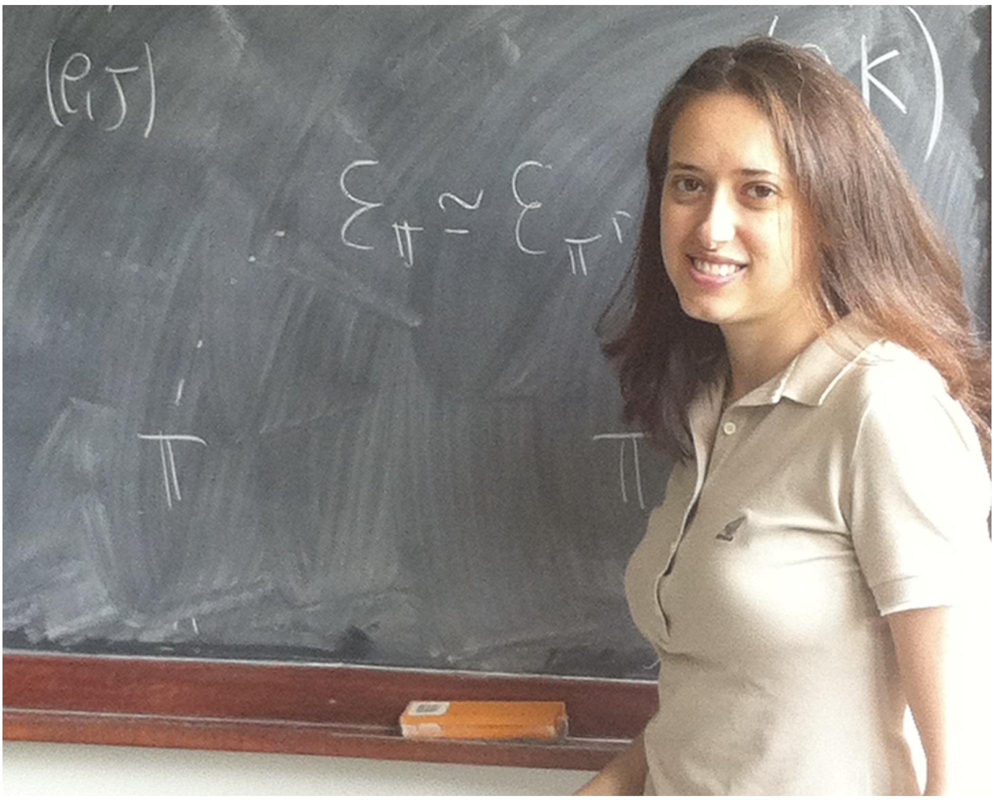


**Biographical Sketch.**
Olivia Caramello is currently president of the Grothendieck Institute (Italy) and associate professor at the University of Insubria in Como, Italy. She was previously (2020-2022) holder of the Israel Gelfand Chair in mathematics at the IHÉS (succeeding Sergiu Klainerman), and has also held other positions at the IHÉS as well as the University Paris-Cité and the Max Planck Institute for Mathematics. She completed her PhD in topos theory, specifically in her program

of using toposes as "bridges", at the University of Cambridge in 2009 under the supervision of Peter Johnstone. (NTC took category theory and topos theory from Johnstone and Caramello, respectively, in 2010 and 2011.)

In this interview, Professor Caramello describes this program, in which classifying toposes of geometric theories can be used to transfer mathematical results between unrelated areas of mathematics. She also shares her perspective on the place in and importance to mathematics of logic in general and toposes in particular – logic, she tells us, is like "X-rays" into mathematical structures, revealing the skeleton that is syntax and allowing one to gain a deep understanding of the nature of these structures. She talks much about Grothendieck and his perspective on mathematics, in particular the "childish" attitude that he said is needed to truly appreciate such abstractions as topos theory, as well as his comments on the controversial nature of this subject. Following Grothendieck, she also emphasizes the importance of seeking a true understanding of mathematical results.

*Editorial Note. Footnotes are by NTC except as noted. Footnotes marked "OC" are asides from the original interview. We wish to thank Professor Caramello for her extensive editorial contributions to the original draft of this interview.*

**NTC**: *Have you been in China before?*

**OC**: I have been in Shanghai for just a couple of days, in I think 2017 on the occasion of an anniversary celebration for Alain Connes that was organized by one of his former students at Fudan University.[1] But it was a short visit so I couldn't take the time to explore.

But I was already impressed by the technological development of Shanghai.

**NTC**: *I was too, when I was in Shanghai for three weeks during 2021. If people from the U.S. could come to Shanghai for a couple days it would completely change their view of China.*

**[Laughter.]**

**NTC**: *When did you arrive in China this time?*

**OC**: I gave a talk at the satellite conference on algebraic and arithmetic geometry just before the start of the congress. I presented my work unifying Fraïssé theory and Galois theory thanks to toposes, with applications to motivic toposes, or the construction of toposes that classify cohomology theories, in particular, $\ell$-adic cohomologies.[2] It provides an approach to the problem of independence of $\ell$ for $\ell$-adic cohomology, and also possibly a new way of dealing with cohomology constructively.

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[1] *Noncommutative Geometry: State of the Art and Future Prospects.* See [13].

[2] The slides for this talk are available online, see [8].

Also Laurent Lafforgue gave his own talk on our current joint work, which is on testing the conjectures on motives that I had formulated back in 2015 in the context of $\ell$-adic cohomology [see [6]], starting from degree zero. For the moment what we have achieved (also with Gonçalo Tabuada) is a proof of the conjectures for 0-motives, but we plan to go on. If we manage to go to higher dimensions, we will have a completely new approach to cohomology with a lot of consequences.

**NTC**: *If I understand right, motives are an attempt to factor a whole bunch of stuff through ...*

**OC**: They are a kind of universal object.

**NTC**: *So also like some kind of unifying notion.*

**OC**: Exactly. We want to marry the two most fundamental notions in Grothendieck's work: toposes and motives. Grothendieck explicitly wrote that the two most important concepts in his mathematical work were toposes and motives, but he didn't make any connection between them. What we actually plan to do is to use toposes to build motives. It would be a striking application of the unifying power of toposes, something that was already glimpsed by Grothendieck, but which remained largely unexploited as it was just an intuition. The theory of toposes as "bridges", which I have been developing since the times of my PhD studies, provides a set of techniques that allow to effectively use toposes as unifying spaces in mathematics [see, e.g., [5]]. So now we have conceptual and technical tools to start realizing Grothendieck's dream.

One might wonder why this dream was not pursued previously – after all, toposes were introduced more than 50 years ago. Actually, topos theory stimulated a lot of hostility in the mathematical community, probably precisely because of this unifying character that can disturb the specialists of different fields. I personally witnessed the fear of some specialists when someone from another domain gets into their field bringing a whole new set of unexpected ideas and techniques. This overspecialized attitude to knowledge has created a kind of irrational hostility towards the subject that has persisted for several years; in fact, Grothendieck himself complains extensively about that in his book *Récoltes et semailles* [see [10]].[3]

**NTC**: *We previously interviewed another scientist here, Jintai Ding [see [9]]. He was working in quantum algebras and then switched to cryptography; he said he thinks people didn't like it because he came from nowhere into their area, and made some big discoveries.*

**OC**: Indeed, I received this kind of reaction from a number of specialists, so definitely this is one of the reasons. There are also many other reasons. Certainly, there has been a widespread lack of understanding of the global vision

[3] *Récoltes et semailles* is this huge, very deep and interesting text of reflections by Grothendieck on his mathematical work. It is currently being translated into English; it was officially published in February 2022 in France. For many years it circulated among mathematicians in PDF form but it was not an official publication. It's a two-volume work for a total of 1900 pages. – OC. [Grothendieck's literary output was prodigious; an archive of his writings at the University of Montpellier made available after his death (see [1]) contains an estimated 28,000 pages. – NTC.]

that toposes incarnate in mathematics. In fact, Grothendieck speculates a lot about this in *Récoltes et semailles*, without actually reaching a definite conclusion, but suggesting different interpretations also of psychoanalytic nature. For instance, he refers to some kind of *innocence* that many mature mathematicians might lack, the innocence to see the richness of things and appreciate the naturality of certain constructions. He likens toposes to childish concepts – he really uses the word "childish" to qualify them. In a sense, you need to have a certain innocence in order to –

**NTC**: *I see, I see.*

**OC**: – you shouldn't look at knowledge as something you can dominate; you should have a global receptive attitude to the richness of things. He also remarked that that vision is not something which was so appreciated in his epoch, and that with time it would go worse and worse – he also predicted this kind of development; which I can see was more or less correct because with hyperspecialization, cultivating a global conception of knowledge becomes more and more difficult.

Of course, there are interesting exceptions to this; some visionary top mathematicians still have such a global approach, but on average mathematicians have a more narrow attitude than in the past. I particularly welcomed the choice of naming this congress "Basic Science", as it indicates openness to any field and emphasizes the importance of mutual interactions notably between mathematics, physics, and computer science.

We recently organized a conference in Paris entitled "Visions in mathematics: from Grothendieck to the present day".[4] It was important for us to raise this debate on the notion of vision, because nowadays people that practice mathematics in a global, visionary way are becoming more and more rare, but on the other hand it is really at the intersection of different fields that one can find the most interesting development opportunities. So we want to encourage the new generation not to get too focused on tiny things, and try to identify universal concepts and common themes that may relate different fields of knowledge and nourish the interdisciplinary dialogue.

Topos theory is not only interesting as a unifying subject that can shed light into mathematics as a whole, but also at the methodological level, since the principles which it embodies can be extracted so as to make sense even beyond mathematics, in many other fields. You see, the concept of bridge object, the concept of invariant – all of this makes sense in any field of knowledge. Having a mathematical incarnation of all of this, as provided by topos theory, thus allows you also to test the validity of some philosophical ideas. It's a kind of meta-mathematics that is carried out within mathematics itself; indeed, technically it is a very sophisticated mathematical subject, but conceptually it embodies some fundamental philosophical ideas. This, I think, is extremely important for the interdisciplinary dialogue and also for possible developments

[4] The videos are available on the YouTube channel of the Grothendieck Institute [see [2]]. – OC

in connection with computer science and artificial intelligence. For instance, now a big debate is how we should design the new artificial intelligence systems in a way that, somehow, matches more the way our brain actually thinks, the way our intelligence works. Right now we are still far from that kind of intelligence which is human-like, because the deep learning techniques are based on real analysis. They encode a kind of numerical approach to the world, while the way our brain thinks is mostly topological: we look at shapes, and we distinguish shapes according to some invariants. Our brains really have invariants wired into them, even though we might not be so conscious of this. But there are a lot of a priori structures in our brain that determine the way we look at reality, and so far the artificial intelligence systems are based on a completely different kind of mathematics: they are based on numbers rather than geometric shapes. So, you see, a major development that could arise, stimulated by topos theory, is new foundations – completely new foundations for deep learning, for information theory, based on topological or geometrical notions and giving an adequate place to invariants. Toposes are kind of universal invariants in mathematics, in the sense that most of the fundamental invariants in mathematics factor through toposes; so if you want to investigate invariants in a systematic way, you have to work in a topos-theoretic setting. In particular, if you want to really formalize the way our brain works, which is certainly linked to invariants, the mathematical formalization you need should be a theory of invariants. This is one reason why toposes are currently stimulating interest among engineers and computer scientists working in artificial intelligence.

Another reason is their relevance in connection with the development of a semantic theory of information, which is another central subject. Nowadays one realizes that it is important to think about information not merely in a syntactic way – like a sequence of bits – but in a semantic way, trying to extract the essence of a message (for instance) that you want to transfer from one person to another. Then the question comes, Which kind of mathematical structure can formalize the essence of a message? Basically, the notion of classifying topos does precisely this. If you have a mathematical theory formalized in a certain formal language (technically, you need the theory to satisfy certain conditions, but they are very general) – given such a theory you can actually embody its mathematical content by a topos, which is called its classifying topos. Actually this object really embodies the essence of the theory, in the sense that it is invariant with respect to all the different ways in which the theory can be presented.

**NTC**: *You mean with respect to some kind of syntactic transformation?*

**OC**: Not only: you have to think in terms of the syntax-semantics duality: you can have different syntaxes that talk about the same semantics. It's like different languages that we might use to describe what happens in the real world: we can talk in Chinese or in English or in Italian, and each of these languages provides a different syntactic presentation for what is going on in the world. In mathematics what we have is the formal analogue of this: you can have different mathematical theories, possibly belonging to different branches of mathematics, that describe in different languages the same structures; but if you want to extract the semantic content, actually the object that does this job for you is the classifying topos. It's a kind of DNA of the theory that removes all the inessential aspects of the theory by extracting its fundamental kernel, the "essence" of the theory, to use Grothendieck's words.[5] The work of categorical logicians in the 70s – Makkai, Reyes, Joyal – identified the widest logical framework within which all the theories that you could formulate would admit classifying toposes – maybe you remember from the topos theory course some of these materials.

**NTC**: *I'm familiar with some of these names still. With reference to classical Tarskian semantics: as I understand it, Tarskian semantics is more just a matter of reexpressing the syntax within set theory, while this is more an extracting –*

**OC**: The first step is generalize Tarskian semantics to an arbitrary topos –

**NTC**: *Right, that I do remember a little bit. What I remember from topos theory is that I liked intersections: they were just pullbacks! I did not like unions because they were something extremely complicated.*

**OC**: One first needs to give a categorical description of the set-theoretical constructions that are involved in the Tarskian interpretation. This serves for defining the notion of model of a theory inside a topos. Since you allow yourself to consider models of theories in arbitrary toposes, then comes the problem of classifying those models, and you get this marvelous result that says that there is always (if the theory is of that kind I mentioned) a classifying object, which is the classifying topos. It's very illuminating to realize that these classifying objects don't exist in the limited set-theoretic setting: you have to enlarge the context to the whole world of toposes to obtain them.

**NTC**: *What do you mean exactly by saying they don't exist?*

**OC**: They don't exist in the sense that the universal models of theories lie in their classifying toposes, not in the set-theoretic setting: they are models internal to toposes, not ordinary set-based models. In order to get this symmetry result you really need to enlarge your setting. If you only consider the set-based models, basically from a geometric point of view it corresponds to just looking at the points of the classifying topos. But in general a topos is not determined by its points. It's something much richer, much more fundamental. If you want to find the context in which really the unification between the syntax and semantics of the theory takes place, you have to work inside the universal model – and the universal model doesn't live within sets, it lives in the classifying topos. It's a bit like when you go from the real numbers to the complex plane in order to find more symmetries with respect to the problem

[5] Grothendieck already stated that toposes have this crucial feature of being able to capture an essence of mathematical situations "most distant from each other coming from one region or another of the vast universe of mathematical things." – OC

of solving, say, polynomial equations in one variable: the fundamental theorem of algebra is a kind of symmetry result that relates the degree of the polynomial with the number of roots, counted with their multiplicities. But that result works for $\mathbb{C}$, not for $\mathbb{R}$, so in order to get that result you need to enlarge from $\mathbb{R}$ to $\mathbb{C}$. It is a recurrent theme in mathematics that in order to find symmetries you have to enlarge your view – you have to complete the concrete context in which you work with respect to "imaginary objects". In the history of mathematics, every introduction of new imaginary objects at the beginning stimulated a lot of resistance, because of course you jump into a more abstract world; every time you enlarge the view you make an abstraction leap that can be difficult to accept from a psychological point of view. This also connects with the theme of the hostility against topos theory. When you construct a topos from, say, a theory or a site or whatever particular object, basically you complete that object with respect to all the imaginary concepts that can be derived from it. It's really cool to be able to do that, but you get a gigantic entity; you get a whole mathematical universe and so people might be frightened by that, because they cannot concretely represent a whole topos. Even particular objects of a topos can be difficult to represent concretely. So you really make a very big leap into abstraction. But the advantage of doing that is that computations become much easier and more natural in the extended context because of the presence of symmetries. So what is useful to do is a kind of "back and forth" where you start with a concrete context then you go into the imaginary world of toposes, you do your computations there, and then you try to come back to the concreteness – you try to understand the result of those computations, which take place naturally there, in more concrete terms. This is the core of the bridge technique, you see. We can think of toposes as kinds of mountains in the mathematical landscape, from the top of which you can have a beautiful view of all that is around. If you want to connect two different territories that are separated by a mountain, what you have to do is reach the top of the mountain. It is that perspective which allows you to see how the territories are related, how they compare with each other and how you can go from one to another. It is having this image in mind that, in the last slide of my talk on the unified Fraïssé-Galois theory, I drew a mountain representing the kind of toposes involved, with Galois on the right and Fraïssé on the left. Fraïssé was a logician working in model theory, and he developed his theory motivated by purely logical considerations, while Galois of course worked in algebra. Apparently there is no connection between their works. But in fact when you look at these theories from a topos-theoretic viewpoint you see that actually Fraïssé was trying to climb the mountain from a certain path and Galois was climbing from another path – a completely different one – but they were just climbing the same mountain, and so if you take the point of view of the mountain you can relate them with each other. You can construct any Fraïssé context from a Galois context and conversely, so the two theories are essentially equivalent. They are just two different points of view that you can have on certain classes of toposes.

**NTC**: *Are there examples then where you – like you were talking about, you start from a concrete setting and then you go up to the abstract setting and do your computations and then I noticed you use the word "try" to come down; are there times where you can't come back down or trying to come back to concrete doesn't –*

**OC**: Well of course the question of whether you might come back or not depends on the complexity of the invariants under consideration and the feasibility of the relevant computations. There are some invariants that can be computed even automatically, according to routine recipes, while there are others that can be extremely difficult to compute. Cohomology, in particular, because we lack completely constructive foundations, tends to be quite difficult to compute, even in particular cases, not to mention in general. Even realizing whether a cohomology group is zero or nonzero in some special cases is hard. So, you see, because we can have a great variety of invariants, and every invariant will generate bridges, the complexity of these bridges can vary a lot, from relatively simple to extremely deep. Since the "arches" of bridges are given by characterizations of invariants in terms of different topos representations, depending on the complexity of the invariant the bridge can be more or less easy to establish, more or less deep, more or less complex.

Generally speaking, the relationship between toposes and their presentations is quite natural in the sense that at least for large classes of invariants and large classes of presentations we can actually obtain such characterizations, so, also on the basis of the experience that we have accumulated, we know that, in principle, we can carry out these calculations. But of course their complexity can change a lot from case to case. It might happen, for instance, that you cannot really exit the bridge in the most natural way that you would dream of, in which case you can try to modify the invariant so that it becomes more computable, or relax some conditions so that maybe you end up no longer with a necessary and sufficient condition but only a sufficient condition, etc.. Typically, even when you find yourself in trouble with the computations, you can still adjust things to extract some information.

**NTC**: *I see, I see. So it's maybe like, on the mountain metaphor, you try to come all the way down, but you find you're stuck on a cliff and you have to stop right there.*

**OC**: Yes, exactly. But at least, you see, the conceptual architecture is very clear. You know where you should go. Of course, if a certain problem is complex, you cannot eliminate the complexity, but you can organize it in conceptual frameworks providing a clear picture of what is going on. The most compelling architectures are those that separate the mechanical, essentially routine, part of the calculations from the part which requires some creative effort. Note that there is an element of canonicity in the computation of invariants: going down the mountain corresponds, in a sense, to following a natural flow, while climbing up requires a significant effort and some imagination to identify the most suitable paths.

**NTC**: *So the step of coming back is more –*

**OC**: More canonical.

**NTC**: *More algorithmic, in some sense.*

**OC**: Yes. Definitely. Of course it can also be very complex, but you have, generally speaking, recipes that allow you to come back; so climbing is definitely more complex in terms of creative input that you might need to provide. Climbing means, when you start from a concrete mathematical context, to be able to identify one or more toposes which capture the essential features of the problem you are interested in, and some invariants defined on them that relate to it via their characterizations; so, it is a non-canonical process, as you need some imagination to identify them. On the other hand, once you get to the topos level, you are on top of the mountain, and then to get down of course you have to be careful not to fall –

[**Laughter.**]

**OC**: – but it is much easier – you don't have to put a real creative effort.

**NTC**: *I see. So speaking of algorithmic things: something I thought of when you were talking about potential applications to AI is the lack of a computational basis for category theory or topos theory, in that one would need some sort of algorithmic representation. Is this something anyone has worked on? or is that perhaps more of an apparent than actual obstacle?*

**OC**: A striking aspect of toposes is that they are computationally extremely effective, so we could devise even computer programs that perform these computations for us. In classical mathematics very often you might be stuck because, for instance, you want to construct a quotient that doesn't exist, or perform some other operation which isn't possible, or construct a function space that doesn't exist; in a topos, these kinds of difficulties disappear. So, in a sense, a topos is a computational paradise. In order to make the most of these possibilities, we should train computers to perform these kinds of very symbolic, very abstract computations, and certainly we don't have this at the present moment. We need a very important and long-term investment at the foundational level to teach computers how to reason not quite in a numerical way but in a more geometric, logical way. Actually, the computer itself is a logical machine, rather than a numerical machine.

**NTC**: *Right, exactly – computers do numbers sort of by accident, right? How do we add [on a computer]? We use logic gates.*

**OC**: So, you see, even if you just want to use computers for numerical calculations, it already goes beyond the original logical conceptions of computers. It has been done because of course analysis – real analysis in particular – has found spectacular applications, and so it has become very central in mathematics. Also, you see, in terms of formalization it's of a lower logical complexity than topology. From a logical point of view, topology is a kind of second-order subject: if you think of the classical notion of topological space, you have points and open sets and they don't lie at the same level in the logical hierarchy. Also, in the point-free conception of geometry which is provided by topos theory, the notion of site is a second-order one, because you have a category and a Grothendieck topology. A category is a first-order notion, but a site requires this notion of covering for objects of the category by families of arrows, and this is second order. Of course, from the point of view of logical complexity, it is more cumbersome to handle. I am not a computer scientist so I cannot speak on the difficulties that we could have in reshaping even possibly the hardware of computers in order to fit more naturally with such new topological foundations. What can be said is that recently there has been a great investment in the direction of type theory, for instance, notably stimulated by Voevodksy's program of univalent foundations of mathematics. The kind of logical system that they rely on is type theory, which is a higher-order logical system, and there exist already computer implementations of all of this. But we need to invest also in the direction of automated theorem proving, and also really think about how we want deep learning to work at the foundational level.

Yesterday there was a very interesting talk by Adi Shamir[6] at the Congress about the fragility of deep learning algorithms. It was extremely interesting because, in exposing these fragilities, he was also proposing a theory for understanding why the so-called adversarial examples could exist. The theory of neural networks is still quite mysterious from a logical viewpoint, so he was trying to offer an explanation for this kind of phenomena and actually his explanation highlighted the fragility of a purely numerical approach. I don't think you can really overcome these problems if not by changing the foundations. Because the data, in the current systems, are unlabelled: there is no structure, it is just numbers, and numbers by themselves are necessarily fragile. For instance, numerically based systems tolerate many strange operations at the level of pixels that don't make any sense for us. They are not meaningful but they are accepted by the foundations. The only way to eliminate completely those kinds of problems would be, I think, to have foundations which are of structured, topological nature. If you think in topological terms, any modification at the level of pixels which destroys the structure will be deemed unacceptable and will therefore not affect the model.

You see, the development of abstract software architectures has always gone in the direction of preventing strange meaningless things from happening. A very basic example with which mathematicians are familiar is the LaTeX system: it makes it difficult to write typographically ugly texts –

**[Laughter.]**

**OC**: – and the same should be with deep learning. This kind of tweaking they make [for building adversarial examples] is ugly and meaningless, so it

[6] See [19]; see also the overview article by Lynn Heller in this issue for a summary.

should be, in a way, forbidden by construction. Data should already be structured and labelled, and if they are pre-treated according to some topological and logical metrics, they will not have this kind of fragility.

Also in terms of metric, there was this main issue of the proximity of two completely different concepts: from the point of view of the system they were too near – they were almost infinitesimally near. He made the example of the rational numbers inside the real line: they are densely distributed, and so near to each point represented there is another belonging to the complementary class, that is, in that case, representing something completely different. Of course, for AI this doesn't make any sense because the metrics of our brain don't work at all like that, since we think in terms of shapes, structures and invariants. The example that was provided was to distinguish between a cat and guacamole. We should think about what we, as humans, do for discriminating betweem the two. We certainly do not look at the single pixels; rather, we compute invariants and distinguish according to them; one relevant invariant is, for instance, the color in this case. So why should the computer do it if this is really not the way we think? I think that nowadays people in the artificial intelligence community are starting to realize the weaknesses of purely numerical foundations. Of course they can achieve impressive results for certain tasks – I'm not denying this – but at the same time there are fundamental flaws that are very difficult to correct without changing the foundations. So it's a challenge. Because of course developing new foundations requires a great investment – a long-term investment – but the consequences can be very important, also at the level of energy consumption. Because if you train machines in a very concrete way the systems you get will not be very resilient: they will not be scalable, they will not be flexible, and so for every different task according to which you want to optimize you will have to retrain from scratch; while if you devise a more abstract architecture, you can hope to –

**NTC**: *Avoid having to do this too many times.*

**OC**: Exactly. So it's kind of a trade-off between these different aspects that AI researchers have to evaluate. But I think in the long term these topological approaches will impose themselves, sooner or later there will come the time for topological deep learning. At the moment all of this is not yet achieved, but I don't have any doubts as to the correctness of the direction.

**NTC**: *Right, right. Actually I did find a book [see [17]] once from probably about 50 years – 40-50 years ago in which a couple people did do some research in implementing computational category theory within a functional programming language. They were using ML, which is related to OCaml, which is what Coq was written in.*

**OC**: The links between category theory and computer science have been strong since the very beginning of category theory. There is a lot of literature about this. But with respect to neural networks, the aspect of category theory that seems to me the most important is the compositionality of categories. Categories are compositional structures at many levels, and, you see, deep learning – what really brought this kind of spectacular development in the accuracy of optimizations is the compositional, rather than superpositional, character of neural networks: the fact that you compose several times. From a categorical viewpoint it's like composition of arrows in some category.

**NTC**: *Interesting. When I first saw that book on computational category theory, my first thought was, How do you compute anything in a category – how do you make it algorithmic? And then as I was reading it more one of the things they mentioned was that you have to give a description of the category – this category is a sort of infinite object in some sense that fits inside of a computer. As far as I have been able to gather (though I haven't read the book in any great detail) they do that using this idea that in functional programming languages you can represent an infinite structure within a computer – it's sort of like a computational thing, so more like a function rather than writing out every single term explicitly.*[7]

**OC**: Well, we should go from numerical computations to symbolic computation, and when you compute with symbols they can represent even infinite-dimensional entities.

**NTC**: *Exactly. Speaking of things not going too badly reminds me of the Banach-Tarski paradox. I read an article [see [20]]: someone was working in a topos model, and the paradox doesn't occur – somehow you can't split it fine enough –*

**OC**: That is a wonderful example: Olivier Leroy, who was a student of Grothendieck, wrote this marvelous paper [see [14]] in which he shows how to overcome this paradox by replacing the notion of a subspace with the notion of sublocale, which is a much more robust notion. A sublocale is like the point-free analogue of a subspace. Technically you define sublocales by working in the opposite category of frames; they are the surjective frame homomorphisms. So of course every subspace gives rise to a sublocale, but the converse is not true. So you might have things appearing in the localic setting that don't come from topology. Because the topological approach is based on the notion of point; in particular, you can have non-trivial toposes that don't have any points, so if you just look at the points you would see nothing, but the topos is there. And in fact, Olivier Leroy, in the title of his paper, refers to the hidden intersections that one doesn't see with the point-set approach.

**NTC**: *I remember [Simpson] talked about glue, bits of glue that you can't quite cut apart, so that's why.*

**OC**: In fact, what happens is that you can have intersections that are empty according to the point-set view but are non-empty according to the point-free view. That's the key point which allows you to overcome the paradox. It's a wonderful example of the possibility of topos theory to solve the paradoxes which are due to non-ideal foundations. In the same spirit, we could fruitfully apply topos theory to solve many other problems related to paradoxes: for

[7] What NTC had in mind here was lazy evaluation of infinite lists in Haskell.

instance, constructing non-standard measures that do not exist in the classical set-theoretic foundations, giving constructive foundations for non-standard analysis,[8] or. more generally, building new settings where different kinds of infinitesimals constructively exist. The advantage of working in a topos-theoretic setting is that instead of having a fixed set-theoretic foundation in which you have to stock and fit everything, you can choose, and change, the universe according to your needs. Of course this gives you much greater flexibility and the possibility of devising new worlds in which what you want to build naturally exists by construction.

*[At this point the interview moved to another location and Koji Shimizu joined.]*

**KS**: *Many people know toposes from étale cohomology or crystalline cohomology, but also, for example, from the work of Jacob Lurie in higher topos theory, or from logic, which are all places where topos theory is very important. I think if you want to learn about topos theory there are lots of entry points, and I want to see the overview.*

**OC**: Because topos theory is entirely written in categorical language, an essential prerequisite is a basic familiarity with the language of category theory. But the other fundamental prerequisite is logic, and this unfortunately is a subject that is not widely taught in university degrees in mathematics. There are universities in which there are not even logic courses. I find this very disappointing and worrying, because logic is not only important for technical reasons, but especially for conceptual reasons, in order to acquire a multidimensional perception of mathematics which goes beyond the flat conception one is confined to if one doesn't think in terms of the duality between syntax and semantics. If there is one fundamental feature of logic – the cornerstone of logic – and I am speaking of logic in the singular, but in fact logic is a plurality because, of course, depending on the context you can devise a new logic –

**NTC**: *I remember this was kind of a mental implosion when I was in Cambridge, and I learned there was not just classical logic, but other logics and my mind blew.*

**OC**: Yes, we have first-order logic, intuitionistic logic, type theory, linear logic, etc. But what is common to all of these different logics? Well, it is the fundamental distinction between syntax and semantics, which can be crucial for the working mathematician. Because if you don't have this distinction, you are bound, somehow, to a flat view, not allowing you to discriminate between the syntactic and the semantic aspects arising in any mathematical situation; they will be intertwined and you will not be able to separate them. Knowing logic allows you to have a kind of X-ray vision of mathematics: you can see the skeleton of structures, which is essentially the underlying syntax, and how the flesh, that is, the semantics, is organized around the skeleton.

[8] See [18] for some discussion related to this.

I can give a basic example: take Lagrange's Theorem in group theory, which gives a relation between the cardinality of a finite group and the cardinality of a given subgroup of it. In the statement of the result, as well as in the proof, there is a combination of syntactic and semantic ingredients: the syntactic ingredients are those that have to do with the notion of group, which can be axiomatized in first-order logic; the semantic aspects have to do with the notion of cardinality that refers to the ambient set theory in which one works, and you have this interplay between them. And because of this interplay you don't quite understand the level of generality of this result: in particular, it is not quite clear whether you could have an analogue of that result in other settings, like going beyond set theory – replacing set theory with an arbitrary topos, or even a category with suitable properties. It is because you are stuck with this hybrid mixture of syntax and semantics that you cannot articulate. So learning logic can help the working mathematician to read mathematics in a much more refined way, which can lead to generalizations of results, to a deeper understanding of how a theorem is obtained, what is the underlying conceptual architecture, etc.

I can also give another illustration of the power of logic. Grothendieck made a comparison between groups and toposes which is technically correct but very reductive. He observed that, as any group can be presented in many different ways by using generators and relations, so a topos can be presented in many ways by using sites. If you say this it doesn't really illuminate a lot about the unifying power of toposes across different mathematical branches. Now if I say the same thing in the language of logic, which is, "You can have different theories possibly belonging to different areas of mathematics that have the same classifying topos", you immediately understand why toposes are so important and potentially useful for all mathematical disciplines.

Grothendieck had already perceived, or at least imagined, the unifying power of toposes; he suggestively talks about that in *Récoltes et semailles*. But he didn't quite go beyond that in terms of providing techniques that could allow toposes to be actually used as unifying "bridges" across different areas, probably because he underestimated the importance of the existence of multiple presentations for a given topos in relation to that goal. My suspicion is that it was because he didn't know logic himself, or at least he did not think about toposes from a logical point of view.

**KS**: *That's an interesting comment because as a number theorist when we learn about toposes or sites, right – étale sites or crystalline sites – there are several variants, but we usually say, It doesn't matter, because they give rise to the same topos. So that's how I feel, like –*

**OC**: Indeed, you don't suspect that there could be a site for that topos of completely different nature, coming from analysis, or from logic or algebra – you don't suspect it, and neither did Grothendieck, probably – I mean he had a loose idea of that because he talked about the possibility of building toposes from a great variety of different mathematical situations, but, you see, he did not go to the point of talking about bridges or transfers of knowledge.

What has motivated me in developing the theory of toposes as bridges was the logical point of view, which allows you to really understand the significance and wide applicability of these concepts in light of the duality between syntax and semantics.

I wrote a paper [see [3]] together with Laurent Lafforgue and Luca Barbieri-Viale in 2015 about Nori motives regarded from a logical viewpoint. Actually there was a big problem with Nori motives, which was whether his construction that is based on the algebraic Tannakian formalism could make sense in the infinite-dimensional setting, that is, if you could build this category – this conjectured category of mixed motives, not just starting from, say, Betti homology which is finite dimensional over $\mathbb{Q}$, but starting from any, possibly infinite-dimensional (co-)homological functor. Because Tannaka theory breaks down beyond finite dimensionality, if you want to remain "inside of algebra" there was no way to solve the problem. We decided to take a logical viewpoint, and we were able to achieve the generalization.

In fact, logic allows you to overcome walls that can hardly be surpassed in the other mathematical disciplines. This is because the mathematical ontology of logic is larger than that, say, of algebra or analysis or whatever particular mathematical domain. In logic, even the contradiction is an object that exists in itself, while in any other field of mathematics, if a theory is contradictory it doesn't admit a realization, so you just do not see it if you take a classical semantic viewpoint. In logic you have a wider view, and you can really exploit the flexibility that you have at the level of syntax to construct objects with particular features. So, from a technical viewpoint, logic is much more flexible than most mathematical disciplines. In the world of syntax, you can manipulate things very easily. So you can go from the abelian to the nonabelian just by removing an axiom: very simple, you don't have to care about anything –

**NTC**: *I understand.*

**[Laughter.]**

**OC**: And similarly you can go from the commutative to the non-commutative. From a logical point of view, these are kind of insignificant distinctions – I mean from a purely formal viewpoint. So the idea is to operate at the syntactic level, and then try to algebraically structure the information that you have presented in axiomatic form, relying on the existence of canonical ways to do this. For instance, the construction of the classifying topos of a theory is a perfectly canonical process. So if you start, say, from a commutative theory, and you want to make it non-commutative, well, you remove the commutative axiom and then try to still build, in a formally similar way, a classifying topos; this will be the "right" algebraic object for the non-commutative theory. Going through logic allows you to do that. If you remain inside algebra, it might be impossible or hardly feasible. Still, many mathematicians don't even suspect the existence of this hidden logical reality.

For instance, Nori built his category of motives as a category of finite-dimensional comodules over some endomorphism coalgebra. This construction is purely algebraic; you don't suspect that there is some logic there. But we showed in our paper that this category is equivalent to the exact completion of the regular syntactic category of the theory attached to the cohomological functor, which is a perfectly logical construction! A construction which, unlike the algebraic one, makes sense even in the infinite-dimensional setting. It was a kind of breakthrough, as nobody expected that.

I really like exploiting the flexibility of logic and toposes to construct new mathematical worlds with particular properties. Earlier we talked about the possibility of using toposes to solve some paradoxes, such as the Banach-Tarski paradox, or problems linked to the non-existence of measures with particular properties, or nonstandard analysis, infinitesimals, etc.; there is great potential for research in this direction.

In any concrete mathematical situation, it is good to introduce formal languages to describe the structures and properties you want to investigate. Suppose, for instance, that you are able to logically describe something that you want to construct. Then you know that your desired object exists at least at the syntactic level. This, of course, doesn't mean that it has, say, a nice algebraic realization, but you can use the classifying topos construction to turn that logical description into a maximally structured algebraic entity. Actually, whenever you build a topos you get a category which is complete and cocomplete, so algebraically it is a very rich object. Then you can start investigating this topos from multiple points of view, for instance you can look for algebraic, geometric or topological representations for it. These will provide different concrete incarnations of your original logical description.

So I think I have given you an idea of the importance of learning logic, not really necessarily at the technical level but at least for being able to articulate, when you read any piece of mathematics, the interplay between the syntactic and the semantic elements which goes on in it. This interplay actually concerns every mathematical paper, but people are not necessarily aware of that. A very active research field nowadays is constructive mathematics. When you want to rewrite mathematics in a constructive way you are obliged to try to extract the syntactic essence of things and so you have to make –

**NTC**: *If you don't mind if I interrupt for a second – you said constructive mathematics: that can mean more than one thing sometimes; what do you mean exactly?*

**OC**: Syntactic differences can collapse when you work in a particular semantic setting. When you make a constructive analysis of the proof of some mathematical statement, you realize in particular about the existence of a plurality of different constructive formalizations that are inequivalent within constructive mathematics, but become equivalent when you quotient with respect to the classical set-theoretic foundation.[9] So it gives you a much deeper understanding of what goes on.

Also, the advantage of identifying the syntactic aspects is that once you have lifted your result at the syntactic level then you can look at other interpretations of the same result in other contexts: you are no longer limited to the particular foundation you started with. You can have analogues of your theory in other settings, while if you remain at the semantic level you are stuck there. If one wants to develop mathematics in a dynamical way, it is fundamental to identify where the invariants lie and what are the unifying aspects that allow you to go from one context to another; in this respect, syntax is crucial. More generally, it is important, whenever you investigate a particular mathematical problem, to try to capture its essential features by means of topos invariants. This will allow you, as we were saying earlier, to lift to the top of a mountain, from which you will then be able to descend in other directions. Such a lifting process might require some creative effort – it's not automatic – while going down from the top of the mountain corresponds to computing invariants in terms of different presentations of the given topos: each presentation will give a different path, and these computations are essentially canonical, though they can be more or less complex depending on the invariant under consideration. This reflects our experience of the difficulty of climbing and the relative facility of going down. So, in a sense, a topos-theoretic outlook on mathematics allows you to also separate the routine part of a mathematical proof – the purely algorithmic or mechanical part – from the key conceptual ingredients, which are those that allow you to make the lift. Grothendieck said that toposes can capture an essence of mathematical situations most diverse from each other; actually, it is this way of capturing an essence that allows you to climb a mountain in the sense I just explained. What you have to do is to find, given a problem, a topos or a family of toposes that embody its essential features. Of course, with some experience this task will become easier and easier, but still, as any kind of lifting operation, it will never be something fully canonical. In fact, the behaviour of topos invariants in terms of different presentations is a deep and sophisticated subject. Still, we dispose of certain methods that can help us identify the "right" toposes and invariants. On the other hand, once you are on the top of the mountain then, in a sense, things become essentially canonical, and so you can, for instance, obtain new ways for computing your given invariant by considering different presentations for the same topos. You could, for instance, compute a cohomology group by using, say, logical, algebraic, or geometric techniques, depending on the existence of logical, algebraic, or geometric representations of the given topos.

**KS**: *So as a question: how do you just dive into this kind of logic, for say first-year college students, or a working mathematician?*

[9] We refer readers who are not familiar with this notion to [15].

**OC**: It doesn't take a lot – you just have to learn the very basics of categorical logic, which can be done relatively quickly: the basic definitions and fundamental results can be learned in a couple of weeks. What might require more time is the kind of practice of identifying the hidden syntactic structures in proofs I was referring to earlier. In fact, what typically happens once you start learning logic – at the beginning it feels like a very formal, very symbolic approach to mathematics that is meaningless: because, of course, the point of syntax is to be meaningless –

[**Laughter.**]

**NTC**: *Right, exactly – that's the difference between syntax and semantics, right?*

**OC**: – yes, so the first reaction can be this, but as you begin reinterpreting pieces of classical mathematics in logical terms you start making sense of the usefulness of this. For me, logic is a bit like X-rays for mathematics: you don't see mathematical skeleta very easily, but of course in medicine understanding the skeleton is essential, you start from that, typically. Or it's like going at the roots of trees: when you walk in a forest, you see many trees, one here, one there, and in the middle no tree. You wonder why. If you want to understand the reasons, you have to go at the level of roots. Also, if you want to plant a new tree you have to operate at the level of roots: by placing some seeds in the ground, you might be able to build a tree exactly in the position you want, but if you don't go to that level how can you do it? You will have maybe to do it in a brute force way, you take one tree and you try to –

[**Laughter.**]

**OC**: – but many mathematicians do mathematics in a brutal way.

**NTC**: *Yeah, for sure, for sure.*

**OC**: Because they try to force things.

**NTC**: *Sometimes you really do feel that – I think even most mathematicians that don't have any logic feel sometimes that, say, "this proof is not natural"; even if one doesn't know what one means by natural one is still doing something weird.*

**OC**: Yes, exactly. So, if you really want to control what you do and to have a conceptual understanding of the various steps, logic can play a crucial role. This is true also in relation with the computational content of proofs. Constructive mathematics has developed a lot in the past decades, because of the connection with computer science.

**NTC**: *When you say constructive, does that include Errett Bishop's program [see [4]] or is it more formalist?*

**OC**: I am thinking especially of the possibility of regarding constructive proofs as programs.

**NTC**: *Right, right.*

**OC**: You can extract the computational essence of a proof, of course provided that the proof is made by using constructive principles, so you will not accept the law of excluded middle, or the axiom of choice – and all principles that assert the existence of something in particular without actually exhibiting it. The idea is that, if you want to prove that something exists, you should build it: it should correspond to a program that produces it for you. That's the philosophy of constructive mathematics. The thing is that, even though most of contemporary mathematics has been developed in a non-constructive way, most of the fundamental results in mathematics admit constructivizations, which in fact very often provide new insights and a much deeper understanding. In that way you can actually extract the computational content of a proof, so a proof becomes, in this sense, a program. In fact, doing abstract computer science or constructive mathematics is essentially the same conceptually. This is quite important, also, because nowadays computers are so powerful that they can also help us in mathematical exploration. Also, you realize that from a physical point of view non-constructive principles are really very weird. Think about the law of excluded middle, for instance: it tells you that if you have an ambient set and a subset then you can, in a sense, cut along the border of the subset so as to separate what is inside from what is outside – but suppose that you have one particle inside and one particle outside and that there is some link between them; if you cut, you will destroy the structure. Now quantum theory tells us that you cannot exclude, a priori, that even two particles that are very far away might be related; so, as you can see, it doesn't really make any sense from a physical viewpoint.

**NTC**: *Right, right.*

**OC**: You can actually wonder why mathematicians have used this principle so much. It has been a very, very bad habit that needs to be corrected as soon as possible.

**[Laughter.]**

**OC**: Actually many of the paradoxes that have been discovered so far have to do with this law. Also principles such as the axiom of choice, which tell you that such a function exists but without building it – what's the meaning of all of that? Do we have a physical interpretation? No, because if we had it, it would likely provide some constructive proof of the axiom inside ZF, which cannot exist.[10]

**NTC**: *Right, right, right.*

**OC**: The way I see future developments in mathematics is really in the direction of constructive developments – constructive foundations and computational studies which extract the content of mathematical proofs. In all of this, logic will play a crucial role. I hope that in a forthcoming future the standard mathematics education curriculum will include at least one course in logic; it could actually be whatever kind of logic – what I think matters the most is really to expose any future mathematician to the fundamental distinction

[10] Errett Bishop – see, e.g., [4], p. 12 – evidently felt that the axiom of choice *per se* was not only acceptable, but actually tautologically true, in constructive mathematics, and that the constructive objection to say Zorn's Lemma was due to other issues.

between syntax and semantics, which is the universal aspect common to all the different kinds of logics. I really hope that it will become a standard part of any mathematics course of study; otherwise the new generation will miss a dimension that can greatly enrich them as mathematicians.

So, yes, to conclude the answer to your first question, I think these are the two main prerequisites for learning topos theory: category theory, and logic. And, of course, one also needs some mathematical maturity, meaning the ability of speaking different mathematical languages, switching from algebra to topology and geometry etc., a taste for connecting different things and developing mathematics in a very dynamical way.

**KS**: *Do you have a textbook or papers you would recommend on logic – do you have any favourite books?*

**OC**: In my website there is a list of reading recommendations that you can follow. There are some nice books for learning about the subject. Concerning categorical logic, the most modern treatment of the basic stuff that dates back to the 70s is the book *Sketches of an Elephant* by Peter Johnstone[11]. My book *Theories, Sites, Toposes* [see [7]] is more advanced but it provides, in the first two chapters, the basic topos-theoretic background which is necessary for understanding its contents. This part, which consists of a total of about sixty pages, presents the main concepts that you should get familiar with: the basics of category theory and categorical logic and some fundamental results in topos theory; it's not really a lot, I mean technically speaking – of course, again, it's a matter of maturity and experience, so the difficulty is more conceptual than technical.

It can also be of psychological nature, as being able to appreciate this diversity of languages, rather than being afraid of it, requires some special attitude, which nowadays is not so encouraged, mainly due to over-specialisation of knowledge and the pressure that unfortunately is exerted a lot on the young generation to produce results very quickly. All of this can lead to not taking enough time to cultivate an intimate relation with one's subject of study, which is essential for developing that kind of maturity. I think that the biggest danger in the training of young mathematicians nowadays lies in their not having enough time to construct such a personal relationship with mathematics, especially if they are encouraged to pursue sporty exploits rather than a deep understanding, which definitely requires more time and effort. Shortcuts can be very appealing because they allow you to be the first, for instance, to solve something, maybe even without quite understanding. Grothendieck is for me a perfect example of an uncompromising mathematician at the level of the quality of understanding: for him understanding always came first. He was not happy to take any shortcuts – if he felt that there was something that needed further thought, he would go into it. That, of course, resulted in this huge scientific production, because when he picked up the pen he would go on without

[11] This is, of course, a classic in topos theory. See [12].

reservation. But in the long term, it is that kind of mathematics that leaves the most permanent mark.

So I would say that we should strive to resist the pressures of contemporary society to be fast and take shortcuts to be competitive and try to arrive first. For me, the main motivation as a mathematician has always been to understand, I've never published a paper just for the sake of publishing it or adding a line to my CV – as, unfortunately, is so often the case today. With my students, I try to make sure that they develop a sense of quality of their research work, that they are very strict with themselves about what they judge satisfactory in terms of quality and depth of understanding. But I can see that it is very, very demanding for them; in the end they succeed but it's not for everyone, of course – you need to afford being uncompromising, it takes an additional effort.

**KS**: *Do you have any personal tricks or strategy to just dive into your work, for example taking walks, or turning off the PC, or not getting distracted; are there any practices?*

**OC**: Yes, I think that walking is certainly a very good practice. Also, I would say, trying to nurture your mind with as many different intellectual stimuli as you can think of. For me, for instance, music has been always a big part of my life; even if I play less than in the past, I still listen to a lot of music. I also try to cultivate my sense of beauty and aesthetics, which I think is crucial for mathematics because you have to be guided by aesthetic principles. This kind of training in perceiving the beauty of things is something that you can practice by going to museums or experiencing art in any of its forms. I just try to cultivate myself and to feed my mind with a great variety of different enriching things. I also rely on the unconscious a lot; for me unconscious learning is quite important. I would like to encourage people to sleep, to rest, not because of laziness, but to give their brain the possibility of performing this kind of unconscious learning, which in my case can be quite powerful. Most of my learning takes place unconsciously, I would say, but in order for this kind of learning to be effective I think that you have to give very clear direction to the brain so that it can work actually without you. So I am a kind of experiment in artificial intelligence myself.

[**Laughter.**]

**NTC**: *We talk about autonomous robots, but we already have autonomous brains, we just don't know it yet.*

**OC**: Yes, indeed. I have always been very interested in this kind of unconscious learning; also concerning musical pieces – since my childhood and teenage years I have trained a lot musically by relying on the unconscious. It has always worked very well for me, and I have realized that it works well for mathematics too. It happens very often that when I get up some solution to a problem I had been working on the day before arrives naturally because of this unconscious learning. I think that for this to happen you have to provide a lot of well-selected inputs to the brain – structured inputs – and to have a very clear global vision of what you want to achieve. So I would suggest people to work very intensely but also to relax and not be tense about their relationship with their subject, not to get "stuck" on something. I like to tell my students that the best mathematics is mathematics that writes itself, in the sense that it is canonical mathematics. If you feel that you are in the dark, that you don't know where to go, and then you start making guesses, it's not a good situation. In such a scenario, I would say, just take a rest and try to gently move around, to reach a place where you see more light, where you are not obliged to make random guesses. I think that it is important to truly respect our mind and not to be desperate looking for truth; if you don't see, most of the time it is not your fault, it just means that geographically you are in a territory which doesn't allow you to look beyond a certain distance, or that the fruit that you want to pick is too far away. The important feeling that I think one should have in doing research is that of a progress in understanding. All the time that is spent with the aim of getting a deep understanding is never lost, because it will be useful for your personal experience, and also for others if you describe to them the journey you have made. I would say, don't try to guess in mathematics; guesses, I don't think they can lead to anything relevant. Look for deep understanding and, if at one point you feel that you are in the dark, take this feeling seriously and don't try to force. Because being in the dark doesn't mean that there is a problem with you, it's like a physical reality, if you do not see, you just have to move in a direction where you perceive some light. It is very important, I think, to respect our fundamental intuitions, to believe in our perceptions, of course not in a blind way, because we should always check our interpretations of reality against reality itself, but in the sense that if we have a feeling,for instance, of non-naturality or non-canonicity, as mentioned earlier, we should take it very seriously. It is important to listen to those uncomfortable feelings, because sometimes from a feeling of discomfort a completely new theory can emerge, starting from that.

When I read mathematical papers, most of the time I feel quite bad because I don't sense that they are at the right level of generality, that they illuminate the deep reasons why the results are true, etc.. So I am used to feeling bad about mathematics; but from those negative feelings I can often build something new in order to overcome that condition –

**NTC**: *Right.*

**OC**: Mathematics is a personal adventure. I think it's crucial to develop a personal relationship with your subject; as you develop relations with people, so you can also develop a relation with your subject. Grothendieck used to say, I question things for them to respond to me. That is the kind of feeling that one should arrive at: the feeling that concepts are not inanimate, that they can talk to you in a way that is very subtle, and so if you want to hear the voice of things (that is an expression by Grothendieck himself) you should put yourself in a kind of very open, receptive attitude – in a relaxed spirit, of course, but very attentive and very concentrated. Receptiveness to the richness of things is what matters the most; one should not have the feeling of fighting against things. The best mathematics is mathematics that comes naturally. If you feel like you are making an effort, that you are guessing, that you are trying that and that, I don't think you are on the right path. This suggestion is also what Grothendieck, I think, would give: he always tried to avoid shortcuts and partial understanding. Again, it's not something that everyone can practice, you need a lot of self-discipline, and also the ability to work very, very hard,[12] because, of course, if you don't want to take shortcuts, that means you have to work harder. I don't think it's advice that can be applied to everyone. Hopefully it could be, but in practice one has a lot of time constraints and so reality is different from the ideal. But I think the ideal shouldn't be forgotten.

**KS**: *That's very enlightening, thank you.*

**NTC**: *Yes, we had a wonderful time talking with you.*

**OC**: Thanks, me too.

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