Grothendieck toposes as unifying 'bridges' in Mathematics

Mémoire pour l'obtention de l'habilitation à diriger des recherches*

Olivia Caramello

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1 Introduction

Since the beginning of her Ph.D. studies, the author's research has focused on investigating the prospective role of Grothendieck toposes as unifying concepts in Mathematics and Logic.

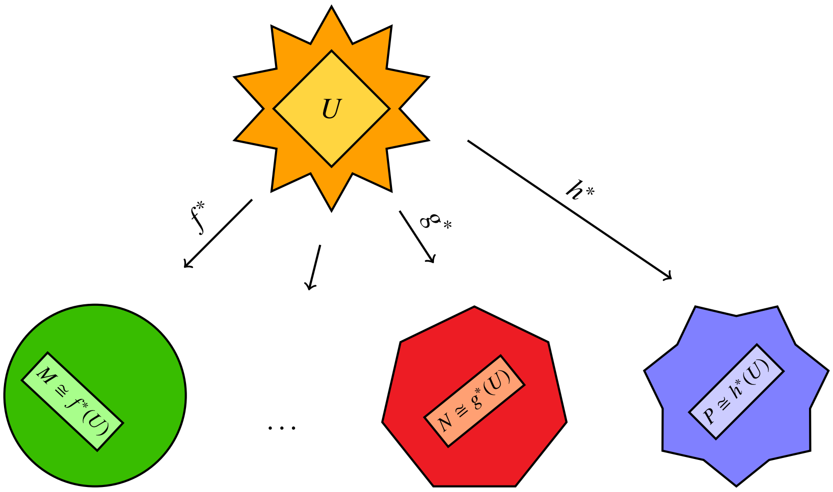


Figure 1: Classifying topos

Specifically, we have built on the following two fundamental facts:

(1) Any first-order (geometric) mathematical theory admits a (unique up to equivalence) *classifying topos* which contains a *universal model* of the theory. It is indeed possible to associate to any theory its set-theoretic models and, more generally, its models in any Grothendieck topos; the existence of the classifying topos means that any model of the theory can be obtained, uniquely up to isomorphism, as a pullback of the universal model along a (unique) morphism of toposes.

(2) Conversely, any Grothendieck topos can be regarded as the classifying topos of a first-order mathematical theory, and in fact of (infinitely) many such theories, which can possibly be completely different from each other. Two mathematical theories having the same classifying topos (up to equivalence) are said to be *Morita-equivalent*.

The idea of regarding Grothendieck toposes from the point of view of the structures that they classify dates back to A. Grothendieck and his student M. Hakim, who characterized in her book *Topos annelés et schémas relatifs* [48] four toposes arising in algebraic geometry, notably including the Zariski topos, as the classifiers of certain special kinds of rings. Later, Lawvere's work on the *Functorial Semantics of Algebraic Theories* [57] implicitly showed that all finite algebraic theories are classified by presheaf toposes. The introduction of geometric logic, that is, the logic that is preserved under inverse images of geometric functors, is due to the Montréal school of categorical logic and topos theory active in the seventies, more specifically to G. Reyes, A. Joyal and M. Makkai. Its importance is evidenced by the fact that every geometric theory admits a classifying topos and that, conversely, every Grothendieck topos is the classifying topos of some geometric theory. After the publication, in 1977, of the monograph *First-order categorical logic* by Makkai and Reyes [62], the theory of classifying toposes, in spite of its promising beginnings, stood essentially undeveloped; very few papers on the subject appeared in the following years and, as a result, most mathematicians remained unaware of the existence and potential usefulness of this fundamental notion.

Fact (1) shows that first-order mathematical theories are most naturally investigated by adopting the point of view of classifying toposes rather than that provided by their set-based models. Indeed, the collection of set-based models of a geometric theory does not in general yield a faithful constructive representation of the theory (there are examples of non-contradictory infinitary geometric theories without any set-based models) and even for finitary theories one has to appeal to non-constructive principles such as the axiom of choice to ensure that the notion of validity of sequents in all the models of the theory coincides with the notion of provability in the theory (Gödel's completeness theorem); on the other hand, the universal model of a geometric theory lying in its classifying topos realizes by itself a constructive integration of the syntax and semantics of the theory since the sequents which are valid in it are precisely (and constructively) the ones which are provable in the theory.

As to fact (2), Grothendieck himself had stressed that completely different sites can give rise to equivalent toposes, but the induced notion of Morita-equivalence of mathematical theories had never been investigated in a systematic way.

Facts (1) and (2) show that a Grothendieck topos can be thought of as a mathematical object which condenses in itself the semantics of a mathematical theory, representing the body of properties of the theory which do not depend on its linguistic presentation.

This raises the natural question of whether Grothendieck toposes could effectively serve as sorts of unifying 'bridges' for transferring concepts and results between mathematical theories which have a common 'semantical core' but a different linguistic presentation. The results obtained in the author's Ph.D. thesis, as well as those of later papers, have provided substantial technical evidence for a positive answer to this question.

In fact, the general techniques resulting from this new view of 'toposes as bridges' originally introduced in [17], besides leading to the solution to long-standing problems in Categorical Logic (cf. sections 2.2, 2.3 and 3.2), have generated many non-trivial applications in distinct mathematical fields, including Model Theory (cf. sections 3.3.1 and 2.3), Proof Theory (cf. section 2.2), Topology (cf. sections 3.1 and 4.1), Algebraic Geometry (cf. sections 2.1.3 and 3.3.3), Algebra (cf. sections 3.3.2, 4.1, 4.2 and 4.3) and Functional Analysis (cf. section 4.1.4), and the potential of this theory has just started to be explored.

In this document, we shall explain the general principles underlying this new view of toposes as unifying 'bridges' and a few selected results already obtained by implementing the resulting techniques in different mathematical contexts.

2 General theory

Many important dualities and equivalences in Mathematics can be naturally interpreted in terms of equivalences between the classifying toposes of different theories; on the other hand, Topos Theory itself is a primary source of Morita-equivalences. In fact, the notion of Morita-equivalence formalizes in many situations the feeling of 'looking at the same thing in different ways', which explains why it is ubiquitous in Mathematics.

Grothendieck toposes can serve as 'bridges' for transferring properties between Morita-equivalent theories in the following sense. The existence of different presentations for the 'semantical core' of a given mathematical theory translates into the existence of different representations (technically speaking, sites) for its classifying topos. Topos-theoretic invariants (i.e., properties or constructions on toposes which are stable under categorical equivalence), appropriately characterized as properties of the sites of definitions of the topos, can then be used to transfer properties between the two representations of the classifying topos and hence between the two theories.

For example, adopting this viewpoint, Deligne's theorem asserting that every coherent topos has enough points can be seen to be equivalent to Gödel's completeness theorem in Logic. Other examples involving sites and theories of various nature can be found below.

This method creates unifying 'bridges' in Mathematics in the sense that properties (resp. constructions) arising in the context of mathematical theories which have a common 'semantical core' but a different 'linguistic presentation' (technically speaking, Morita-equivalent theories) come to be seen as different manifestations of a unique property (resp. construction) lying at the topos-theoretic level.

The way through which concrete results are generated by an application of this technique is inherently 'upside-down' with respect to the more traditional approaches in which one starts with simple ingredients and proceeds to combine them to build more complicated structures. Indeed, this method takes as primitive ingredients rich and sophisticated mathematical entities, namely Morita-equivalences and topos-theoretic invariants, and extracts from them concrete information relevant for classical mathematics (the fact that these 'primitive ingredients' are intrinsically complex does not nonetheless imply that it is difficult to obtain such objects from the current mathematical practice; quite the contrary in fact).

An important aspect of the 'bridge-building' technique is, besides its level of generality (indeed, it can be applied to first-order mathematical theories of essentially any kind), the fact that it is amenable to computations, due to the very well-behaved (although highly non-trivial) nature of the representation theory of Grothendieck toposes in terms of sites. In fact, this technique allows one to generate insights in different mathematical fields in a 'semi-automatic way', that is without making any arbitrary, 'non-canonical' choice. I should hasten to point out that not all of the results generated in this way are 'interesting mathematical theorems'; many of them can be rather 'weird' according to usual mathematical standards, even though they might well be quite deep.

2.1 The 'bridge-building' technique

This section is based on [17] and section 2.2. of [8].

Recall that the classifying topos $\mathcal{E}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ (also denoted by $\mathbf{Set}[\mathbb{T}]$) can always be built as the topos of sheaves $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ on the geometric syntactic site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$: the syntactic category $\mathcal{C}_{\mathbb{T}}$ has as objects the geometric formulae-in-context $\{\vec{x} \,.\, \phi\}$ over Σ (up to 'renaming' equivalence) and as arrows $\{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, \psi\}$ (where the contexts $\vec{x}$ and $\vec{y}$ are supposed to be disjoint without loss of generality) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\vec{x}, \vec{y})$ which are $\mathbb{T}$-provably functional i.e. such that the sequents

$$(\phi \vdash_{\vec{x}} (\exists \vec{y})\theta),$$
$$(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi), \text{ and}$$
$$((\theta \wedge \theta[\vec{z}/\vec{y}]) \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{y} = \vec{z}))$$

are provable in $\mathbb{T}$. The syntactic topology $J_{\mathbb{T}}$ is the canonical topology on $\mathcal{C}_{\mathbb{T}}$, i.e. the topology whose covering sieves are those which contain small covering families.

The 'bridge-building' technique allows one to construct topos-theoretic 'bridges' connecting distinct mathematical theories with each other.

Specifically, if $\mathbb{T}$ and $\mathbb{T}'$ are two Morita-equivalent theories (that is, geometric theories classified by the same topos), their common classifying topos can be used as a 'bridge' for transferring information between them:

$$\begin{tikzcd} & \mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'} \arrow[dr, dashed, bend left] & \\ \mathbb{T} \arrow[ur, dashed, bend left] & & \mathbb{T}' \end{tikzcd}$$

The transfer of information between $\mathbb{T}$ and $\mathbb{T}'$ takes place by expressing topos-theoretic *invariants* (that is, properties or constructions on toposes which are stable under categorical equivalence) defined on their common classifying topos directly in terms of the theories $\mathbb{T}$ and $\mathbb{T}'$. This is done by associating to each of the two theories a site of definition for its classifying topos (for example, the geometric syntactic site) and then considering topos-theoretic invariants on the classifying topos from the points of view of the two sites of definition. More precisely, suppose that $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ are two sites of definition for the same topos, and that I is a topos-theoretic invariant. Then one can seek *site characterizations* for I, that is, in the case I is a property (the case of I being a 'construction' admits an analogous treatment), logical equivalences of the kind 'the topos $\mathcal{E}$ satisfies I if and only if $(\mathcal{C}, J)$ satisfies a property $P_{(\mathcal{C},J)}$ (written in the language of the site $(\mathcal{C}, J)$)' and, similarly for $(\mathcal{D}, K)$, logical equivalences of the kind 'the topos $\mathbf{Sh}(\mathcal{D}, K)$ satisfies

I if and only if $(\mathcal{D}, K)$ satisfies a property $Q_{(\mathcal{D},K)}$':

Invariant I across the Morita-equivalence

site characterization for I $\quad$ $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{D}, K)$ $\quad$ *site characterization for I*

$(\mathcal{C}, J)$ $\qquad$ $(\mathcal{D}, K)$

Property $P_{(\mathcal{C},J)}$ $\qquad$ *Property $Q_{(\mathcal{D},K)}$*

Clearly, such characterizations immediately lead to a logical equivalence between the properties $P_{(\mathcal{C},J)}$ and $Q_{(\mathcal{D},K)}$, which can thus be seen as different manifestations of a unique property, namely I, in the context of the two different sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$.

In fact, one does not necessarily need 'if-and-only-if' site characterizations in order to build 'bridges': in order to establish an implication between a property $P_{(\mathcal{C},J)}$ of a site $(\mathcal{C}, J)$ and a property $Q_{(\mathcal{D},K)}$ of another site of definition $(\mathcal{D}, K)$ of the same topos, it suffices to find an invariant I such that $P_{(\mathcal{C},J)}$ implies I on $\mathbf{Sh}(\mathcal{C}, J)$ and I on $\mathbf{Sh}(\mathcal{D}, K)$ implies $Q_{(\mathcal{D},K)}$.

The 'bridge' technique allows one to interpret and study many dualities and equivalences arising in different fields of mathematics by means of the investigation of how topos-theoretic invariants characterize in terms of sites. In other words, the representation theory of Grothendieck toposes becomes a sort of 'metatheory of mathematical duality', which makes it possible to effectively compare distinct mathematical theories with each other and transfer knowledge between them. In the following sections we discuss more in detail the subject of *Morita-equivalences*, which play in our context the role of 'decks' of our 'bridges', and of *site characterizations* for topos-theoretic invariants, which constitute their 'arches'.

Incidentally, it should be noted that this method could be generalized to the case of 'bridges' whose deck is given by some kind of relationship between toposes which is not necessarily an equivalence, in the presence of properties or constructions of toposes which are invariant with respect to such a relation. Nonetheless, the advantage of focusing on Morita-equivalences is twofold; on one hand, it is convenient because, due to the fact that every property expressed in categorical language is automatically invariant with respect to categorical equivalence, we dispose of an unlimited number of invariants readily available to consider, whilst on the other hand, it realizes a *unification* of 'concrete' properties of different theories by interpreting them as different *manifestations* of a unique property lying at the topos-theoretic level.

2.1.1 *Decks of 'bridges'*: Morita-equivalences

Let us first recall from [54] the following classical definition.

Definition 2.1. Two geometric theories $\mathbb{T}$ and $\mathbb{T}'$ are said to be *Morita-equivalent* if they have equivalent classifying toposes, equivalently if they have equivalent categories of models in every Grothendieck topos $\mathscr{E}$, naturally in $\mathscr{E}$, that is for each Grothendieck topos $\mathscr{E}$ there is an equivalence of categories

$$\tau_{\mathscr{E}} : \mathbb{T}\text{-mod}(\mathscr{E}) \to \mathbb{T}'\text{-mod}(\mathscr{E})$$

such that for any geometric morphism $f : \mathscr{F} \to \mathscr{E}$ the following diagram commutes (up to isomorphism):

$$\begin{tikzcd}
\mathbb{T}\text{-mod}(\mathscr{E}) \arrow[r, "\tau_{\mathscr{E}}"] \arrow[d, "f^*"'] & \mathbb{T}'\text{-mod}(\mathscr{E}) \arrow[d, "f^*"] \\
\mathbb{T}\text{-mod}(\mathscr{F}) \arrow[r, "\tau_{\mathscr{F}}"'] & \mathbb{T}'\text{-mod}(\mathscr{F})
\end{tikzcd}$$

Note that ‘to be Morita-equivalent to each other’ defines an *equivalence relation* of the collection of all geometric theories.

Given the level of technical sophistication of this definition, it is reasonable to wonder if Morita-equivalences naturally arise in Mathematics and, in case, if there are systematic ways for ‘generating’ them. The following remarks are meant to show that the answer to both questions is positive.

- If two geometric theories $\mathbb{T}$ and $\mathbb{T}'$ have equivalent categories of models in the category **Set** then, provided that the given categorical equivalence is established by only using constructive logic (that is, by avoiding in particular the law of excluded middle and the axiom of choice) and geometric constructions (that is, by only using set-theoretic constructions which involve finite limits and small colimits, equivalently which admit a syntactic formulation involving only equalities, finite conjunctions, (possibly) infinitary disjunctions and existential quantifications), it is reasonable to expect the original equivalence to ‘lift’ to a Morita-equivalence between $\mathbb{T}$ and $\mathbb{T}'$. Indeed, a Grothendieck topos behaves logically as a ‘generalized universe of sets’ in which one can perform most of the classical set-theoretic arguments and constructions, with the only significant exception of those requiring non-constructive principles. So we can naturally expect to be able to generalize the original equivalence between the categories of set-based models of the two theories to the case of models in arbitrary Grothendieck toposes; moreover, the fact that the constructions involved in the definition of the equivalence are geometric ensures that the above-mentioned naturality condition for Morita-equivalences is satisfied (since geometric constructions are preserved by inverse image functors of geometric morphisms). As examples of ‘lifting’ of naturally arising categorical equivalences to Morita-equivalences we mention the Morita-equivalence between MV-algebras and abelian ℓ-groups with strong unit (cf. [25]) and that between abelian ℓ-groups and perfect MV-algebras (cf. [26]). We shall review these equivalences in section 4.3.1.

- Two cartesian (in particular, finitary algebraic) theories $\mathbb{T}$ and $\mathbb{T}'$ have equivalent categories of models in **Set** if and only if they are Morita-equivalent (or, equivalently, if and only if their cartesian syntactic categories are equivalent).

- If two geometric (resp. regular, coherent) theories have equivalent geometric (resp. regular, coherent) syntactic categories (i.e., they are bi-interpretable) then they are Morita-equivalent. This follows at once from the fact that the 'logical' topologies in the syntactic sites are defined intrinsically in terms of the categorical structure present on the relevant syntactic categories. Anyway, as it can be naturally expected, the most interesting Morita-equivalences do not arise from bi-interpretations.

  In particular, if two finitary first-order theories are bi-interpretable (in the sense of classical Model Theory) then their Morleyizations are Morita-equivalent (recall that the Morleyization of a finitary first-order theory $\mathbb{T}$ is coherent theory canonically associated to it whose set-based models can be identified with those of $\mathbb{T}$).

- Two associative rings with unit are Morita-equivalent (in the classical, ring-theoretic, sense) if and only if the algebraic theories axiomatizing the (left) modules over them are Morita-equivalent (in the topos-theoretic sense). Indeed, by the first remark above, these theories are Morita-equivalent if and only if their categories of set-based models are equivalent, that is if and only if the categories of (left) modules over the two rings are equivalent. Specifically, for each ring R the theory axiomatizing its (left) R-modules can be defined as the theory obtained from the algebraic theory of abelian groups by adding one unary function symbol for each element of the ring and writing down the obvious equational axioms which express the conditions in the definition of R-module.

- Other notions of Morita-equivalence for various kinds of algebraic or geometric structures considered in the literature can be reformulated as equivalences between different representations of the same topos, and hence as Morita-equivalences between different geometric theories. For instance:

  - Two topological groups are Morita-equivalent (in the sense of [65]) if and only if their toposes of continuous actions are equivalent. A natural analogue of this notion for topological and localic groupoids has been studied by several authors and a summary of the main results in contained in section C5.3 of [54].
  - Two small categories are Morita-equivalent (in the sense of [41]) if and only if the corresponding presheaf toposes are equivalent, that is if and only if their Cauchy completions (also called Karoubian completions) are equivalent (cf. [7]).

  - Two inverse semigroups are Morita-equivalent (in the sense of [73] or, equivalently, of [43]) if and only if their classifying toposes (as defined in [44]) are equivalent (cf. [43]).

- Categorical dualities or equivalences between 'concrete' categories can often be seen as arising from the process of 'functorializing' Morita-equivalences which express structural relationships between each pair of objects corresponding to each other under the given duality or equivalence (cf. for example [18], [21] and [22]). In fact, the theory of geometric morphisms of toposes provides various natural ways of 'functorializing' bunches of Morita-equivalences (cf. section 4.1).

- Different sites of definition for a given topos can be interpreted logically as Morita-equivalences between different theories; in fact, the converse also holds, in the sense that any Morita-equivalence gives canonically rise to two different sites of definition of the common classifying topos. The representation theory of Grothendieck toposes in terms of sites and, more generally, any technique that one may employ for obtaining a different site of definition or representation for a given topos (such as, for instance, the Comparison Lemma of [2]) thus constitutes a tool for generating Morita-equivalences.

- The usual notions of *spectra* for mathematical structures can be naturally interpreted in terms of classifying toposes, and the resulting sheaf representations as arising from Morita-equivalences between an 'algebraic' and a 'topological' representation of such toposes. More specifically, Cole's general theory of spectra [33] (cf. also section 6.5 of [53] for a succinct overview of this theory) is based on the construction of suitable classifying toposes. Coste introduced in [37] alternative representations of such classifying toposes, identifying in particular simple sets of conditions under which they can be represented as toposes of sheaves on a topological space. He then derived from the equivalence between two of these representations, one of essentially algebraic nature and the other of topological nature, a criterion for the canonical homomorphism from the given structure to the global sections of the associated structure sheaf to be an isomorphism.

- The notion of Morita-equivalences materializes in many situations the intuitive feeling of 'looking at the same thing in different ways', meaning, for instance, describing the same structure(s) in different languages or constructing a given object in different ways. Concrete examples of this general remark can be found for instance in [18] and [22], where the different constructions of the Zariski spectrum of a ring, of the Gelfand spectrum of a C^*-algebra, and of the Stone-Cech compactification of a topological space are interpreted as Morita-equivalences between different theories (cf. sections 4.1.1 and 4.1.4 below).

- Different ways of looking at a given mathematical theory can often be formalized as Morita-equivalences. Indeed, different ways of describing the structures axiomatized by a given theory can often give rise to a theory written in a different language whose models (in any Grothendieck topos) can be identified, in a natural way, with those of the former theory and which is therefore Morita-equivalent to it.

- A geometric theory *alone* generates an infinite number of Morita-equivalences, via its 'internal dynamics'. In fact, any way of looking at a geometric theory as an extension of a geometric theory written over its signature provides a different representation of its classifying topos, as a subtopos of the classifying topos of the latter theory (cf. Theorem 2.5 below).

- Different separating sets of objects for a given topos give rise to different sites of definition for it; indeed, for any separating set of objects $\mathcal{C}$ of a Grothendieck topos, we have an equivalence $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}|_{\mathcal{C}})$, where $J_{\mathcal{E}}|_{\mathcal{C}}$ is the Grothendieck topology on $\mathcal{C}$ induced by the canonical topology $J_{\mathcal{E}}$ on $\mathcal{E}$. In particular, for any topological space X and any basis $\mathcal{B}$ for it, we have an equivalence $\mathbf{Sh}(X) \simeq \mathbf{Sh}(\mathcal{B}, J_{\mathbf{Sh}(X)}|_{\mathcal{B}})$ (cf. sections 4.1.1 and 4.1.4 for examples of dualities arising from Morita-equivalences of this form in which the induced topologies can be characterized intrinsically by means of topos-theoretic invariants).

2.1.2 *Arches of 'bridges'*: Site characterizations

As we remarked above, the 'arches' of topos-theoretic 'bridges' should be provided by site characterizations for topos-theoretic invariants, that is results connecting invariant properties (resp. constructions) on toposes and properties (resp. constructions) of their sites of definition (written in their respective languages).

It thus becomes crucial to investigate the behaviour of topos-theoretic invariants with respect to sites. As a matter of fact, such behaviour is often very natural, in the sense that topos-theoretic invariants generally admit natural site characterizations. For instance, 'if and only if' characterizations for a wide class of geometric invariants of toposes, notably including the property of a topos to be atomic (resp. locally connected, localic, equivalent to a presheaf topos, compact, two-valued) were obtained in [19] (cf. section 3.1 below).

On the other hand, it was shown in [23] that a wide class of logically-inspired invariants of topos, obtained by interpreting first-order formulae written in the language of Heyting algebras, admit elementary 'if and only if' site characterizations. Moreover, as we shall see in section 3.1, several notable invariants of subtoposes admit natural site chacterizations as well as explicit logical descriptions in terms of the theories classified by them. Topos-theoretic invariants relevant in Algebraic Geometry and Homotopy Theory, such as for example the cohomology and homotopy groups of toposes, also admit, at least in many important cases, natural characterizations in terms of sites.

It should be noted that, whilst it is often possible to obtain, by using topos-theoretic methods, site characterizations for topos-theoretic invariants holding for large classes of sites, such criteria can be highly non-trivial as far as their mathematical depth is concerned (since the representation theory of toposes is by all means a non-trivial subject). Therefore, when combined with specific Morita-equivalences to form 'bridges', they can lead to deep results on the relevant theories, especially when the given Morita-equivalence is a non-trivial one. These insights can actually be quite surprising, when observed from a concrete point of view (that is, from the point of view of the two theories related by the Morita-equivalence), since a given topos-theoretic invariant may manifest itself in very different ways in the context of different sites (cf. section 3.1 below).

The 'centrality' of topos-theoretic invariants in Mathematics is well illustrated by the fact that, in spite of their apparent remoteness from the more 'concrete' objects of study in mathematics, once translated at the level of sites or theories, they often specialize to construction of natural mathematical or logical interest. Besides homotopy and cohomology groups of toposes, whose 'concrete' instantiations in the context of topological spaces and schemes have been of central importance in Topology and Algebraic Geometry throughout the last decades, a great deal of other invariants, including those which might seem at first sight too abstract to be connected to any problem of natural mathematical interest, can be profitably used to shed light on classical theories. For example, even an abstract logically-inspired construction such as the *DeMorganization* of a topos, introduced in [11] as the largest dense subtopos of a given topos satisfying De Morgan's law, was shown in [12] to yield, when applied to a specific topos such as the classifying topos of the coherent theory of fields, the classifying topos of a very natural mathematical theory, namely the theory of fields of finite characteristic which are algebraic over their prime fields. The author's papers contain several other examples, some of which will be presented below.

It should be noted that the arches of our 'bridges' need not necessarily be 'symmetric', that is arising from the instantiation in the context of two given sites of a unique site characterization holding for both of them. As an example, take the property of a topos to be coherent: this property does not admit an 'elementary' 'if and only if' site characterization holding for all sites, but it admits such a site characterization holding for all trivial sites (i.e. sites whose underlying Grothendieck topology is trivial) and an implicative characterization of the kind 'if a theory is coherent, then its classifying topos is coherent'. These characterizations can for instance be combined to obtain a 'bridge' yielding a result on coherent theories classified by a presheaf topos.

The level of mathematical 'depth' of the results obtained by applying the 'bridge' technique can vary enormously, depending on the complexity of the site characterizations and of the given Morita-equivalence. Still, as we shall see at various points of this text, even very simple invariants applied to easily established Morita-equivalences can yield surprising insights which would be hardly attainable, or not even imaginable, otherwise.

Lastly, it is worth noting that sites are by no means the only mathematical objects that one can use for representing toposes. For instance, Grothendieck toposes can be represented by using groupoids (either topological or localic) or quantales. It clearly makes sense to apply the 'bridge' technique, described above for sites, also in the case of these representations, replacing the site characterizations for the given invariant with appropriate characterizations of it in terms of the mathematical objects used for representing the relevant topos.

2.1.3 The flexibility of Logic: a two-level view

Logical theories are, in themselves, non-structured objects. Any formal expression (independently of any consideration of truth or falsity) is an object of study of Logic. It is therefore very easy to generate logical objects, in particular logical theories: every family of axioms defines a theory.

So the mathematical ontology of logic is much larger than that of any other field of mathematics. Nonetheless, the theory of classifying toposes allows one to structurate any logical context in a maximal way: the classifying topos $\mathcal{E}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ is a topological object which embodies exactly the semantic context of $\mathbb{T}$ and nothing else.

There is therefore a duality between

- the non-structured level of syntax: the theories $\mathbb{T}$,
- the (categorically) structured level of semantics universally represented by classifying toposes $\mathcal{E}_{\mathbb{T}}$.

Modifying an algebraic structure is in general delicate. On the contrary, modifying a logical theory by adding, removing or altering the axioms or some elements of its language, is very easy.

So, there is a correspondence between logical notions and operations on the one hand, and notions and operations on toposes on the other hand. This allows one to exploit the flexibility of logic to fabricate structures satisfying a desired set of properties.

For instance, as argued in [19] and [21], many structures presented by generators and relations can be built by using classifying toposes or relevant syntactic categories embedding in them. In fact, we can informally regard the sorts and symbols in the signature of the theory as defining the 'generators' and the axioms of the theory as providing the 'relations'. This intuition is actually made precise in [18], where it is shown that any model of an 'infinitary ordered Horn theory' presented by generators and relations can be built as the syntactic category of a $\mathcal{S}$-theory (a generalization of the notion of first-order theory introduced in [18], which involves generalized connectives specified by a set $\mathcal{S}$).

In [18] and [21] we built, following this approach, various kinds of preordered structures presented by generators and relations, *e.g.* the free frame on a complete join semilattice (this was a question posed by A. Döring), free Boolean algebras on distributive lattices, coherent posets, meet-semilattices or disjunctively distributive lattices (as we shall see in section 4.1.3, these constructions play a crucial role in the construction of Priestley-type dualities for this kind of structures). Indeed, for these kind of structures the $\mathcal{S}$-theories associated with them belongs to fragments of geometric logic, so that the 'bridge' technique can be exploited in connection with the existence of multiple representations of the correspoding classifying toposes for finding concrete descriptions for poset structures presented by generators and relations, a notoriously difficult problem which frequently arises in practice. See section 4.1.1 below for some illustrations of this point.

More recently, in [3], we showed that Nori's category of effective homological motives can be constructed as the effectivization of the regular syntactic category of a regular theory obtained by taking the set of regular sequents over a language associated to Nori's diagram which are satisfied by Betti homology. Actually, we generalized Nori motives by constructing an abelian R-linear category associated with any representation $T : D \to A_R$ of a diagram D in the category A_R of R-modules (or more generally, in any abelian R-linear category A_R) which satisfies a universal property implying Nori's one.

The first step in this construction consists in associating to the representation T the so-called regular theory $\mathrm{Th}(T)$ of T. The language L_D of this theory consists in sorts associated to the objects of the diagram, function symbols associated to the arrows of the diagram as well as to the R-linear structure operations (addition and multiplication by elements of R) and constants corresponding to the zero elements of such module structures. The axioms of the theory consists of all the regular sequences written over L_D which are satisfied by the representation T.

The second step consists in associating to the regular theory of T its regular syntactic category. This syntactic category is regular, as it is associated to a regular theory, and it is additive and R-linear by construction, but it is not abelian as it lacks quotients.

The third step consists in replacing this syntactic regular category by its effectivization, a construction which formally adds quotients of equivalence relations in a way which admits a fully explicit description. The resulting category, call it $\mathcal{C}_{\mathbb{T}_T}$, is abelian, R-linear and our representation T factors as the composite of an exact and faithful functor $F_T : \mathcal{C}_{\mathbb{T}_T} \to R\text{-}\mathsf{mod}$ with the representation $\tilde{T} : D \to \mathcal{C}_{\mathbb{T}_T}$ corresponding to the 'tautological' model of T in $\mathcal{C}_{\mathbb{T}_T}$. This category notably satisfies the universal 'Nori-type' factorisation property:

Theorem 2.2 (Theorem 2.5 [3]). *Let R be a ring, D a diagram and $T : D \to R\text{-}\mathsf{mod}$ a representation. Then the category $\mathcal{C}_{\mathbb{T}_T}$ is abelian and R-linear and, together with the representation $\tilde{T}$ and the functor F_T defined above, satisfies the following universal property: $T = F_T \circ \tilde{T}$ and this factorisation is universal in the sense that for any factorisation $T = F \circ S$, where $F : \mathcal{A} \to R\text{-}\mathsf{mod}$ is an exact and faithful functor defined on an R-linear abelian category $\mathcal{A}$ and $S : D \to \mathcal{A}$ is a representation of D in $\mathcal{A}$, there exists a unique exact (and faithful) functor $F_S : \mathcal{C}_T \to \mathcal{A}$ such that the following diagram commutes:*

$$
\begin{tikzcd}
 & C_T \arrow[d, dashed, "F_S"] \arrow[ddr, "F_T"] & \\
 & \mathcal{A} \arrow[dr, "F"] & \\
D \arrow[uur, "\tilde{T}"] \arrow[ur, "S"] \arrow[rr, "T"] & & R\text{-}\mathsf{mod}
\end{tikzcd}
$$

If R is Noetherian and T takes values in the category $R\text{-}\mathsf{mod}_{\mathrm{f}}$ of finite-type R-modules then F_T takes values in $R\text{-}\mathsf{mod}_{\mathrm{f}}$ as well.

The generality of this construction allows one to associate an R-linear abelian category of "mixed motives" to any homology or cohomology functor with coefficients in a field or a ring which contains R. For instance, each of the usual cohomology or homology theories, such as Betti, ℓ-adic, p-adic, De Rham, gives rise in this way to a $\mathbb{Q}$-linear abelian category of mixed motives. The question thus naturally arises as to under which conditions these categories are equivalent to each other. The following result provides an answer to it:

Theorem 2.3. *(i) Given two representations $T : D \to R\text{-}\mathsf{mod}$ and $T' : D' \to R'\text{-}\mathsf{mod}$ (or more generally, any pair of representations of diagrams in effective regular categories), the categories $C_{\mathbb{T}_T}$ and $C_{\mathbb{T}_{T'}}$ are equivalent if and only if the theories $\mathrm{Th}(T)$ and $\mathrm{Th}(T')$ are Morita-equivalent.*

(ii) Let R be a ring, D a diagram and $T, T' : D \to R\text{-}\mathsf{mod}$ representations of D. Then we have an equivalence of categories $\xi : C_{\mathbb{T}_T} \to C_{\mathbb{T}_{T'}}$ making the diagram

$$
\begin{tikzcd}
 & C_{\mathbb{T}_T} \arrow[d, dashed, "\xi"] \\
D \arrow[ur, "\tilde{T}"] \arrow[r, "\tilde{T}'"] & C_{\mathbb{T}_{T'}}
\end{tikzcd}
$$

commute if and only if $\mathrm{Th}(T) = \mathrm{Th}(T')$, equivalently if and only if for any tuples

$$\langle s_1, \ldots, s_k \rangle : d_1, \ldots, d_n, e_1, \ldots, e_m \to c_1, \ldots, c_k$$

and

$$\langle s'_1, \ldots, s'_{k'} \rangle : d_1, \ldots, d_n, a_1, \ldots, a_l \to c'_1, \ldots, c'_{k'}$$

of R-linear combinations of edges in D,

$$p^T_{\vec{d},\vec{e}}(Ker(T(\langle s_1, \ldots, s_k \rangle)) \subseteq p^T_{\vec{d},\vec{a}}(Ker(T(\langle s'_1, \ldots, s'_{k'} \rangle))$$

*as subobjects of $T(d_1) \times \cdots \times T(d_n)$ in R-*mod *if and only if*

$$p^{T'}_{\vec{d},\vec{e}}(Ker(T'(\langle s_1, \ldots, s_k \rangle))) \subseteq p^{T'}_{\vec{d},\vec{a}}(Ker(T'(\langle s'_1, \ldots, s'_{k'} \rangle)))$$

*as subobjects of $T'(d_1) \times \cdots \times T'(d_n)$ in R-*mod *(where the $p^{T}_{\vec{d},\vec{e}}$, $p^{T}_{\vec{d},\vec{a}}$, $p^{T'}_{\vec{d},\vec{e}}$ and $p^{T'}_{\vec{d},\vec{a}}$ are the natural projection functors associated to forgetting the second family of components).*

This theorem has the following implications for motives:

Corollary 2.4 (cf. Corollary 3.6 [3])**.** *Let K be an arbitrary base field and $R = \mathbb{Q}$ as above. Let D be a diagram built starting from the category of finite-type schemes over K (for instance, Nori's diagram).*

(i) Let T be a representation of D in the category of vector spaces over $\mathbb{Q}$ which is induced by a cohomological functor with coefficients of characteristic 0 *(Betti if $K \subseteq \mathbb{C}$, De Rham if* $\mathrm{car}(K) = 0$*, ℓ-adic if $\ell \neq$* $\mathrm{car}(K)$*, p-adic if $p =$* $\mathrm{car}(K), \ldots$*).*

*Then T factors canonically and universally through the $\mathbb{Q}$-abelian category $\mathcal{C}_{\mathbb{T}_T}$ and the $\mathbb{Q}$-linear exact and faithful functor $\mathcal{C}_{\mathbb{T}_T} \to \mathbb{Q}$-*vect.

(ii) Consider a family $\{T\}$ of representations

$$T : D \to \mathbb{Q}\text{-}vect$$

defined by "good" cohomological functors such as the above-mentioned ones. Then the following conditions are equivalent:

- *The cohomological functors $T \in \{T\}$ factor through a category of motives in the usual sense, that is, there exists a $\mathbb{Q}$-linear abelian category $\mathcal{M}$, endowed with a representation $D \to \mathcal{M}$, such that each $T \in \{T\}$ factors as the composite of $D \to \mathcal{M}$ with an exact and faithful functor*

$$\mathcal{M} \to \mathbb{Q}\text{-}vect\ .$$

- *The categories $\mathcal{C}_{\mathbb{T}_T}$ associated with the different $T \in \{T\}$, endowed with the representation $\tilde{T} : D \to \mathcal{C}_{\mathbb{T}_T}$, are equivalent.*
- *The regular theories $\mathbb{T}_T$ of the representations $T \in \{T\}$ are identical.*

The above-mentioned corollary can be interpreted as follows: if (mixed) motives actually exist then they have a logical nature. Indeed, what the different cohomological functors must have in common with each other in order for a category of (mixed) motives (in Grothendieck's sense) to exist is their associated regular theories. If this is the case, the category of motives can be built from any of these cohomological functors (independently from all the others).

Note that the vanishing conditions of the spaces associated by T to a given object of D or, more generally, the vanishing conditions of the subspaces defined as the kernels of the homomorphisms associated by T to a linear combination of composites of edges in D, for instance the subspaces $\mathrm{Ker}(T(P(u)))$ associated with an endomorphism u and a polynomial P with coefficients in $\mathbb{Q}$, are expressible in regular logic over L_D. On the other hand, regular logic is not strong enough for saying anything on the dimensions of the spaces and subspaces over their natural coefficient fields which vary. In fact, for investigating the problem of the "independence from ℓ" we have developed another framework based of full first-order logic and atomic two-valued toposes (see section 3.3.3 below).

It is interesting to compare our construction of Nori-type categories via syntactic categories with the Tannakian formalism on which the classical Nori construction is based. Nori's original construction of a universal R-linear abelian category associated with a representation $T : D \to R\text{-}\mathsf{mod}_{\mathrm{f}}$ of a diagram D in the category of finite-type R-modules over a Noetherian ring R consists in taking the (filtered) colimit

$$C_T = \mathrm{colim}_{F \subseteq D \text{ finite}} \mathrm{End}(T|_F)\text{-mod}_{\mathrm{fin}}$$

over all the finite subdiagrams of D, with the canonical representation of D in it given by:

$$d \rightsquigarrow T(d)$$

and

$$(f : d \to d') \rightsquigarrow T(f) : T(d) \to T(d') \,.$$

If R is a field, this category is easily seen to be equivalent to the category $\mathrm{Comod}_{\mathrm{fin}}(\mathrm{End}^{\vee}(T)$ of finite-dimensional comodules over the coalgebra $\mathrm{End}^{\vee}(T)$ (recall that this coalgebra is defined in such a way that its dual can be identified with the endomorphism algebra $\mathrm{End}(T)$, but the dual of $\mathrm{End}(T)$ does not necessarily coincide with $\mathrm{End}^{\vee}(T)$ unless the diagram D is finite). The canonical representation $i_D : D \to \mathrm{Comod}_{\mathrm{fin}}(\mathrm{End}^{\vee}(T))$ is given by the assignment $d \rightsquigarrow T(d)$, where $T(d)$ is endowed with the canonical structure of comodule over $\mathrm{End}^{\vee}(T)$.

We thus obtain an equivalence of categories

$$\chi : \mathcal{C}_{\mathbb{T}_T} \to \mathrm{Comod}_{\mathrm{fin}}(\mathrm{End}^{\vee}(T))$$

compatible with the canonical representations $\tilde{T}$ of D in the category $\mathcal{C}_{\mathbb{T}_T}$ and i_D of D in the category $\mathrm{Comod}_{\mathrm{fin}}(\mathrm{End}^{\vee}(T))$:

$$\begin{tikzcd} & \mathcal{C}_{\mathbb{T}_T} \arrow[d, dashed, "\chi"] \\ \mathrm{D} \arrow[ur, "\tilde{T}"] \arrow[r, "i_D"] & \mathrm{Comod}_{\mathrm{fin}}(\mathrm{End}^{\vee}(T)) \end{tikzcd}$$

This equivalence allows one in particular to get a very explicit concrete description of the objects and arrows of the category $\mathrm{Comod}_{\mathrm{fin}}(\mathrm{End}^{\vee}(T))$ in terms of the objects and arrows coming from the diagram D under the representation i_D. Indeed, in [3] we obtain the following result: every object A of $\mathrm{Comod}_{\mathrm{fin}}(\mathrm{End}^{\vee}(T))$ fits in an exact sequence

$$0 \to K' \rightarrowtail K \to A \to 0,$$

where K and K' sit in a diagram of the form

$$\begin{tikzcd}
K = \mathrm{Ker}(i_D(\vec{s})) \arrow[r, hook] & i_D(a_1)\times\cdots\times i_D(a_n) \arrow[r, "i_D(\vec{s})"] & i_D(c_1)\times\cdots\times i_D(c_k) \\
K' \arrow[u, tail] \arrow[ur, tail] & & \\
\mathrm{Ker}(i_D(\vec{t})) \arrow[r, hook] \arrow[u] & i_D(a_1)\times\cdots\times i_D(a_n)\times i_D(b_1)\times\cdots\times i_D(b_m) \arrow[r, "i_D(\vec{t})"] \arrow[uu, "\pi"] & i_D(d_1)\times\cdots\times i_D(d_r)
\end{tikzcd}$$

where all the objects a_i, b_j, c_m, d_l are in D and all the terms in $\vec{s}$ and in $\vec{t}$ are over L_D.

This improves a result of Arapura (Lemma 2.2.5 [1]) that every object of the category $\mathrm{Comod}_{\mathrm{fin}}(\mathrm{End}^{\vee}(T))$ is a subquotient of a finite direct sum of objects of the form $i_D(d)$ (for $d \in D$).

Another key feature of our logical approach to motives is that, unlike the Tannakian approach which is intrinsically based on the linear nature of the objects under consideration and on the finite dimensionality assumptions, it naturally generalizes to non-abelian settings: indeed, the R-linear abelian structure on our category $\mathcal{C}_{\mathbb{T}_T}$ is a byproduct of the fact that the signature of the theory $\mathcal{C}_{\mathbb{T}_T}$ and its axioms formalize the notion of R-linear structure on each sort; with a different 'non-linear' choice of the signature and the axioms of the theory, abelianity and R-linearity are no longer ensured but the resulting category still satisfies the same kind of universal property. Our logical approach via syntactic categories thus unifies the linear with the non-linear, allowing one to pass from one to the other by simply varying the choice of the signature and of the axioms of the theory.

2.1.4 The duality between real and imaginary

There is a precise logical sense in which sites (or mathematical theories viewed as presentations) can be regarded as 'real' objects and the toposes naturally associated to them as 'imaginary' objects; indeed, the topos of sheaves $\mathbf{Sh}(\mathcal{C}, J)$ on a site $(\mathcal{C}, J)$ is a mathematical universe which naturally completes $\mathcal{C}$ in a maximal way: unlike $\mathcal{C}$, $\mathbf{Sh}(\mathcal{C}, J)$ always possesses arbitrary set-indexed limits and colimits, exponentials and even a subobject classifier: the finite limits which exist in $\mathcal{C}$ are preserved by the canonical functor $\mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$, and every J-covering family is sent by this functor to an epimorphic family. Every object of the topos $\mathbf{Sh}(\mathcal{C}, J)$ is a 'definable' coproduct of objects coming from $\mathcal{C}$, that is, an imaginary in the model-theoretic sense. As when passing from the real line to the complex plane by means of the addition of the imaginary i one gains a much better understanding of the symmetries of the solutions of polynomial equations even with real coefficients, so passing from a site to the associated topos allows one to benefit from the rich invariant theory for toposes and the possibility of effectively computing on them, features which are unavailable in the context of sites.

In practice, the application of the technique of 'toposes as bridges' often proceeds as follows:

(1) One starts with an equivalence, a duality or an elementary correspondence in the 'real' world of concrete mathematical theories (or of sites) and their set-based models.

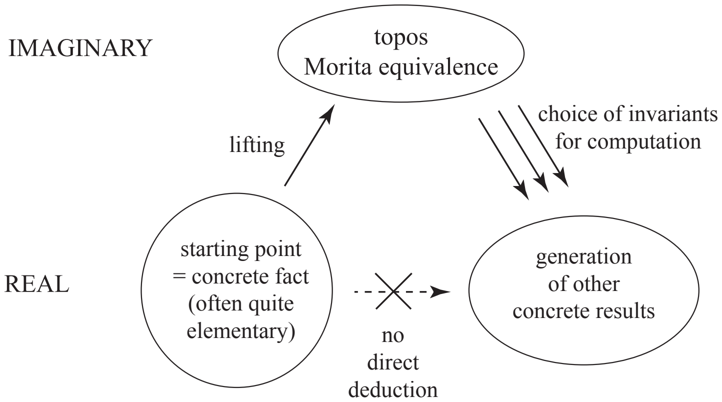
(2) One lifts this equivalence or correspondence to a Morita-equivalence in the 'imaginary' world of toposes

$$\mathcal{E} \simeq \mathcal{E}',$$

which usually means that the original equivalence or correspondence can be deduced from $\mathcal{E} \simeq \mathcal{E}'$ by means of the choice of a certain invariant, for instance that of the points of the toposes.

(3) One considers other topos-theoretic invariants and calculates or expresses them in terms of sites of theories of presentations of $\mathcal{E}$ and $\mathcal{E}'$, thus obtaining other equivalences, dualities or concrete correspondences. It turns out that these results are usually surprising and, in general, cannot be directly deduced from the original concrete equivalence or relation from which one had started. As an illustration of this last remark, take the syntactic criterion for a theory $\mathbb{T}$ to be of presheaf type (Theorem 3.3 below); this implies that the classifying topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$ is equivalent to the topos $[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$, which in turn implies, by an application of the 'bridge' technique, the definability theorem for $\mathbb{T}$ (Theorem 2.15 below).

We can schematically represent this way of applying the 'bridge' technique in the form of an ascent followed by a descent between two levels, the 'real' one of concrete mathematics and the 'imaginary' one of toposes:



Starting from a topos or a Morita-equivalence, the calculation or expression in terms of sites or theories of presentation of topos-theoretic invariants is often technically non-trivial but feasible. On the contrary, trying to go in the other direction from a very sophisticated concrete mathematical result to a Morita-equivalence which could generate it is in general very difficult, if not impossible.

In other, more metaphorical, words, this methodology generates a 'rain' of results falling in a territory surrounding a given problem whose essential aspects have been encoded by means of suitable topos-theoretic invariants. It is difficult to predict exactly where the single drops will fall, but, as the rain will eventually cover more and more of the wet space, so the application of this methodology is liable to bring a lot of concrete insights on aspects related to the original problem which could eventually lead to its solution.

2.1.5 A theory of 'structural translations'

The view underlying the methodology 'toposes as bridges' described above consists in regarding a topos as an object which, together with all its different representations, embodies a great amount of relationships existing between the different theories classified by it. Any topos-theoretic invariant behaves like a 'pair of glasses' which allows one to discern certain information which is 'hidden' in a given Morita-equivalence. Toposes can thus act as 'universal translators' across different mathematical theories which share the same classifying topos (or which have classifying toposes which are related to each other to the extent that suitable invariants can be transferred from one to the other).

From a technical point of view, the main reason for the effectiveness of the 'bridge' technique is two-fold: on one hand, as we have argued in section 2.1.2, topos-theoretic invariants usually manifest themselves in significantly different ways in the context of different sites; on the other, due to the very well-behaved nature of the representation theory of Grothendieck toposes in terms of sites, the site characterizations formally expressing such relationships are essentially canonical and can often be derived by means of rather mechanical 'calculations'.

Unlike the traditional, 'dictionary-oriented' method of translation based on a 'renaming', according to a given 'dictionary', of the primitive constituents of the information as expressed in a given language, the 'invariant-oriented' translations realized by topos-theoretic 'bridges' consist in 'structural unravelings' of appropriate invariants across different representations of the toposes involved, rather than through the use of an explicit description of the Morita-equivalence serving as 'dictionary'. In fact, for the transfer of 'global' properties of toposes, it is only the *existence* of a Morita-equivalence that really matters, rather than its explicit description, since, by its very definition, a topos-theoretic invariant is stable under *any* categorical equivalence. If one wants to establish more 'specific' results, one can use invariant properties of objects of toposes rather than properties of the whole topos, in which case an explicit description of the Morita-equivalence is of course needed, but for investigating most of the 'global' properties of theories this is not at all necessary.

We have already hinted above to the fact that there is an strong element of *automatism* implicit in the 'bridge' technique. In fact, in order to obtain insights on the Morita-equivalence under consideration, in many cases one can just readily apply to it general characterizations connecting properties of sites and topos-theoretic invariants. Still, the results generated in this way are in general non-trivial; in some cases they can be rather 'weird' according to the usual mathematical standards (although they might still be quite deep) but, with a careful choice of Morita-equivalences and invariants, one can easily get interesting and natural mathematical results. In fact, a lot of information that is not visible with the usual 'glasses' is revealed by the application of this machinery.

The range of applicability of the 'bridge' technique is very broad within mathematics, by the very generality of the notion of topos (and of that of geometric theory). Through this method, results are generated transversally to the various mathematical fields, in a 'uniform' way which is determined by the form of the toposes involved and by the invariants considered on them. Notice that this way of doing mathematics is inherently 'upside-down': instead of starting with simple ingredients and combining them to build more complicated structures, one assumes as primitive ingredients rich and sophisticated (meta-)mathematical entities, namely Morita-equivalences and topos-theoretic invariants, and proceeds to extracting from them 'concrete' information relevant for classical mathematics.

2.2 *Theories, Sites, Toposes*

The research monograph *Theories, Sites, Toposes: relating and studying mathematical theories through topos-theoretic 'bridges'* [8] (to appear for Oxford University Press) introduces a set of methods and techniques for studying mathematical theories and relating them to each other through the use of Grothendieck toposes. The theoretical development is complemented by a number of examples and applications in different mathematical areas which illustrate the wide-ranging impact and benefits of a topos-theoretic outlook on Mathematics.

The contents of the different chapters can be briefly summarized as follows.

In Chapter 1 we provide the topos-theoretic background necessary for understanding the contents of the book. The presentation is self-contained and only assumes a basic familiarity with the language of category theory. We start by reviewing the basic theory of Grothendieck toposes, including the fundamental equivalence between geometric morphisms and flat functors. We then present first-order logic and its interpretation in categories having 'enough' structure. Lastly, we discuss the key concept of syntactic category of a first-order theory, which is used in Chapter 2 for constructing classifying toposes of geometric theories.

Chapter 2 consists of two parts. In the first part we review the fundamental notion of classifying topos of a geometric theory and discuss the appropriate kinds of interpretations between theories which induce morphisms between the associate classifying toposes; the theoretical presentation is accompanied by a few concrete examples of classifying toposes of theories naturally arising in Mathematics. We also establish a characterization theorem for universal models of geometric theories inside classifying toposes. In the second part we explain the general unifying technique 'toposes as bridges'. This technique, which allows to extract 'concrete' information from the existence of different representations for the classifying topos of a geometric theory, is systematically exploited in the course of the book to establish theoretical results as well as applications. The 'decks' of topos-theoretic 'bridges' are normally given by Morita-equivalences, while the 'arches' are given by site characterizations of topos-theoretic invariants.

In Chapter 3 we establish a duality theorem providing, for each geometric theory, a natural bijection between its geometric theory extensions (also called 'quotients') and the subtoposes of its classifying topos:

Theorem 2.5. *Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the quotients of $\mathbb{T}$ (up to syntactic equivalence) and the subtoposes of the classifying topos* $\mathbf{Set}[\mathbb{T}]$ *of* $\mathbb{T}$.

We provide two different proofs of this theorem, one relying on the theory of classifying toposes and the other, of purely syntactic nature, based on a proof-theoretic interpretation of the notion of Grothendieck topology. More specifically, we establish a proof-theoretic equivalence between the classical system of geometric logic over a geometric theory $\mathbb{T}$ and a new proof system for sieves in the geometric syntactic category $C_{\mathbb{T}}$ of $\mathbb{T}$ whose inference rules are given by the pullback stability and transitivity rules for Grothendieck topologies. This equivalence is interesting since the latter proof system turns out to be computationally better-behaved than the former; indeed, checking that a sieve belongs to the Grothendieck topology generated by a given family of sieves is often technically easier than proving that a geometric sequent is provable in a given theory. This equivalence of deduction systems can therefore be used for shedding light on axiomatization prob-

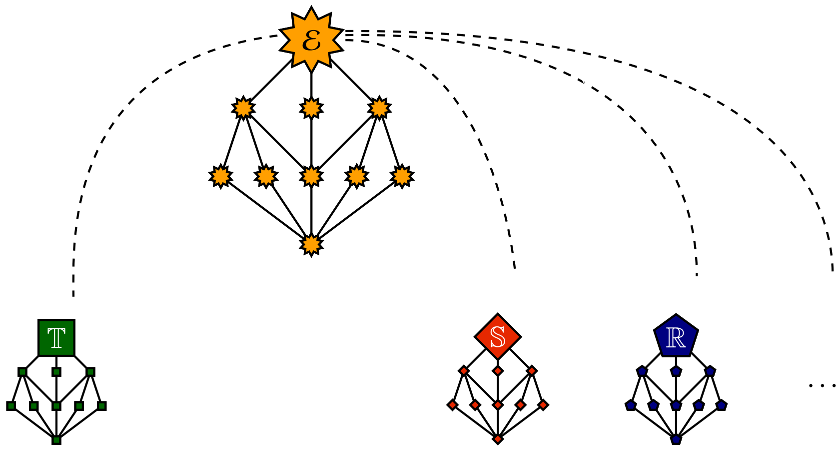


Figure 2: Lattices of theories

lems for geometric theories. We use it in particular for proving a deduction theorem for geometric logic, which we obtain by means of a calculation on Grothendieck topologies.

In Chapter 4, by using the duality theorem established in Chapter 3, we transfer many ideas and concepts of elementary topos theory to geometric logic. Specifically, we analyze notions such as the coHeyting algebra structure on the lattice of subtoposes of a given topos, open, closed, quasi-closed subtoposes, the dense-closed factorization of a geometric inclusion, coherent subtoposes, subtoposes with enough points, the surjection-inclusion factorization of a geometric morphism, skeletal inclusions, atoms in the lattice of subtoposes of a given topos, the Booleanization and DeMorganization of a topos. We also obtain explicit descriptions of the Heyting operation between Grothendieck topologies on a given category and of the Grothendieck topology generated by a given collection of sieves, and we establish a number of results about the problem of 'relativizing' a local operator with respect to a given subtopos.

It turns out that the logical notions arising by translating across this duality notable concepts and constructions in topos theory are of natural mathematical interest. For instance, the collection of (syntactic equivalence classes of) quotients of a given geometric theory has the structure of a lattice (in fact, a Heyting algebra) with respect to a natural notion of ordering of theories, while open (respectively, closed) subtoposes correspond via the duality to quotients obtained by adding sequents of the form $(\top \vdash_{[]} \phi)$ (respectively, $(\phi \vdash_{[]} \bot)$), where ϕ is a geometric sentence over the signature of the theory. Also, the surjection-inclusion factorization of a geometric morphism has a natural semantic interpretation (in terms of the geometric theory Th(M) of a given model M, i.e. of the theory consisting of all the geometric sequents which are satisfied in M):

Theorem 2.6. *Let $\mathbb{T}$ be a geometric theory over a signature Σ and $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism into the classifying topos $\mathcal{E}$ for $\mathbb{T}$, corresponding to a $\mathbb{T}$-model M in $\mathcal{F}$ via the universal property of the classifying topos of $\mathbb{T}$. Then the topos $\mathcal{E}'$ in the surjection-inclusion factorization $\mathcal{F} \twoheadrightarrow \mathcal{E}' \hookrightarrow \mathcal{E}$ of f classifies the quotient $Th(M)$ of $\mathbb{T}$ via Theorem 2.5.*

Subtoposes obtained by topos-theoretic constructions such as the Booleanization or DeMorganization of a topos have natural logical counterparts as well (cf. section 3.3.1), which often specialize, in the case of important mathematical theories, to 'quotients' of genuine mathematical interest; for example, the Booleanization of the theory of linear orders is the theory of dense linear orders without endpoints (cf. section 3.3.1), the DeMorganization of the (coherent) theory of fields is the geometric theory of fields of finite characteristic in which every element is algebraic over the prime field (cf. [12]), the Booleanization of the (coherent) theory of fields is the theory of algebraically closed fields of finite characteristic in which every element is algebraic over the prime field (cf. [12]).

Chapter 5 is devoted to flat functors in relation to classifying toposes. In the first section we establish some general results about colimits of internal diagrams in toposes, in the second we develop a general theory of extensions of flat functors along geometric morphisms of toposes, and in the third we discuss, following [9], a way of representing flat functors by using a suitable internalized version of the Yoneda lemma. These general results will be instrumental for establishing in Chapter 6 the main characterization theorem for theories of presheaf type.

Theories of presheaf type occupy a central role in Topos Theory for a number of reasons:

- Every small category $\mathcal{C}$ can be seen, up to Cauchy-completion, as the category of finitely presentable models of a theory of presheaf type (namely, the theory of flat functors on $\mathcal{C}^{\text{op}}$).

- As every Grothendieck topos is a subtopos of some presheaf topos, so every geometric theory is a quotient of some theory of presheaf type (cf. Theorem 2.5).

- Every finitary algebraic theory (and more generally, any cartesian theory) is of presheaf type.

- The class of theories of presheaf type contains, besides all cartesian theories, many other interesting mathematical theories pertaining to different fields of mathematics (for instance, the coherent theory of linear orders or the geometric theory of algebraic extensions of a given field, cf. Chapter 9).

- The 'bridge technique' of Chapter 2 can be fruitfully applied in the context of theories of presheaf type due to the fact that the classifying topos of any such theory admits (at least) two quite different representations, one of semantic nature (namely, set-valued functors on the category of finitely presentable models of the theory) and one of syntactic nature (namely, sheaves on the syntactic site of the theory).

The subject of theories of presheaf type has a long history, starting with the book [48] by Hakim, which first introduced the point of view of classifying toposes in the context of the theory of commutative rings with unit and its quotients. The subsequent pionereeing work [57] by Lawvere led to the discovery that any finitary algebraic theory is of presheaf type, classified by the topos of presheaves on the opposite of its category of finitely presentable models (cf. [55]). This result was later generalized to cartesian (or essentially algebraic) theories as well as to universal Horn theories (cf. [6]). At the same time, new examples of non-cartesian theories of presheaf type were discovered (cf. for instance [5] for a long, but by no means exhaustive, list of examples), and partial results in connection to the problem of characterizing the class of theories of presheaf type emerged; for instance, [52], [5] and [74] contain different sets of sufficient conditions for a theory to be of presheaf type.

In Chapter 6 we carry out a systematic investigation of the class of theories of presheaf type (i.e., classified by a presheaf topos). After establishing a number of general results about them, notably including a definability theorem (cf. Theorem 2.15 below), we prove a fully constructive characterization theorem providing necessary and sufficient conditions for a theory to be of presheaf type, expressed in terms of the models of the theory in arbitrary Grothendieck toposes. This theorem, whose general statement is quite abstract, admits several ramifications and simpler corollaries which can be effectively applied in practice to test whether a given theory is classified by a presheaf topos as well as for generating new examples of theories of presheaf type. It also subsumes all the above-mentioned results previously obtained on this topic.

In Chapter 7 we introduce the concept of expansion of a geometric theory and develop some basic theory about it. An *expansion* of a geometric theory $\mathbb{T}$ over a signature Σ is a geometric theory obtained from $\mathbb{T}$ by adding sorts, relation or function symbols to Σ and geometric axioms over the resulting extended signature. For any expansion $\mathbb{T}'$ of a geometric theory $\mathbb{T}$, there is a canonical induced morphism of classifying toposes $p_{\mathbb{T}}^{\mathbb{T}'} : \mathbf{Set}[\mathbb{T}'] \to \mathbf{Set}[\mathbb{T}]$. We prove in particular the following

Theorem 2.7. *(i) Let $p : \mathcal{E} \to \mathbf{Set}[\mathbb{T}]$ be a geometric morphism to the classifying topos of a geometric theory $\mathbb{T}$. Then p is, up to isomorphism, of the form $p_{\mathbb{T}}^{\mathbb{T}'}$ for some geometric expansion $\mathbb{T}'$ of $\mathbb{T}$.*

(ii) The hyperconnected-localic factorization of the geometric morphism $p_{\mathbb{T}}^{\mathbb{T}'} : \mathbf{Set}[\mathbb{T}'] \to \mathbf{Set}[\mathbb{T}]$ is given by $p_{\mathbb{T}}^{\mathbb{T}''} \circ p_{\mathbb{T}''}^{\mathbb{T}'}$, where $\mathbb{T}''$ is the intermediate expansion of $\mathbb{T}$ obtained by adding to the signature Σ of $\mathbb{T}$ no new sorts and a relation symbol for each ($\mathbb{T}'$-provable class of) geometric formula over the signature of $\mathbb{T}'$ in a context only involving the sorts of Σ, and all the sequents

over this extended signature which are provable in $\mathbb{T}'$ (where $\mathbb{T}'$ is identified with its expansion with these additional definable relation symbols).

This shows that an important geometric topos-theoretic construction such as the hyperconnected-localic factorization has a very natural logical counterpart.

We then investigate the preservation, by 'faithful interpretations' of theories, of each of the conditions in the characterization theorem for theories of presheaf type established in Chapter 6, obtaining results of the form 'under appropriate conditions, a geometric theory in which a theory of presheaf type faithfully interprets is again of presheaf type'. In this context, we also investigate the possibility of expanding a given geometric theory $\mathbb{T}$ to a theory of presheaf type classified by the topos $[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$; these techniques are applied in particular in [29] (cf. section 3.3.3 below). In passing, we establish the following criterion for a theory to be of presheaf type:

Theorem 2.8. *Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then $\mathbb{T}$ is of presheaf type if and only if the following conditions are satisfied:*

(i) Every finitely presentable model is presented by a geometric formula over Σ.

(ii) Every property of finite tuples of elements of a finitely presentable $\mathbb{T}$-model which is preserved by $\mathbb{T}$-model homomorphisms is definable (in finitely presentable $\mathbb{T}$-models) by a geometric formula over Σ.

(iii) The finitely presentable $\mathbb{T}$-models are jointly conservative for $\mathbb{T}$.

In Chapter 8 we study the quotients of a given theory of presheaf type by means of Grothendieck topologies that can be naturally attached to them, establishing a 'semantic' representation for the classifying topos of such a quotient as a subtopos of the classifying topos of the given theory of presheaf type. More specifically, we prove the following

Theorem 2.9. *Let $\mathbb{T}'$ be a quotient of a theory of presheaf type $\mathbb{T}$. Then the subtopos of the classifying topos of $\mathbb{T}$ corresponding to $\mathbb{T}'$ via the duality of Theorem 2.5 can be identified with the subtopos*

$$\mathbf{Sh}(f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op}, J) \simeq [f.p.\mathbb{T}\text{-}mod(\mathbf{Set}), \mathbf{Set}],$$

where the Grothendieck topology J is defined as follows. If $\mathbb{T}'$ is obtained from $\mathbb{T}$ by adding axioms σ of the form $(\phi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{y_i})\theta_i)$, where, for any $i \in I$, $[\theta_i] : \{\vec{y_i} . \psi_i\} \to \{\vec{x} . \phi\}$ is an arrow in $\mathcal{C}_{\mathbb{T}}$ and $\phi(\vec{x})$, $\psi(\vec{y_i})$ are formulae presenting respectively $\mathbb{T}$-models $M_{\{\vec{x}.\phi\}}$ and $M_{\{\vec{y_i}.\psi_i\}}$ (note that, by Theorems 2.5 and 3.3, every quotient of $\mathbb{T}$ has an axiomatization of this kind) then J is generated by the sieves S_σ on $M_{\{\vec{x}.\phi\}}$ in the category f.p.$\mathbb{T}$-mod$(\mathbf{Set})$ generated by the arrows s_i defined as follows: for each $i \in I$, $[[\theta_i]]_{M_{\{\vec{y_i}.\psi_i\}}}$ is the graph of a morphism $[[\vec{y_i} . \psi_i]]_{M_{\{\vec{y_i}.\psi_i\}}} \to [[\vec{x} . \phi]]_{M_{\{\vec{y_i}.\psi_i\}}}$; then the image of the generators of $M_{\{\vec{y_i}.\psi_i\}}$ via this morphism is an element of $[[\vec{x} . \phi]]_{M_{\{\vec{y_i}.\psi_i\}}}$ and this in turn determines, by definition of $M_{\{\vec{x}.\phi\}}$, a unique arrow $s_i : M_{\{\vec{x}.\phi\}} \to M_{\{\vec{y_i}.\psi_i\}}$ in $\mathbb{T}$-mod$(\mathbf{Set})$.

We also show that the models of $\mathbb{T}'$ can be characterized among the models of $\mathbb{T}$ as those which satisfy a key property of J-homogeneity; this is a notion which specializes, if J is the atomic topology, to the notion of (weakly) homogeneous model in classical Model Theory (cf. section 3.3.1 below).

Theorem 2.9 is also useful for proof-theoretic purposes since it allows one to reformulate conditions of provability of geometric sequents over the signature of $\mathbb{T}$ in a quotient of $\mathbb{T}$ in terms of the condition that the sieves corresponding to them belong to the Grothendieck topology associated with the given quotient. It turns out that calculations are generally much easier with Grothendieck topologies than with Hilbert-style axiomatizations; as an illustration of this remark, we compute in a section of Chapter 8 an explicit axiomatization for the meet of two quotients of the theory of commutative rings with unit (in the sense of Chapter 4): the theory of local rings and that of integral domains.

Always in this chapter, we identify a number of sufficient conditions for the quotient of a theory of presheaf type to be again of presheaf type, including a finality property of the category of models of the quotient with respect to the category of models of the theory, and a rigidity property of the Grothendieck topology associated with the quotient. Among these results, we mention in particular the following

Theorem 2.10. *Let $\mathbb{T}$ be a theory of presheaf type and $\mathcal{A}$ a full subcategory of f.p.$\mathbb{T}$-mod(**Set**). Then the theory $\mathbb{T}_{\mathcal{A}}$ of $\mathbb{T}$ consisting of all the geometric sequents over the signature of $\mathbb{T}$ which are valid in all models in $\mathcal{A}$ is of presheaf type classified by the topos $[\mathcal{A}, \mathbf{Set}]$; in particular, every finitely presentable $\mathbb{T}_{\mathcal{A}}$-model is a retract of a model in $\mathcal{A}$.*

This theorem arises from the following 'bridge':

$$
\begin{tikzcd}
 & & [\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \arrow[r, phantom, "\simeq"] & \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \arrow[drr, dashed, bend left] & & \\
 & & {[\mathcal{A}, \mathbf{Set}]} \arrow[u, hook] \arrow[r, phantom, "\simeq"] & \mathbf{Sh}(\mathcal{C}_{\mathbb{T}_{\mathcal{A}}}, J_{\mathbb{T}_{\mathcal{A}}}) \arrow[u, hook] & & \\
\mathcal{A} \arrow[d, hook] & & & & & \mathbb{T} \arrow[d] \\
\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}) \arrow[uuurr, dashed, bend left] & & & & & \mathbb{T}_{\mathcal{A}}
\end{tikzcd}
$$

This result turns out to be very useful for identifying new theories are of presheaf type; moreover, under some natural assumptions that are frequently verified in practice, it is shown in Chapter 8 that one can give an explicit axiomatization of the theory $\mathbb{T}_{\mathcal{A}}$. More specifically, the following result holds:

Theorem 2.11. *Let $\mathbb{T}$ be a theory of presheaf type and $\mathcal{K}$ a full subcategory of the category set-based $\mathbb{T}$-models such that every $\mathbb{T}$-model in $\mathcal{K}$ is both finitely presentable and finitely generated (with respect to the same generators). Then the following sequents, where we denote by $\mathcal{P}$ the set of (representatives of) geometric formulae over Σ which present a $\mathbb{T}$-model in $\mathcal{K}$, added to the axioms of $\mathbb{T}$, yield an axiomatization of the theory $\mathbb{T}_{\mathcal{K}}$ of Theorem 2.10:*

(i) *the sequent*

$$(\top \vdash_{[]} \bigvee_{\phi(\vec{x})\in\mathcal{P}} (\exists\vec{x})\phi(\vec{x}));$$

(ii) *for any formulae* $\phi(\vec{x})$ *and* $\psi(\vec{y})$ *in* $\mathcal{P}$*, where* $\vec{x} = (x_1^{A_1}, \ldots, x_n^{A_n})$ *and* $\vec{y} = (y_1^{B_1}, \ldots, y_m^{B_m})$*, the sequent*

$$(\phi(\vec{x}) \wedge \psi(\vec{y}) \vdash_{\vec{x},\vec{y}} \bigvee_{\substack{\chi(\vec{z})\in\mathcal{P}, t_1^{A_1}(\vec{z}),\ldots,t_n^{A_n}(\vec{z})\\ s_1^{B_1}(\vec{z}),\ldots,s_m^{B_m}(\vec{z})}} (\exists\vec{z})(\chi(\vec{z}) \wedge \bigwedge_{\substack{i\in\{1,\ldots,n\},\\ j\in\{1,\ldots,m\}}} (x_i = t_i(\vec{z}) \wedge y_j = s_j(\vec{z})))),$$

where the disjunction is taken over all the formulae $\chi(\vec{z})$ *in* $\mathcal{P}$ *and all the tuples of terms* $t_1^{A_1}(\vec{z}), \ldots, t_n^{A_n}(\vec{z})$ *and* $s_1^{B_1}(\vec{z}), \ldots, s_m^{B_m}(\vec{z})$ *such that*

$$(t_1^{A_1}(\vec{\xi_\chi}), \ldots, t_n^{A_n}(\vec{\xi_\chi})) \in [[\vec{x}\ .\ \phi]]_{M_{\{\vec{z},\chi\}}}$$

and

$$(s_1^{B_1}(\vec{\xi_\chi}), \ldots, s_m^{B_m}(\vec{\xi_\chi})) \in [[\vec{y}\ .\ \psi]]_{M_{\{\vec{z},\chi\}}};$$

(iii) *for any formulae* $\phi(\vec{x})$ *and* $\psi(\vec{y})$ *in* $\mathcal{P}$*, where* $\vec{x} = (x_1^{A_1}, \ldots, x_n^{A_n})$ *and* $\vec{y} = (y_1^{B_1}, \ldots, y_m^{B_m})$*, and any terms* $t_1^{A_1}(\vec{y}), s_1^{A_1}(\vec{y}), \ldots, t_n^{A_n}(\vec{y}), s_n^{A_n}(\vec{y})$*, the sequent*

$$(\bigwedge_{i\in\{1,\ldots,n\}} (t_i(\vec{y}) = s_i(\vec{y})) \wedge \phi(t_1/x_1, \ldots, t_n/x_n) \wedge \phi(s_1/x_1, \ldots, s_n/x_n) \wedge \psi(\vec{y})$$

$$\vdash_{\vec{y}} \bigvee_{\chi(\vec{z})\in\mathcal{P}, u_1^{B_1}(\vec{z}),\ldots,u_m^{B_m}(\vec{z})} ((\exists\vec{z})(\chi(\vec{z}) \wedge \bigwedge_{j\in\{1,\ldots,m\}} (y_j = u_j(\vec{z}))),$$

where the disjunction is taken over all the formulae $\chi(\vec{z})$ *in* $\mathcal{P}$ *and all the sequences of terms* $u_1^{B_1}(\vec{z}), \ldots, u_m^{B_m}(\vec{z})$ *such that* $(u_1^{B_1}(\vec{\xi_\chi}), \ldots, u_m^{B_m}(\vec{\xi_\chi})) \in [[\vec{y}\ .\ \psi]]_{M_{\{\vec{z},\chi\}}}$ *and* $t_i(u_1(\vec{\xi_\chi}), \ldots, u_m(\vec{\xi_\chi})) = s_i(u_1(\vec{\xi_\chi}), \ldots, u_m(\vec{\xi_\chi}))$ *in* $M_{\{\vec{z},\chi\}}$ *for all* $i \in \{1, \ldots, n\}$*;*

(iv) *for any sort A over* Σ*, the sequent*

$$(\top \vdash_{x_A} \bigvee_{\chi(\vec{z})\in\mathcal{P}, t^A(\vec{z})} (\exists\vec{z})(\chi(\vec{z}) \wedge x = t(\vec{z}))),$$

where the the disjunction is taken over all the formulae $\chi(\vec{z})$ *in* $\mathcal{P}$ *and all the terms* $t^A(\vec{z})$*;*

(v) *for any sort A over* Σ*, any formulae* $\phi(\vec{x})$ *and* $\psi(\vec{y})$ *in* $\mathcal{P}$*, where* $\vec{x} = (x_1^{A_1}, \ldots, x_n^{A_n})$ *and* $\vec{y} = (y_1^{B_1}, \ldots, y_m^{B_m})$*, and any terms* $t^A(\vec{x})$ *and* $s^A(\vec{y})$*, the sequent*

$$(\phi(\vec{x}) \wedge \psi(\vec{y}) \wedge t(\vec{x}) = s(\vec{y}) \vdash_{\vec{x},\vec{y}} \bigvee_{\substack{\chi(\vec{z})\in\mathcal{P}, p_1^{A_1}(\vec{z}),\ldots,p_n^{A_n}(\vec{z})\\ q_1^{B_1}(\vec{z}),\ldots,q_m^{B_m}(\vec{z})}} (\exists \vec{z})(\chi(\vec{z}) \wedge \bigwedge_{\substack{i\in\{1,\ldots,n\},\\ j\in\{1,\ldots,m\}}} (x_i = p_i(\vec{z}) \wedge y_j = q_j(\vec{z}))),$$

where the disjunction is taken over all the formulae $\chi(\vec{z})$ in $\mathcal{P}$ and all the tuples of terms $p_1^{A_1}(\vec{z}), \ldots, p_n^{A_n}(\vec{z})$ and $q_1^{B_1}(\vec{z}), \ldots, q_m^{B_m}(\vec{z})$ such that

$$(p_1^{A_1}(\vec{\xi_\chi}), \ldots, p_n^{A_n}(\vec{\xi_\chi})) \in [[\vec{x} \,.\, \phi]]_{M_{\{\vec{z}.\chi\}}},$$

$$(q_1^{B_1}(\vec{\xi_\chi}), \ldots, q_m^{B_m}(\vec{\xi_\chi})) \in [[\vec{y} \,.\, \psi]]_{M_{\{\vec{z}.\chi\}}}$$

and

$$t(p_1(\vec{\xi_\chi}), \ldots, p_n(\vec{\xi_\chi})) = s(q_1(\vec{\xi_\chi}), \ldots, q_m(\vec{\xi_\chi}))$$

in $M_{\{\vec{z}.\chi\}}$.

As we shall see in section 4.3.3, these results have been applied in [24] to describe geometric theories classified by Connes' cyclic topos (cf. [34]), Connes-Consani's epicyclic topos (cf. [35]) and Connes-Consani's arithmetic topos (cf. [36]). They have also been applied in [27] to obtain a simple axiomatization for the geometric theory of finite MV-chains (cf. section 4.3.2 below).

Another result from Chapter 8, which we apply in [27] (cf. section 4.3.2 below), reads as follows. Recall that a Grothendieck topology on a small category $\mathcal{C}$ is said to be *rigid* if every object of $\mathcal{C}$ has a J-covering sieve generated by J-irreducible objects (i.e., by objects not admitting any non-trivial J-covering sieves).

Theorem 2.12. *Let $\mathbb{T}'$ be a quotient of a theory of presheaf type $\mathbb{T}$ corresponding to a Grothendieck topology J on the category f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{op}$ under the duality of Theorem 2.5. Suppose that $\mathbb{T}'$ is itself of presheaf type. Then every finitely presentable $\mathbb{T}'$-model is finitely presentable also as a $\mathbb{T}$-model if and only if the topology J is rigid.*

This theorem arises from the following 'bridge':

essential geometric inclusion

$$[\text{f.p.}\mathbb{T}'\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J) \hookrightarrow [\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$$

f.p.$\mathbb{T}'$-mod$(\mathbf{Set})$ — J

included in f.p.$\mathbb{T}$-mod$(\mathbf{Set})$ — *rigid*

Chapter 8 also contains a number of results on classifying toposes of quotients with enough set-based models, and establish, for any such quotient, a characterization of the Grothendieck topology corresponding to it in terms of its category of set-based models. We also discuss coherent quotients and the Grothendieck topologies which correspond to them when considered as quotients of a cartesian theory, namely the finite-type ones, and show that the lattice operations on Grothendieck topologies naturally restrict to the collection of finite-type ones.

In Chapter 9 we discuss some classical, as well as new, examples of theories of presheaf type from the perspective of the theory developed in the previous chapters. We revisit in particular well-known examples of theories of presheaf type, such as the theory of intervals and the geometric theory of finite sets, and introduce new ones, including the theory of algebraic extensions of a given field, the theory of locally finite groups, the theory of vector spaces with linear independence predicates and the theory of lattice-ordered abelian groups with strong unit.

In Chapter 10 we describe some applications of the theory developed in the previous chapters in a variety of different mathematical contexts. The main methodology that we use to generate such applications is the 'bridge technique'. We discuss in particular restrictions of Morita-equivalences to quotients of the two theories involved, give a solution to a problem of Lawvere concerning the boundary operator on subtoposes, establish syntax-semantics 'bridges' for quotients of theories of presheaf type, present topos-theoretic interpretations and generalizations of Fraïssé's theorem in Model Theory on countably categorical theories and of topological Galois theory (cf. sections 3.3.1 and 3.3.2), develop a notion of maximal spectrum of a commutative ring with unit (cf. section 4.1.4) and investigate compactness conditions for geometric theories allowing one to identify theories lying in smaller fragments of geometric logic.

2.3 Universal models and classifying toposes

It is natural to wonder what are the key features which characterize universal models of geometric theories inside their classifying toposes among all the models of the theory.

As classifying toposes, universal models appear in different guises, so it is important to dispose of effective criteria for identifying them and hence to use them for establishing results about the given geometric theory. For example, if $\mathbb{T}$ is a theory of presheaf type then one can explicitly describe a universal model lying in its classifying topos $[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$, as follows.

Theorem 2.13 (Theorem 3.1 [14]). *Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ. Then the Σ-structure $N_{\mathbb{T}}$ in $[f.p.\mathbb{T}\text{-}mod(\mathbf{Set}), \mathbf{Set}]$ which assigns to a sort A the functor $N_{\mathbb{T}}A$ given by $(N_{\mathbb{T}}A)(M) = MA$, to a function symbol $f : A_1 \cdots A_n \to B$ the morphism $N_{\mathbb{T}}A_1 \times \cdots \times N_{\mathbb{T}}A_n \to N_{\mathbb{T}}B$ given by $(N_{\mathbb{T}}f)(M) = Mf$ and to a relation symbol $R \rightarrowtail A_1 \cdots A_n$ the subobject $N_{\mathbb{T}}R \rightarrowtail N_{\mathbb{T}}A_1 \times \cdots \times N_{\mathbb{T}}A_n$ given by $(N_{\mathbb{T}}R)(M) = MR$ (for any $M \in f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$) is a universal model for*

$\mathbb{T}$*; moreover, for any geometric formula* $\phi(\vec{x})$ *over* Σ*, the interpretation* $[[\vec{x}\,.\,\phi]]_{N_{\mathbb{T}}}$ *of* $\phi(\vec{x})$ *in* $N_{\mathbb{T}}$ *is given by* $[[\vec{x}\,.\,\phi]]_{N_{\mathbb{T}}}(M) = [[\vec{x}\,.\,\phi]]_M$ *for any* $M \in$ *f.p.*$\mathbb{T}$*-mod*(**Set**)*. In particular, the finitely presentable* $\mathbb{T}$*-models are jointly conservative for* $\mathbb{T}$*.*

As shown by the following theorem, universal models of geometric theories enjoy a strong form of logical completeness.

Theorem 2.14 (Theorem 2.2 [14])**.** *Let* $\mathbb{T}$ *be a geometric theory over a signature* Σ *and* U *a universal model of* $\mathbb{T}$ *in a Grothendieck topos* $\mathcal{E}$*. Then*

(i) For any subobject $S \rightarrowtail UA_1 \times \cdots \times UA_n$ *in* $\mathcal{E}$*, there exists a geometric formula* $\phi(\vec{x}) = \phi(x_1^{A_1}, \ldots, x_n^{A_n})$ *over* Σ *such that* $S = [[\vec{x}\,.\,\phi]]_U$*.*

(ii) For any arrow $f : [[\vec{x}\,.\,\phi]]_U \to [[\vec{y}\,.\,\psi]]_U$ *in* $\mathcal{E}$*, where* $\phi(\vec{x})$ *and* $\psi(\vec{y})$ *are geometric formulae over* Σ*, there exists a geometric formula* $\theta(\vec{x}, \vec{y})$ *over* Σ *such that the sequents* $(\phi \vdash_{\vec{x}} (\exists y)\theta)$*,* $(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi)$ *and* $(\theta \wedge \theta[\vec{y'}/\vec{y}] \vdash_{\vec{x},\vec{y},\vec{y'}} \vec{y} = \vec{y'})$ *are provable in* $\mathbb{T}$ *and* $[[\vec{x}, \vec{y}\,.\,\theta]]_U$ *is the graph of* f*.*

Combined with Theorem 2.13, this theorem gives the following definability theorem for theories of presheaf type.

Theorem 2.15 (Corollary 3.2 [14])**.** *Let* $\mathbb{T}$ *be a theory of presheaf type and suppose that we are given, for every finitely presentable set-based model* M *of* $\mathbb{T}$*, a subset* R_M *of* M^n *in such a way that every* $\mathbb{T}$*-model homomorphism* $h : M \to N$ *maps* R_M *into* R_N*. Then there exists a geometric formula-in-context* $\phi(x_1, \ldots, x_n)$ *such that* $R_M = [[\vec{x}\,.\,\phi]]_M$ *for each finitely presentable* $\mathbb{T}$*-model* M*.*

Indeed, the theorem arises from the following 'bridge':

$$\begin{array}{ccccc} & & \substack{\textit{Subobject of } UA_1\times\cdots\times UA_n \\ [\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})} & & \\ & \diagup & & \diagdown & \\ \substack{\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \\ \textit{Functorial (covariant) assignment} \\ M \to R_M \subseteq MA_1\times\cdots\times MA_n} & & & & \substack{(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \\ \textit{Geometric formula} \\ \phi(x_1^{A_1},\ldots,x_n^{A_n})} \end{array}$$

(where U is 'the' universal model of $\mathbb{T}$ in its classifying topos).

Notice that this theorem, which applies in particular to every finitary algebraic theory, is by no means straightforward, not even for one particular theory of presheaf type. Whilst our method of proof based on the double representation, semantic and syntactic, of the classifying topos of a theory of presheaf type is very simple and natural, it seems impossible to give a proof without using toposes.

In [14] we also show that associated sheaf functors preserve universal models, from which it follows that a universal model $L_{\mathbb{T}'}$ of a quotient $\mathbb{T}'$ of a theory of presheaf type $\mathbb{T}$ with associated Grothendieck topology J on the category f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ is the image under the associated sheaf functor

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \to \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J)$$

of the universal model $N_{\mathbb{T}}$ of Theorem 2.13. Such an explicit description is particularly useful in the case where J is subcanonical and for each sort A over the signature of $\mathbb{T}$, the formula $\{x^A \,.\, \top\}$ presents a $\mathbb{T}$-model P_A. Indeed, under these hypotheses the representable $\mathrm{Hom}_{\mathrm{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})}(P_A, -)$ is a J-sheaf and hence it coincides with $L_{\mathbb{T}'}A$. Since the inclusion

$$\mathbf{Sh}(\mathrm{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\mathrm{op}}, J) \hookrightarrow [\mathrm{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$$

preserves finite limits, it follows that every cartesian sequent over the signature of $\mathbb{T}$ which is provable in $\mathbb{T}'$ is provable in $\mathbb{T}$. So we have the following

Proposition 2.16. *Let $\mathbb{T}$ be a theory of presheaf type such that for any sort A over its signature the free model on the sort A exists (for instance, a cartesian theory). If the Grothendieck topology J on f.p.$\mathbb{T}$-mod($\mathbf{Set}$)^{op} associated with a quotient $\mathbb{T}'$ of $\mathbb{T}$ is subcanonical then the cartesianizations of $\mathbb{T}$ and $\mathbb{T}'$ coincide.*

This result, which is purely constructive, was applied in [27] to prove that the theory of a Komori variety V is the cartesianization (i.e., the set of cartesian sequents which are provable in the theory) of the theory of local MV-algebras in V (cf. Proposition 5.6 [27]). Such a property normally arises when there is a sheaf representation of models of $\mathbb{T}$ as global sections of sheaves of models of $\mathbb{T}'$ (in the sense that all the stalks are models of $\mathbb{T}'$). What is remarkable about our result is, besides its constructive nature, the fact that it does *not* require the existence of any 'concrete' representation of models of $\mathbb{T}$ in terms of models of $\mathbb{T}'$; it lies at an higher level of abstraction. This is another illustration of the fact that the deep relations between different theories naturally live inside the 'imaginary' entities attached to them such as their classifying toposes and cannot often be contemplated concretely.

The paper [14] also contains many other results and a discussion of how to use universal models to investigate issues of definability by geometric formulae and of satisfiability of the law of excluded middle and De Morgan's law on Grothendieck toposes. These latter laws involve the operation of Heyting negation $\neg$ in a topos (recall that the Heyting negation $\neg m$ of a subobject $m : a \to b$ is the biggest subobject of b which is disjoint from m); indeed, the first amounts to the requirement that for any subobject m, $m \vee \neg m = 1$ while the second to the weaker condition $\neg m \vee \neg\neg m = 1$.

As an illustration of the significance of this operation in toposes, consider the two properties of an element of a commutative ring with unit ϕ_1 to be invertible and ϕ_2 to be nilpotent. We prove in [14] (Proposition 7.1) that these properties are *orthogonal* to each other in the classifying topos of the theory $\mathbb{T}_n$ of non-trivial commutative rings with unit, in the sense that $\neg\phi_1 = \phi_2$ and $\neg\phi_2 = \phi_1$ in the universal model of $\mathbb{T}_n$ in its classifying topos. This fact, besides being conceptually pleasing as it formalizes the vague intuition of orthogonality that one might have regarding these properties in relation to one another, has various technical implications. To give an example, we recall the following

Proposition 2.17 (Proposition 6.6. [14]). *Let $\mathbb{T}$ be a theory of presheaf type with universal model $N_{\mathbb{T}}$ in* $[f.p.\mathbb{T}\text{-}mod(\mathbf{Set}), \mathbf{Set}]$ *as in Theorem 2.13 and J a Grothendieck topology on the category* $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op}$ *such that every J-covering sieve is non-empty. Then for any geometric formula $\phi(x_1^{A_1}, \ldots, x_n^{A_n})$ with the property that there exists a subobject E of $N_{\mathbb{T}}A_1 \times \cdots \times N_{\mathbb{T}}A_n$ such that $[[\vec{x} \,.\, \phi]]_{N_{\mathbb{T}}} = \neg E$, we have that for any $M \in f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ and $\vec{a} \in MA_1 \times \cdots \times MA_n$, if $\{f : M \to N \text{ in } f.p.\mathbb{T}\text{-}mod(\mathbf{Set}) \mid f(\vec{a}) \in [[\vec{x} \,.\, \phi]]_N\} \in J(M)$ then $\vec{a} \in [[\vec{x} \,.\, \phi]]_M$.*

This proposition can be applied for instance to the big Zariski site, taking $\mathbb{T}$ to be the theory of non-trivial commutative rings with unit and J to be the Grothendieck topology on the opposite of the category of non-trivial finitely generated rings induced by the Zariski topology, for establishing the locality of properties ϕ satisfying the hypotheses of the proposition, i.e. the property that if ϕ is satisfied by the images of an element a under the localization maps $A \to A[{s_i}^{-1}]$ for a set of elements $s_1, \ldots, s_n$ such that $(s_1, \ldots, s_n) = 1$ then it is satisfied by a.

For example, by observing that the properties ϕ_1 and ϕ_2 are orthogonal to each other and hence that each of them satisfies the hypothesis of the proposition, one recovers at once the following well-known algebraic results: for any set $\{s_1, \ldots, s_n\}$ of non-nilpotent elements of A which is not contained in any proper ideal of A, for any $a \in A$, a is invertible (resp. nilpotent) in A if and only if for each $i = 1, \ldots, n$, the image of a in $A[{s_i}^{-1}]$ is invertible (resp. nilpotent). These algebraic facts can of course also be proved 'concretely' by using algebraic manipulation, but our proposition is conceptually transparent and, because of its generality, it can be applied to treat a great variety of other problems which might otherwise seem to be unrelated to each other.

Another formal consequence of the orthogonality between ϕ_1 and ϕ_2 is the fact that the property of an element of a (finitely generated) commutative ring with unit to be neither invertible nor nilpotent is not definable by a geometric formula over the signature of this theory.

Let us now turn to the problem of intrinsically characterizing the pairs $(\mathcal{E}, U)$ such that $\mathcal{E}$ is the classifying topos of a theory $\mathbb{T}$ and U is a universal model of $\mathbb{T}$ in $\mathcal{E}$.

Given a geometric theory $\mathbb{T}$, one can give conditions on a pair $(\mathcal{E}, M)$ consisting of a Grothendieck topos $\mathcal{E}$ and a model M of $\mathbb{T}$ in $\mathcal{E}$ for $\mathcal{E}$ to be a classifying topos for $\mathcal{E}$ and M to be a universal $\mathbb{T}$-model in it.

Theorem 2.18. *[8] Let $\mathbb{T}$ be a geometric theory, $\mathcal{E}$ a Grothendieck topos and M a model of $\mathbb{T}$ in $\mathcal{E}$. Then $\mathcal{E}$ is a classifying topos for $\mathbb{T}$ and M is a universal model for $\mathbb{T}$ if and only if the following conditions are satisfied:*

(i) The family $\mathcal{F}$ of objects which can be built from the interpretations in M of the sorts, function symbols and relation symbols over the signature of $\mathbb{T}$ by using geometric logic constructions (i.e. the objects given by the domains of the interpretations in M of geometric formulae over the signature of $\mathbb{T}$) is separating for $\mathcal{E}$.

(ii) *The model M is conservative for $\mathbb{T}$; that is, for any geometric sequent σ over the signature of $\mathbb{T}$, σ is valid in M if and only if it is provable in $\mathbb{T}$.*

(iii) *Any arrow k in $\mathcal{E}$ between objects A and B in the family $\mathcal{F}$ of point (i) is definable; that is, if A (resp. B) is equal to the interpretation of a geometric formula $\phi(\vec{x})$ (resp. $\psi(\vec{y})$) over the signature of $\mathbb{T}$, there exists a $\mathbb{T}$-provably functional formula θ from $\phi(\vec{x})$ to $\psi(\vec{x})$ such that the interpretation of θ in M is equal to the graph of k.*

As it can be expected, Theorem 2.18 can be applied in a variety of different situations. For instance, in [8], it was used in combination with Theorem 2.13 to derive the following criterion for a theory to be of presheaf type:

Theorem 2.19. *Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then $\mathbb{T}$ is of presheaf type if and only if the following conditions are satisfied:*

(i) *Every finitely presentable model is presented by a geometric formula over Σ.*

(ii) *Every property of finite tuples of elements of a finitely presentable $\mathbb{T}$-model which is preserved by $\mathbb{T}$-model homomorphisms is definable (in finitely presentable $\mathbb{T}$-models) by a geometric formula over Σ.*

(iii) *The finitely presentable $\mathbb{T}$-models are jointly conservative for $\mathbb{T}$.*

The following characterization theorem for geometric logic provides necessary and sufficient conditions for a class of structures inside Grothendieck toposes to be the class of models of a geometric theory.

Theorem 2.20 (cf. Theorem 2 [16])**.** *Let Σ be a first-order signature and $\mathcal{S}$ a collection of Σ-structures in Grothendieck toposes closed under isomorphisms of structures. Then $\mathcal{S}$ is the collection of all models in Grothendieck toposes of a geometric theory over Σ if and only if it satisfies the following two conditions:*

(i) *for any geometric morphism $f : \mathcal{F} \to \mathcal{E}$, if $M \in \Sigma\text{-}str(\mathcal{E})$ is in $\mathcal{S}$ then $f^*(M)$ is in $\mathcal{S}$;*

(ii) *for any (set-indexed) jointly surjective family $\{f_i : \mathcal{E}_i \to \mathcal{E} \mid i \in I\}$ of geometric morphisms and any Σ-structure M in $\mathcal{E}$, if $f_i^*(M)$ is in $\mathcal{S}$ for every $i \in I$ then M is in $\mathcal{S}$.*

If $\mathcal{S}$ is the class of models of an infinitary first-order theory $\mathbb{T}$ then condition (ii) of the theorem can be replaced by the following simpler condition: for any surjective geometric morphism $f : \mathcal{F} \to \mathcal{E}$ and any Σ-structure M in $\mathcal{E}$, if $f^*(M)$ is in $\mathcal{S}$ then M is in $\mathcal{S}$. This proves a conjecture of I. Moerdijk.

3 Theory and applications of toposes as 'bridges'

3.1 Characterization of topos-theoretic invariants

One of the main parts of the 'bridge' technique described in section 2.1 consists in the characterization of topos-theoretic invariants in terms of sites of definition for toposes. In [15], [19] and [23] we have analyzed the behaviour of different kinds of notable invariants in terms of sites, obtaining natural site characterizations for them. It turns out that both topologically inspired invariants (such as the property of a topos to be atomic, locally connected, connected and locally connected, compact, equivalent to a presheaf topos) and logically inspired ones (such as the property of a topos to be Boolean, to satisfy De Morgan's law or other intermediate logics) admit natural site characterizations of the kind 'A topos $\mathbf{Sh}(\mathcal{C}, J)$ satisfies the given invariant I if and only if the site $(\mathcal{C}, J)$ satisfies a given property $P_{(\mathcal{C},J)}$ (expressible without any reference to sheaves)', holding for all sites $(\mathcal{C}, J)$ or for large classes of them. Such characterizations are technically elaborated but, as shown in these papers, the calculations leading to them are feasible in practice and, in some cases, even 'automatic', in the sense that one can design general methodologies working for large classes of invariants which can be implemented for carrying them out. More specifically, we showed in [23] that the property of satisfaction of any first-order sentence in the theory of Heyting algebra yields, when applied to the internal Heyting algebra to the topos given by its subobject classifier Ω, an invariant admitting site characterizations of the above kind holding for any site. Such characterizations can be obtained by mechanically applying the well-known explicit formulae describing the internal Heyting structure on the subobject classifier Ω of a topos $\mathbf{Sh}(\mathcal{C}, J)$ in terms of the site $(\mathcal{C}, J)$. Examples of these characterizations are the following (recall that a J-closed sieve is a sieve S such that for any arrow f, if $f^*(S)$ is J-covering then $f \in S$):

- **To be Boolean** $((\forall x)(x \vee \neg x = 1))$. Let

  $$R_a = \{f : b \to a \mid \emptyset \in J(b)\} \, .$$

  for each object a of $\mathcal{C}$. Then the topos $\mathbf{Sh}(\mathcal{C}, J)$ is Boolean if and only if for each J-closed sieve S on an object a of $\mathcal{C}$, the sieve

  $$\{f : b \to a \mid f \in S \quad \text{or} \quad f^*(S) = R_b\}$$

  is J-covering (cf. [11]).

- **To be De Morgan** $((\forall x)(\neg x \vee \neg\neg x = 1))$. Using the above notation, the topos $\mathbf{Sh}(\mathcal{C}, J)$ is De Morgan if and only if for each J-closed sieve S on an object a of $\mathcal{C}$, the sieve

  $$\{f : b \to a \mid f^*(S) = R_b \quad \text{or} \quad \forall\, g : c \to b,\ g^*(f^*(S)) = R_c \Rightarrow g \in R_b\}$$

  is J-covering (cf. [11]).

- **To satisfy Gödel-Dummett's logic** $((\forall x)(\forall y)((x \Rightarrow y \vee y \Rightarrow x) = 1))$. A topos $\mathbf{Sh}(C, J)$ satisfies Gödel-Dummett's logic if and only if for every J-closed sieves S and R on an object c of C, the sieve

  $$\{f : b \to a \mid f^*(S) \subseteq f^*(R) \quad \text{or} \quad f^*(R) \subseteq f^*(S)\}$$

  is J-covering (cf. [23]).

Another important logically inspired topos-theoretic invariant admitting a natural site characterization is the property of a topos to be **two-valued**, which amounts to the validity of the formula $(\forall x)(x = 0 \vee x = 1)$ in the 'external' Heyting algebra of the subterminal objects in the topos. A topos $\mathbf{Sh}(C, J)$ is two-valued if and only if the only J-closed ideals on C are $\mathrm{ob}(C)$ and the set of objects of C which are J-covered by the empty sieve (recall that a J-closed ideal on C is a subset I of the set $\mathrm{ob}(C)$ of objects of C such that for any arrow $f : a \to b$, if $b \in I$ then $a \in I$, and for any J-covering sieve S on an object c, if $\mathrm{dom}(f) \in I$ for all $f \in S$ then $c \in I$).

Very interestingly, for different sites of definition (C, J) and $(\mathcal{D}, K)$ of a given topos, the 'concrete' properties $P_{(C,J)}$ and $P_{(\mathcal{D},K)}$ can be completely different, despite being manifestations of a *unique* abstract property, namely I, in the context of different sites. This gives rise to a veritable mathematical morphogenesis, which may be exploited for investigating concrete problems since, given a property of the form $P_{(C,J)}$, one can investigate it by means of the logically equivalent, but completely different-looking, property $P_{(\mathcal{D},K)}$. For instance, in the case of the topos of sheaves $\mathbf{Sh}(X)$ on a topological space X, the above-mentioned criteria specialize to the following ones:

$\mathbf{Sh}(X)$ is two-valued (resp. Boolean, De Morgan, satisfies Gödel-Dummett's logic) if and only if X has exactly two open sets (resp., every open set of X is closed, the closure of every open set of X is open - i.e. X is extremally disconnected -, the closure of every open set of X is extremally disconnected).

On the other hand, in the case of the topos of presheaves $[C^{\mathrm{op}}, \mathbf{Set}]$ on a category C they yield the following results:

$[C^{\mathrm{op}}, \mathbf{Set}]$ is two-valued (resp. is Boolean, De Morgan, satisfies Gödel-Dummett's logic) if and only if C is strongly connected - i.e. for any two objects a and b there exist arrows $a \to b$ and $b \to a$ - (resp. a groupoid, satisfies the amalgamation property, satisfies the property that for any two arrows with common codomain, the former factors through the latter or vice versa).

The above criteria can for instance be used, in connection with a Morita-equivalence of the form $\mathbf{Sh}(X) \simeq [C^{\mathrm{op}}, \mathbf{Set}]$, for translating properties of X into properties of C and conversely. A simple example of a Morita-equivance of this form is provided by the Alexandrov space $\mathcal{A}_{\mathcal{P}}$ associated with a preorder $\mathcal{P}$ (i.e., the topological space whose open sets are precisely the upper sets of $\mathcal{P}$). In this case, we have an equivalence

$$[\mathcal{P}, \mathbf{Set}] \simeq \mathbf{Sh}(\mathcal{A}_{\mathcal{P}})$$

(cf. section 4.1 below) and an application of the 'bridge' technique yields at once the following results: for any preorder $\mathcal{P}$, $\mathcal{P}$ is discrete (resp. satisfies the amalgamation property, the property that for any elements a, b, c such that $a \leq c$ and $b \leq c$, either $a \leq b$ or $b \leq a$) if and only if the space $\mathcal{A}_{\mathcal{P}}$ has the trivial topology (resp. is extremally disconnected, every open set of it is extremally disconnected).

As another illustration of the mathematical morphogenesis realized by the calculation of topos-theoretic invariants in terms of different sites of definition, take the property of a topos to be two-valued. As we shall see in section 3.3.1, the general criterion obtained above specializes, in the case of an atomic site $(\mathcal{C}^{\text{op}}, J_{\text{at}})$ (where J_{at} is the atomic topology), to the property of $\mathcal{C}$ to be non-empty and to satisfy the joint embedding property, and in the case of the syntactic site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of a geometric theory $\mathbb{T}$ to the property of $\mathbb{T}$ to be (geometrically) complete, i.e. to the property that every geometric sentence over its signature is provably true or provably false in the theory, but not both. In the presence of Morita-equivalences of the form $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$, we thus obtain the following equivalence: the category $\mathcal{C}$ is non-empty and satisfies the joint embedding property if and only if the theory $\mathbb{T}$ is (geometrically) complete. Notice that the property of (geometric) completeness is in general a rather hard property to verify, while the joint embedding property is often more tractable.

We have already mentioned that it is possible to design general methodologies for obtaining site characterizations for topos-theoretic invariants. In [15], a general metatheorem applicable to a wide class of 'geometric' invariants of toposes was proved, and applied therein to derive such characterizations (for the properties to be localic, atomic, coherent, locally connected, compact, equivalent to a presheaf topos, etc.). As an example of such characterizations, we report the one for the invariant property of a topos to be atomic:

Theorem 3.1 (Theorem 4.4 [15]). *Let $(\mathcal{C}, J)$ be a site. Let (f) denote the sieve on cod(f) generated by an arrow f in $\mathcal{C}$, i.e. the set of arrows with codomain cod(f) which factor through f. Then the topos $\mathbf{Sh}(\mathcal{C}, J)$ is atomic if and only if for any object a of $\mathcal{C}$ there exists a J-covering sieve on a generated by arrows $f : b \to a$ such that:*

- $\emptyset \notin J(b)$,
- *for any arrow $g : b' \to a$ such that $g^*((f)) \in J(b')$, we have $\emptyset \in J(b')$ or $f^*((g)) \in J(b)$.*

Once again, we can contemplate the phenomenon of 'differentiation from the unity' consisting in the fact that a topos-theoretic invariant property often manifests itself in completely different ways in the context of different sites. Indeed, all the atomic sites satisfy the condition of the theorem, and on the other hand, in the case of the classifying topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of a geometric theory $\mathbb{T}$, the criterion of the theorem tells that this topos is atomic if and only the theory $\mathbb{T}$ is atomic in the model-theoretic sense, i.e. every geometric formula-in-context over the signature of the theory is $\mathbb{T}$-provably equivalent to a disjunction of geometric formulae in the same context which are $\mathbb{T}$-complete (recall that a formula-in-context $\phi(\vec{x})$ is $\mathbb{T}$-complete if the sequent $(\phi \vdash_{\vec{x}} \bot)$ is not provable in $\mathbb{T}$, and for every geometric formula χ in the same context either $(\chi \wedge \phi \vdash_{\vec{x}} \bot)$ or $(\phi \vdash_{\vec{x}} \chi)$ is provable in $\mathbb{T}$). We shall survey an application of the 'bridge' technique involving these site characterizations in section 3.3.1 below.

Another important topos-theoretic invariant is the property of a topos to be equivalent to a presheaf topos. In [15] a general criterion for a geometric theory to be classified by a presheaf topos was established. To describe it, we need to introduce the following

Definition 3.2. Let $\mathbb{T}$ be a geometric theory over a signature Σ. A geometric formula-in-context $\{\vec{x} \,.\, \phi\}$ is said to be $\mathbb{T}$-*irreducible* if for any family $\{\theta_i \mid i \in I\}$ of $\mathbb{T}$-provably functional geometric formulae $\{\vec{x_i}, \vec{x} \,.\, \theta_i\}$ from $\{\vec{x_i} \,.\, \phi_i\}$ to $\{\vec{x} \,.\, \phi\}$ such that $(\phi \vdash_{\vec{x}} \bigvee\limits_{i \in I} (\exists \vec{x_i})\theta_i)$ is provable in $\mathbb{T}$, there exist $i \in I$ and a $\mathbb{T}$-provably functional geometric formula $\{\vec{x}, \vec{x_i} \,.\, \theta'\}$ from $\{\vec{x} \,.\, \phi\}$ to $\{\vec{x_i} \,.\, \phi_i\}$ such that the composite arrow $[\theta_i] \circ [\theta']$ in $\mathcal{C}_{\mathbb{T}}$ is equal to the identity on $\{\vec{x} \,.\, \phi\}$ (equivalently, the bi-sequent $(\phi(\vec{x}) \wedge \vec{x} = \vec{x'} \dashv\vdash_{\vec{x},\vec{x'}} (\exists \vec{x_i})(\theta'(\vec{x'}, \vec{x_i}) \wedge \theta_i(\vec{x_i}, \vec{x})))$ is provable in $\mathbb{T}$).

This notion is motivated by the fact that a Grothendieck topos is equivalent to a presheaf topos if and only if it has a separating set of irreducible objects (recall that an object of a Grothendieck topos is said to be *irreducible* if every epimorphic family on it in the topos contains a split epimorphism), and a geometric formula-in-context $\{\vec{x} \,.\, \phi\}$ is $\mathbb{T}$-irreducible if and only if its image under the Yoneda embedding $\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is an irreducible object of the topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$.

Theorem 3.3 (Corollary 3.15 [15]). *Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then $\mathbb{T}$ is of presheaf type if and only if there exists a collection $\mathcal{F}$ of geometric formulae-in-context over Σ satisfying the following properties:*

(i) for any geometric formula $\{\vec{y} \,.\, \psi\}$ over Σ, there exist objects $\{\vec{x_i} \,.\, \phi_i\}$ in $\mathcal{F}$ (for $i \in I$) and $\mathbb{T}$-provably functional geometric formulae $\{\vec{x_i}, \vec{y} \,.\, \theta_i\}$ from $\{\vec{x_i} \,.\, \phi_i\}$ to $\{\vec{y} \,.\, \psi\}$ such that $(\psi \vdash_{\vec{y}} \bigvee\limits_{i \in I} (\exists \vec{x_i})\theta_i)$ is provable in $\mathbb{T}$;

(ii) every formula $\{\vec{x} \,.\, \phi\}$ in $\mathcal{F}$ is $\mathbb{T}$-irreducible.

In fact, it is shown in [15] that for any theory of presheaf type $\mathbb{T}$, $\mathbb{T}$-irreducible formulae are precisely the syntactic counterparts of the finitely presentable $\mathbb{T}$-models, in the sense that the full subcategory of the syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ on the $\mathbb{T}$-irreducible formulae is dually equivalent to the category of finitely presentable $\mathbb{T}$-models (cf. Theorem 4.10 in section 4.2.2 below).

Other results of [15] include syntactic characterizations of the geometric theories classified by a locally connected (resp., connected and locally connected, compact) topos, as well as criteria for a geometric theory over a given signature to be cartesian (respectively regular, coherent). This paper also contains various results about quotients of theories of presheaf type, obtained by transferring the above-mentioned topos-theoretic invariants across the two different representations

$$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

of the classifying topos of such a theory.

3.2 De Morgan and Boolean toposes

Paper [11] solves the long-standing question of characterizing the class of geometric theories classified by a De Morgan (resp. Boolean) topos, by providing complete syntactic characterizations for both these classes of theories.

To achieve these characterizations, we first establish general site characterizations for the property of a Grothendieck topos to be De Morgan (resp. Boolean) - those which we recalled in section 3.1 - and then refine these criteria to yield simplified descriptions in several cases of interest, including those of syntactic categories of increasing level of complexity.

In the process, we introduce a new (invariant) construction which yields a universal way of making a given topos De Morgan. More specifically, every topos is shown to have a largest dense subtopos satisfying De Morgan's law; we call this topos its *DeMorganization*. This construction represents a natural analogue of the well-known procedure for making a topos Boolean, namely the Booleanization $\mathbf{sh}_{\neg\neg}(\mathcal{E})$ of a topos $\mathcal{E}$, which in fact, as shown in [11], can be characterized as the largest (in fact, unique) dense subtopos satisfying the law of excluded middle. This construction, as well as that of the Booleanization, admits natural site characterizations. For instance, in [11] it is shown that

- The DeMorganization of a presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ coincides with the topos $\mathbf{Sh}(\mathcal{C}, M)$, where M is the *De Morgan topology* on $\mathcal{C}$, i.e. the Grothendieck topology on $\mathcal{C}$ generated by the sieves of the form

  $$M_R := \{f : d \to c \mid f^*(R) = \emptyset \text{ or } f^*(R) \text{ is stably non-empty }\}$$

  for a sieve R in $\mathcal{C}$ (recall that a sieve S on an object c is said to be stably non-empty if for any arrow $g : a \to c$, $g^*(S) \neq \emptyset$);

- The DeMorganization of a topos $\mathbf{Sh}(L)$ of sheaves on a locale L coincides with the topos of sheaves $\mathbf{Sh}(L_m)$ on the sublocale L_m of L given by the quotient of L by the filter generated by the family $\{\neg u \vee \neg\neg u \mid u \in L\}$.

- The DeMorganization of the classifying topos of a geometric theory $\mathbb{T}$ corresponds under the duality of Theorem 2.5 to the *DeMorganization* of $\mathbb{T}$ i.e. to the quotient of $\mathbb{T}$ obtained from $\mathbb{T}$ by adding all the geometric sequents of the form

  $$(\top \vdash_{\vec{x}} \bigvee_{\psi(\vec{x}) \in I_{(\phi(\vec{x}), \phi'(\vec{x}))}} \psi)$$

for any two geometric formulae $\phi(\vec{x})$ and $\phi'(\vec{x})$ such that $(\phi \wedge \phi' \vdash_{\vec{x}} \bot)$ is provable in $\mathbb{T}$, where $I_{(\phi(\vec{x}),\phi'(\vec{x}))}$ is the collection of geometric formulae $\psi(\vec{x})$ such that either $(\psi \wedge \phi \vdash_{\vec{x}} \bot)$ or $(\psi \wedge \phi' \vdash_{\vec{x}} \bot)$ is provable in $\mathbb{T}$.

The *Booleanization* of a geometric theory $\mathbb{T}$, i.e. the quotient of $\mathbb{T}$ which corresponds under the duality of Theorem 2.5 to the Booleanization of its classifying topos, can instead be described, as shown in [11], as the theory obtained from $\mathbb{T}$ by adding the sequent

$$(\top \vdash_{\vec{y}} \psi)$$

for any $\mathbb{T}$-*stably consistent* geometric formula-in-context $\psi(\vec{y})$ (i.e. geometric formula-in-context $\psi(\vec{y})$ such that for any geometric formula $\chi(\vec{y})$ in the same context such that $(\chi \vdash_{\vec{y}} \bot)$ is not provable in $\mathbb{T}$, $(\chi \wedge \psi \vdash_{\vec{y}} \bot)$ is not provable in $\mathbb{T}$).

Of course, in the case of specific theories of natural mathematical interest, it is often possible to establish more natural and 'economical' axiomatizations for the DeMorganization or Booleanization of a geometric theory, by exploiting the specific features of the theory under consideration. For instance, in [12] the DeMorganization of the coherent theory of fields is identified as the geometric theory of fields of finite characteristic which are algebraic over their prime fields, while its Booleanization is the theory of fields of finite characteristic which are algebraic over their prime fields and algebraically closed. These surprising results provide by themselves a clear indication of the 'centrality' of topos-theoretic invariants in mathematics, including those which might appear too abstract to be of any 'concrete' relevance.

The notions of Booleanization and DeMorganization of a topos have been generalized in [23] to a wide class of intermediate logics including, for instance, Smetanich's logic and Gödel-Dummett's logic. As in the case of the Booleanization and DeMorganization, these invariant constructions admit natural definitions in terms of sites, which can be fruitfully exploited in connection with the 'bridge' technique.

3.3 Atomic two-valued toposes

A very important class of toposes is formed by the atomic two-valued ones. As we shall see, these toposes naturally arise in different contexts, including the theory of countably categorical theories (cf. [13]), Fraïssé's construction in Model Theory (cf. [10]) and topological Galois theory (cf. [28]). We shall also briefly review an approach to the independence from ℓ questions for ℓ-adic cohomology based on atomic two-valued toposes, which we introduced in [29].

3.3.1 Fraïssé's construction from a topos-theoretic persective

In this section, which is based on [10], we present a topos-theoretic interpretation and substantial generalization of the well-known result (Theorem 7.4.1(a) in [49]) providing the link between Fraïssé's construction and countably categorical theories. The three concepts involved in the classical Fraïssé's construction (i.e. amalgamation and joint embedding properties, homogeneous structures, and atomicity and completeness of the resulting theory) are seen to correspond precisely to three different ways (resp. of geometric, semantic and syntactic nature) of looking at the same classifying topos, and the technical relationships between them are shown to arise precisely from the expression of topos-theoretic invariant properties of this classifying topos in terms of different sites of definition for it, according to the philosophy 'toposes as bridges'.

The context in which we formulate our topos-theoretic interpretation of Fraïssé's theorem is that of theories of presheaf type, which we reviewed in sections 2.2 and 3.1. Recall that this is a very extensive class of geometric theories notably including all the cartesian theories and many other interesting mathematical theories pertaining to different fields of mathematics.

In order to present our main result, we have to recall the following notions, which are natural categorical generalisations of the concepts involved in the classical Fraïssé's construction.

Definition 3.4. A category $\mathcal{C}$ is said to satisfy the *amalgamation property* (AP) if for every objects $a, b, c \in \mathcal{C}$ and arrows $f : a \to b$, $g : a \to c$ in $\mathcal{C}$ there exists an object $d \in \mathcal{C}$ and arrows $f' : b \to d$, $g' : c \to d$ in $\mathcal{C}$ such that $f' \circ f = g' \circ g$:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, dashed, "f'"] \\ c \arrow[r, dashed, "g'"'] & d \end{tikzcd}$$

The amalgamation property on a category is also called the left Ore condition, and its dual the right Ore condition. If $\mathcal{C}$ satisfies AP then we can equip $\mathcal{C}^{\mathrm{op}}$ with the atomic topology J_{at}, that is the Grothendieck topology whose covering sieves are exactly the non-empty ones. This point will be a fundamental ingredient of our topos-theoretic interpretation of Fraïssé's theorem.

Definition 3.5. A category $\mathcal{C}$ is said to satisfy the *joint embedding property* (JEP) if for every pair of objects $a, b \in \mathcal{C}$ there exists an object $c \in \mathcal{C}$ and arrows $f : a \to c$, $g : b \to c$ in $\mathcal{C}$:

$$\begin{tikzcd} & a \arrow[d, dashed, "f"] \\ b \arrow[r, dashed, "g"'] & c \end{tikzcd}$$

Notice that if $\mathcal{C}$ has a weakly initial object then AP on $\mathcal{C}$ implies JEP on $\mathcal{C}$; however, in general the two notions are quite distinct from each other.

Definition 3.6. Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be an embedding of categories.

(a) An object $u \in \mathcal{D}$ is said to be $\mathcal{C}$*-homogeneous* if for every objects $a, b \in \mathcal{C}$ and arrows $j : a \to b$ in $\mathcal{C}$ and $\chi : a \to u$ in $\mathcal{D}$ there exists an arrow $\tilde{\chi} : b \to u$ in $\mathcal{D}$ such that $\tilde{\chi} \circ j = \chi$:

$$\begin{tikzcd} a \arrow[r, "\chi"] \arrow[d, "j"'] & u \\ b \arrow[ur, dashed, "\tilde{\chi}"'] & \end{tikzcd}$$

(b) An object $u \in \mathcal{D}$ is said to be $\mathcal{C}$*-ultrahomogeneous* if for every objects $a, b \in \mathcal{C}$ and arrows $j : a \to b$ in $\mathcal{C}$ and $\chi_1 : a \to u$, $\chi_2 : b \to u$ in $\mathcal{D}$ there exists an automorphism $\check{j} : u \to u$ such that $\check{j} \circ \chi_1 = \chi_2 \circ j$:

$$\begin{tikzcd} a \arrow[r, "\chi_1"] \arrow[d, "j"'] & u \arrow[d, dashed, "\check{j}"] \\ b \arrow[r, "\chi_2"'] & u \end{tikzcd}$$

(c) An object $u \in \mathcal{D}$ is said to be $\mathcal{C}$*-universal* if for every $a \in \mathcal{C}$ there exists an arrow $\chi : a \to u$ in $\mathcal{D}$:

$$\begin{tikzcd} a \arrow[r, dashed, "\chi"] & u \end{tikzcd}$$

Given a theory of presheaf type $\mathbb{T}$, a homogeneous $\mathbb{T}$-model is a f.p.$\mathbb{T}$-mod(**Set**)-homogeneous object of the category $\mathbb{T}$-mod(**Set**); in other words, a set-based model M of $\mathbb{T}$ is homogeneous if and only if for any arrow $y : c \to M$ in $\mathbb{T}$-mod(**Set**) and any arrow $f : c \to d$ in f.p.$\mathbb{T}$-mod(**Set**) there exists an arrow u in $\mathbb{T}$-mod(**Set**) such that $u \circ f = y$:

$$\begin{tikzcd} c \arrow[r, "y"] \arrow[d, "f"'] & M \\ d \arrow[ur, dashed, "u"'] & \end{tikzcd}$$

The notion of homogeneous $\mathbb{T}$-model is most relevant when all the $\mathbb{T}$-model homomorphisms are injective; the notion of injectivization of a geometric theory, which we introduced in [8], thus plays an important role in this context. Recall that the injectivization of a geometric theory $\mathbb{T}$ is the theory obtained from $\mathbb{T}$ by adding for each sort over its signature a binary predicate and the axioms asserting that it is provably complemented to the equality relation on that sort.

Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ such that its category f.p.$\mathbb{T}$-mod(**Set**) of finitely presentable models satisfies the amalgamation property. Then we can put on the opposite category f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{\mathrm{op}}$ the atomic topology J_{at}, obtaining a subtopos

$$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\mathrm{op}}, J_{\mathrm{at}}) \hookrightarrow [\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$$

of the classifying topos of $\mathbb{T}$, which corresponds by the duality of Theorem 2.5 to a unique quotient $\mathbb{T}'$ of $\mathbb{T}$ classified by it. This quotient can be characterized as the theory over Σ obtained from $\mathbb{T}$ by adding all the sequents of the form $(\psi \vdash_{\vec{y}} (\exists\vec{x})\theta(\vec{x},\vec{y}))$, where $\phi(\vec{x})$ and $\psi(\vec{y})$ are formulae which present a $\mathbb{T}$-model and $\theta(\vec{x},\vec{y})$ is a $\mathbb{T}$-provably functional formula from $\{\vec{x}\,.\,\phi\}$ to $\{\vec{y}\,.\,\psi\}$.

Since $\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}})$ classifies the theory $\mathbb{T}'$, by the syntactic method for constructing classifying toposes, we have a Morita-equivalence

$$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}),$$

which can be used for building 'bridges' between the two sites by considering appropriate topos-theoretic invariants on the classifying topos of $\mathbb{T}'$:

$$\begin{tikzcd}
 & \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) & \\
(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \arrow[ur, dash, dashed] & & (\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \arrow[ul, dash, dashed]
\end{tikzcd}$$

By applying this methodology, we obtain in particular the following

Theorem 3.7 (Theorem 3.8 [10]). *Let $\mathbb{T}$ be a theory of presheaf type such that the category f.p.$\mathbb{T}$-mod*(**Set**) *is non-empty and satisfies the amalgamation and joint embedding properties. Then (any theory Morita-equivalent to) the theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models is complete and atomic; in particular, assuming the axiom of countable choice, any two countable homogeneous $\mathbb{T}$-models in* **Set** *are isomorphic.*

Moreover, every geometric formula which presents a $\mathbb{T}$-model is $\mathbb{T}'$-complete.

This theorem actually arises from a 'triple' bridge (recalling from [13] that, assuming the axiom of countable choice, every geometric theory which is atomic and complete is countably categorical, that is any two of its set-based countable models are isomorphic):

$$\begin{tikzcd}
 & \textit{Geometric morphism from } \mathcal{E} \textit{ to} \\[-20pt]
 & \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) & \\
(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \arrow[ur, dash, dashed] & & (\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \arrow[ul, dash, dashed] \\[-20pt]
\text{homogeneous } \mathbb{T}\text{-model in } \mathcal{E} & & \mathbb{T}'\text{-model in } \mathcal{E}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \textit{Atomic topos} \\[-20pt]
 & \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \arrow[dr, dashed, leftrightarrow] & \\
(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \arrow[ur, dashed] & & (\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \\[-20pt]
\textit{Atomic site i.e.} & & \\[-20pt]
\text{AP on f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}) & & \textit{Atomicity of } \mathbb{T}'
\end{tikzcd}$$

$$\begin{tikzcd}
 & \textit{Two-valued topos} \\[-20pt]
 & \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) & \\
(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \arrow[ur, dash, dashed] & & (\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \arrow[ul, dash, dashed] \\[-20pt]
\text{JEP on f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}) & & \textit{Completeness of } \mathbb{T}'
\end{tikzcd}$$

Notice that if the category f.p.$\mathbb{T}$-mod(**Set**) satisfies AP then each of its connected components satisfies AP as well as JEP. The toposes of sheaves on such subcategories with respect to the atomic topology are precisely the classifying toposes of the *completions* of the theory $\mathbb{T}'$, i.e. of the (complete) quotients of $\mathbb{T}'$ obtained by adding an axiom of the form $(\top \vdash_{[]} \phi)$ for a $\mathbb{T}'$-complete geometric sentence ϕ.

Theorem 3.7 is a vast generalization of the well-known result (Theorem 7.4.1(a) in [49]) allowing one to build countably categorical theories through Fraïssé's method; the classical result can be obtained as a particular case of our theorem when the theory is the quotient of the empty theory over a finite signature corresponding to a uniformly finite collection of finitely presented models of the empty theory satisfying the hereditary property.

Among the most natural contexts of application of Theorem 3.7, we can mention the following:

(a) The theory $\mathbb{I}$ of decidable objects (that is, the injectivization of the empty theory over a one-sorted signature) is of presheaf type, and its category of finitely presentable models is the category **I** of finite sets and injections between them. The theory of homogeneous $\mathbb{I}$-models is classified by the topos $\mathbf{Sh}(\mathbf{I}^{\text{op}}, J_{\text{at}})$, also known as the *Schanuel topos*. Notice that the homogeneous $\mathbb{I}$-models in **Set** are precisely the infinite sets.

(b) The theory of decidable Boolean algebras (that is, the injectivization of the algebraic theory of Boolean algebras) is of presheaf type. Its finitely presentable models are the finite Boolean algebras, while its homogeneous set-based models are the atomless Boolean algebras. The well-known result that any two countable atomless Boolean algebras are isomorphic thus follows from Theorem 3.7 assuming the axiom of countable choice.

(c) The theory of decidable linear orders (that is, the injectivization of the coherent theory of linear orders) is of presheaf type, and its category of finitely presentable models coincides with the category of finite linear orders and order-preserving injections between them. Its homogeneous models are precisely the dense linearly ordered objects without endpoints. Theorem 3.7 thus ensures that the theory of dense linearly ordered objects without endpoints is atomic and complete.

(d) As shown in [12], the (infinitary) geometric theory of fields of finite characteristic which are algebraic over their prime fields is of presheaf type, and the theory of its homogeneous models can be identified with the theory of fields of finite characteristic which are algebraic over their prime fields and algebraically closed; the completions of this theory are obtained precisely by adding, in each case, the axiom fixing the characteristic of the field (i.e., the sequent $(\top \vdash_{[]} p \,.\, 1 = 0)$ for a prime number p). Theorem 3.7 thus implies, assuming the countable axiom of choice, the well-known fact that any two (countable) algebraic closures of a given finite field are isomorphic, while, not assuming any form of the axiom of choice, it still implies a remarkable property, namely the fact that any two algebraic closures of a given finite field satisfy the same first-order sentences written in the language of fields; in fact, this property is true more generally for any base field (the theory of algebraic extensions of a base field is of presheaf type, and is easily seen to satisfy the hypotheses of Theorem 3.7).

If the category f.p.$\mathbb{T}$-mod(**Set**) satisfies AP then the theory $\mathbb{T}'$ can be identified with the *Booleanization* of the theory $\mathbb{T}$ defined in section 3.2.

The paper [10] also contains a number of other results on homogeneous models, obtained by applying the 'bridge' technique to several other topos-theoretic invariants, including a consistency result for homogeneous models in terms of the geometry of the category of finitely presentable models of the basic theory of presheaf type $\mathbb{T}$. The first part of [10] contains a general categorical framework, subsuming all the previous generalizations of Fraïssé's construction, for building universal ultrahomogeneous objects.

Besides containing a model-theoretic characterization of the geometric theories with enough set-based models which are classified by atomic two-valued toposes, the paper [13] contains a number of categorical results of independent interest on atomic toposes.

3.3.2 Topological Galois Theory

Under appropriate hypotheses which are satisfied in a great number of cases, the atomic two-valued toposes considered in section 3.3.1 admit Galois-type representations as toposes of continuous actions of a topological group of automorphisms of a suitable structure. As we shall see, this leads to a framework which generalizes classical Galois theory and also subsumes Grothendieck's theory of Galois categories of [47].

The termimology used in this section is borrowed from section 3.3.1. We shall present the theorems in the setting of theories of presheaf type $\mathbb{T}$, but one can replace in every statement the category f.p.$\mathbb{T}$-mod(**Set**) with an arbitrary small category C, and the category $\mathbb{T}$-mod(**Set**) with the ind-completion of C.

Theorem 3.8 (cf. Corollary 3.7 [28]). *Let $\mathbb{T}$ be a theory of presheaf type such that its category f.p.$\mathbb{T}$-mod(**Set**) of finitely presentable models satisfies AP and JEP, and let M be a f.p.$\mathbb{T}$-mod(**Set**)-universal and f.p.$\mathbb{T}$-mod(**Set**)-ultrahomogeneous model of $\mathbb{T}$. Then we have an equivalence of toposes*

$$\mathbf{Sh}(f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op}, J_{at}) \simeq \mathbf{Cont}(Aut(M)),$$

where $Aut(M)$ is endowed with the topology of pointwise convergence (in which a basis of open neighbourhoods of the identity is given by the sets of the form $\{f : M \cong M \mid f(\vec{a}) = \vec{a}\}$ for any $\vec{a} \in M$), which is induced by the functor

$$F : f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op} \to \mathbf{Cont}(Aut(M))$$

sending any model c of f.p.$\mathbb{T}$*-mod*(**Set**) *to the set* $\mathrm{Hom}_{\mathbb{T}\text{-}mod(\mathbf{Set})}(c, M)$ *(endowed with the obvious action by Aut*(M)*) and any arrow* $f : c \to d$ *in f.p.*$\mathbb{T}$*-mod*(**Set**) *to the Aut*(M)*-equivariant map*

$$- \circ f : \mathrm{Hom}_{\mathbb{T}\text{-}mod(\mathbf{Set})}(d, M) \to \mathrm{Hom}_{\mathbb{T}\text{-}mod(\mathbf{Set})}(c, M) .$$

Notice that there are in general be many different f.p.$\mathbb{T}$-mod(**Set**)-universal and f.p.$\mathbb{T}$-mod(**Set**)-ultrahomogeneous models of $\mathbb{T}$; for any two such models M and M', their topological automorphism groups $Aut(M)$ and $Aut(M')$ are in general non-isomorphic but Theorem 3.8 implies that they are always Morita-equivalent (in the sense that $\mathbf{Cont}(Aut(M)) \simeq \mathbf{Cont}(Aut(M'))$).

Under the hypotheses of Theorem 3.8, let $\mathbb{T}'$ be the theory of homogeneous $\mathbb{T}$-models (as defined in section 3.3.1). Then the model M, endowed with the (continuous) canonical action of $Aut(M)$, is a universal model of $\mathbb{T}'$ in the topos $\mathbf{Cont}(Aut(M))$ (cf. the proof of Theorem 3.1 [28]). So for any tuple $A_1, \ldots, A_n$ of sorts of the signature of $\mathbb{T}$, we have a 'bridge'

$$\begin{tikzcd} & \overset{\scriptstyle Subobject\ of\ UA_1\times\cdots\times UA_n}{\mathbf{Cont}(Aut(M)) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})} & \\ \underset{\substack{Aut(M)\text{-closed subset of}\\ S\subseteq MA_1\times\cdots\times MA_n}}{Aut(M)} \arrow[ur, dash, dashed] & & \underset{\substack{Geometric\ formula\\ \phi(x_1^{A_1},\ldots,x_n^{A_n})}}{(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})} \arrow[ul, dash, dashed] \end{tikzcd}$$

(where U is 'the' universal model of $\mathbb{T}'$ in its classifying topos), which yields the following

Theorem 3.9. *Let* $\mathbb{T}'$ *be the theory of homogeneous* $\mathbb{T}$*-models for a theory* $\mathbb{T}$ *satisfying the hypotheses of Theorem 3.8.*

(i) For any subset $S \subseteq MA_1 \times \cdots \times MA_n$ *which is closed under the action of Aut*(M)*, there exists a (unique up to* $\mathbb{T}'$*-provable equivalence) geometric formula* $\phi(\vec{x})$ *over the signature of* $\mathbb{T}$ *(where* $\vec{x} = (x_1^{A_1}, \ldots, x_n^{A_n})$*) such that* $S = [[\vec{x} \,.\, \phi]]_M$*.*

(ii) For any Aut(M)*-equivariant map* $f : S \to T$ *between invariant subsets S and T as in (i) there exists a (unique up to* $\mathbb{T}'$*-provable equivalence)* $\mathbb{T}'$*-provably functional geometric formula* $\theta(\vec{x}, \vec{y})$ *from* $\phi(\vec{x})$ *to* $\psi(\vec{y})$*, where* $S = [[\vec{x} \,.\, \phi]]_M$ *and* $T = [[\vec{y} \,.\, \psi]]_M$*, whose interpretation* $[[\theta(\vec{x}, \vec{y})]]_M$ *coincides with the graph of f.*

Remarks 3.10. (a) It easily follows from Theorem 3.9 that for any finite tuple $A_1, \ldots, A_n$ of sorts of the signature of the theory $\mathbb{T}$, the orbits of the action of $Aut(M)$ on $MA_1 \times \cdots \times MA_n$ coincide precisely with the interpretations $[[\vec{x} \,.\, \phi]]_M$ in M of $\mathbb{T}$-complete formulae $\phi(\vec{x})$, where $\vec{x} = (x_1^{A_1}, \ldots, x_n^{A_n})$, that is they correspond exactly to the $\mathbb{T}$-provable equivalence classes of $\mathbb{T}$-complete formulae in the context $\vec{x}$.

(b) If in part (ii) of Theorem 3.9 the formulae $\phi(\vec{x})$ and $\psi(\vec{y})$ present respectively $\mathbb{T}$-models $M_{\{\vec{x}.\phi\}}$ and $M_{\{\vec{y}.\psi\}}$ and all the arrows in f.p.$\mathbb{T}$-mod(**Set**) are strict monomorphisms, then the formula θ can be taken to be $\mathbb{T}$-provably functional, and hence to induce (by Theorem 4.10) a $\mathbb{T}$-model homomorphism $z : M_{\{\vec{y}.\psi\}} \to M_{\{\vec{x}.\phi\}}$ such that the map $[[\vec{x}, \vec{y} \,.\, \theta]]_M : [[\vec{x} \,.\, \phi]]_M \to [[\vec{y} \,.\, \psi]]_M$ corresponds, under the identifications $\mathrm{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(M_{\{\vec{x}.\phi\}}, M) \cong [[\vec{x} \,.\, \phi]]_M$ and $\mathrm{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(M_{\{\vec{y}.\psi\}}, M) \cong [[\vec{y} \,.\, \psi]]_M$, to the map $\mathrm{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(z, M)$ (cf. Theorem 3.11 below).

Concerning the existence of models M satisfying the hypotheses of Theorem 3.8, we should remark that ultrahomogeneous structures naturally arise in a great variety of different mathematical contexts. Their existence can be proved either directly through an explicit construction or through abstract logical arguments. A general method for building countable ultrahomogeneous structures is provided by Fraïssé's construction in Model Theory (cf. Chapter 7 of [49]), while the categorical generalization established in [10] allows to construct ultrahomogeneous structures of arbitrary cardinality. As examples of models M satisfying the hypotheses of Theorem 3.8 in relation to the theories of presheaf type considered in section 3.3.1, we mention:

- the set $\mathbb{N}$ of natural numbers, with respect to the theory of decidable objects;
- the ordered set $(\mathbb{Q}, <)$ of rational numbers, with respect to the theory $\mathbb{L}$ of decidably linearly ordered objects;
- the unique countable atomless Boolean algebra, with respect to the theory of decidable Boolean algebras;
- any Galois extension F' of a given field F, with respect to the theory $\mathbb{T}_F^{F'}$ over the signature of the theory of fields which consists of all the geometric sequents which are valid in every finite intermediate extension $F \subseteq L \subseteq F'$.

There is a natural link between ultrahomogeneity and the property of a model of an atomic complete theory to be *special*, which can be exploited to construct models satisfying the hypotheses of Theorem 3.8. Recall that a model M of an atomic complete theory $\mathbb{T}$ is said to be *special* if every $\mathbb{T}$-complete formula $\phi(\vec{x})$ is realized in M and for any tuples $\vec{a}$ and $\vec{b}$ of elements of M which satisfy the same $\mathbb{T}$-complete formulae there is an automorphism $f : M \to M$ of M which sends $\vec{a}$ to $\vec{b}$. Any model M satisfying the hypotheses of Theorem 3.8 is clearly special as a model of the theory of homogeneous $\mathbb{T}$-models. Conversely, given a special model M of an atomic complete theory $\mathbb{S}$, we have, by the Comparison Lemma, an equivalence $\mathbf{Sh}(\mathcal{C}_{\mathbb{S}}, J_{\mathbb{S}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{S}}^{\mathrm{at}}, J_{\mathrm{at}})$, where $\mathcal{C}_{\mathbb{S}}^{\mathrm{at}}$ is the full subcategory of $\mathcal{C}_{\mathbb{S}}$ on the $\mathbb{S}$-complete formulae, which shows that $\mathbb{S}$ is Morita-equivalent to the theory of homogeneous $\mathbb{T}$-models for some theory of presheaf type $\mathbb{T}$ with respect to which the model M satisfies the hypotheses of Theorem 3.8.

Given an arrow $\chi : c \to M$ in $\mathbb{T}$-mod(**Set**), where c is in f.p.$\mathbb{T}$-mod(**Set**), we denote by

$$\mathcal{I}_\chi := \{f : M \cong M \mid f \circ \chi = \chi\}$$

the open subgroup of $Aut(M)$ corresponding to χ.

The following theorem gives necessary and sufficient conditions for the functor F of Theorem 3.8 to be full and faithful.

Theorem 3.11 (cf. Proposition 4.1 [28])**.** *Under the hypotheses of Theorem 3.8, the following conditions are equivalent:*

*(i) Every arrow $f : d \to c$ in f.p.$\mathbb{T}$-mod(**Set**) is a* strict monomorphism *(in the sense that for any arrow $g : e \to c$ in f.p.$\mathbb{T}$-mod(**Set**) such that $h \circ g = k \circ g$ whenever $h \circ f = k \circ f$, g factors uniquely through f):*

$$\begin{tikzcd} d \arrow[r, "f"] & c \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & u \\ e \arrow[u, dashed] \arrow[ur, "g"'] & & \end{tikzcd}$$

(ii) The functor

$$F : f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op} \to \mathbf{Cont}(Aut(M))$$

of Theorem 3.8 is full and faithful.

*(iii) For any models $c, d \in$ f.p.$\mathbb{T}$-mod(**Set**) and any arrows $\chi : c \to M$ and $\xi : d \to M$ in $\mathbb{T}$-mod(**Set**), $\mathcal{I}_\xi \subseteq \mathcal{I}_\chi$ (that is, for any automorphism f of M, $f \circ \xi = \xi$ implies $f \circ \chi = \chi$) if and only if there exists a unique arrow $f : c \to d$ in f.p.$\mathbb{T}$-mod(**Set**) such that $\chi = \xi \circ f$:*

$$\begin{tikzcd} c \arrow[r, "\chi"] \arrow[d, dashed, "f"'] & M \\ d \arrow[ur, "\xi"'] & \end{tikzcd}$$

(iv) The Grothendieck topology J_{at} is subcanonical.

Let Sgr($Aut(M)$) be the preorder category consisting of the subgroups of $Aut(M)$ and the inclusion relations between them, and f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}/M$ the category whose objects are the arrows $c \to M$ in $\mathbb{T}$-mod(**Set**), where c is an object of f.p.$\mathbb{T}$-mod(**Set**), and whose arrows $(\chi : c \to M) \to (\xi : d \to M)$ are the arrows $f : d \to c$ in f.p.$\mathbb{T}$-mod(**Set**) such that $\chi \circ f = \xi$.

Then the functor F of Theorem 3.8 yields a functor

$$\tilde{F} : \text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}/M \to \text{Sgr}(Aut(M))$$

sending any object $\chi : c \to M$ of f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}/M$ to the subgroup $\mathcal{I}_\chi$ of $Aut(M)$.

Note that if the conditions of Theorem 3.11 are satisfied then the category f.p.$\mathbb{T}$-mod($\mathbf{Set}$)$^{\text{op}}/M$ is a preorder. Indeed, it is easy to see that if all the arrows of f.p.$\mathbb{T}$-mod($\mathbf{Set}$) are monic then all the arrows of $\mathbb{T}$-mod($\mathbf{Set}$) are monic as well.

It is natural to wonder under which conditions the functor $\tilde{F}$ yields a bijection between the objects of the category f.p.$\mathbb{T}$-mod($\mathbf{Set}$)$^{\text{op}}/M$ and the open subgroups of the topological group $Aut(M)$, as in the case of classical Galois theory. The following theorem provides an answer to this question.

Theorem 3.12 (cf. Theorem 4.17 [28]). *Assuming that the equivalent conditions of Theorem 3.11 are satisfied, the following conditions are equivalent:*

(i) The functor
$$F : f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op} \to \mathbf{Cont}(Aut(M))$$
is a categorical equivalence onto the full subcategory of $\mathbf{Cont}(Aut(M))$ *on the transitive actions.*

(ii) The map
$$\tilde{F} : f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op}/M \to Sgr(Aut(M))$$
is a bijection onto the set of open subgroups of $Aut(M)$*.*

(iii) The category f.p.$\mathbb{T}$*-mod*($\mathbf{Set}$) *has equalizers, for any object c of f.p.*$\mathbb{T}$*-mod*($\mathbf{Set}$) *there exist arbitrary intersections of subobjects of c in f.p.*$\mathbb{T}$*-mod*($\mathbf{Set}$)*, and for any pair of arrows $h, k : c \to e$ in f.p.*$\mathbb{T}$*-mod*($\mathbf{Set}$) *with equalizer $m : d \rightarrowtail c$ we have that for any pair of arrows $l, n : c \to e'$, $l \circ m = n \circ m$ if and only if there exist an arrow $s : e' \to e''$ such that $(s \circ l, s \circ n)$ belongs to the equivalence relation on* $\mathrm{Hom}_{f.p.\mathbb{T}\text{-}mod(\mathbf{Set})}(c, e'')$ *generated by the relation consisting of the pairs of the form $(t \circ h, t \circ k)$ for an arrow $t : e \to e''$.*

(iv) Every atom of the topos $\mathbf{Sh}(f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op}, J_{at})$ *come from an object c of f.p.*$\mathbb{T}$*-mod*($\mathbf{Set}$)*.*

Remarks 3.13. (a) If the category f.p.$\mathbb{T}$-mod($\mathbf{Set}$) satisfies the hypotheses of Theorem 3.12, it is always possible to 'complete' it to a category satisfying the equivalent conditions of the theorem, and having an equivalent associated topos, by means of an elementary process which constitutes a sort of completion by imaginaries (in the sense of classical Model Theory). This process is described in detail in [28] (cf. the proof of Theorem 4.15 therein) and applies to a number of classical categories such as the category of Boolean algebras and embeddings, the category of finite groups and embeddings, the category of finite graphs and embeddings etc.

(b) For any Galois category $\mathcal{C}$ (in the sense of [47]), the opposite of the full subcategory $\mathcal{C}_{\text{at}}$ of $\mathcal{C}$ on the atomic objects (i.e., the objects which are non-zero and have no proper subobjects) satisfies the hypotheses of Theorem 3.12. Our framework thus generalizes and simplifies that of Grothendieck's Galois theory by working with small atomic sites instead of larger sites such as his Galois categories, which in fact, as proved in [28], are finite coproduct completions of their full subcategories on their atomic objects.

Notice that both Theorem 3.11 and Theorem 3.12 naturally arise from topos-theoretic 'bridges', the former by considering the invariant notion of arrow between two given objects, and the latter by considering the atoms of the given topos. The paper [28] also contains many other results on these Galois-type theories, a number of which are obtained by considering other relevant topos-theoretic invariants in connection with the Morita-equivalence of Theorem 3.8.

3.3.3 Motivic toposes

We have seen in section 2.1.3 (cf. Corollary 2.4) how the notion of syntactic category applies to the question of the existence of categories of (mixed) motives (in Grothendieck's sense) through which all the different ℓ-adic cohomological functors $H^{\bullet}(\bullet, \overline{\mathbb{Q}}_\ell)$ factor.

In the context of Corollary 2.4 of a quiver D defined from the category of finite type schemes over a base field K and of representations

$$T_\ell : D \to \overline{\mathbb{Q}}_\ell\text{-vect}$$

induced by ℓ-adic cohomological functors (for $\ell \neq \mathrm{car}\,(K)$) or possibly p-adic (if $\ell = p = \mathrm{car}\,(K)$), the question of the "independence from ℓ" naturally poses:

- For any object d of the diagram D, is the dimension over $\overline{\mathbb{Q}}_\ell$ of the vector space $T_\ell(d)$ independent from ℓ?

- More generally, for any $\mathbb{Q}$-linear combination of composites

  $$f : d \to d'$$

  of arrows of D, is the dimension over $\overline{\mathbb{Q}}_\ell$ of

  $$\mathrm{Ker}\,(T_\ell(f) : T_\ell(d) \to T_\ell(d'))$$

  independent from ℓ?

This problem is notably difficult since there is no direct canonical way of comparing two different ℓ-adic cohomological functors T_ℓ: indeed, the different coefficient fields $\overline{\mathbb{Q}}_\ell$ of these functors are different and, whilst by using the axiom of choice one can construct certain isomorphisms between them, these isomorphisms are neither canonical nor they preserve the natural topologies of $\overline{\mathbb{Q}}_\ell$.

Still, this problem appears more tractable by adopting a logical perspective. Indeed, complete theories within first-order logic enjoy the remarkable property that all their models satisfy exactly the same first-order properties expressible in the language of the theory, in spite of the fact that they might be completely different and not concretely related to one another. It is therefore natural to imagine that the different ℓ-adic cohomological functors could be models of a first-order complete theory written over a signature related to that considered in the context of Nori's construction. We have encountered in section 3.3.1 a wide class of such theories, and described their classifying toposes in both syntactic and semantic terms. These toposes can be represented in the form $\mathbf{Sh}(\text{f.p.}\mathbb{S}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}})$ where $\mathbb{S}$ is a theory of presheaf type such that the category f.p.$\mathbb{S}$-mod(**Set**) satisfies the amalgamation and joint embedding properties.

As remarked in section 3.3.1, the theory of homogeneous $\mathbb{S}$-models is most interesting when all the arrows of the category f.p.$\mathbb{S}$-mod(**Set**) are monic. If the theory $\mathbb{S}$ contains a predicate which is provably complemented to the equality relation then all the homomorphisms of set-based $\mathbb{S}$-models are injective and every finitely generated model of $\mathbb{I}$ is finitely presentable, the converse holding if the axioms of $\mathbb{S}$ have a general specified form. The T_ℓ would then be homogeneous $\mathbb{S}$-models. If they were also f.p.$\mathbb{S}$-mod(**Set**)-universal and the category f.p.$\mathbb{S}$-mod(**Set**) has an initial object, then every finitely generated $\mathbb{S}$-model would be embeddable in each T_ℓ. This motivates us to give the following

Definition 3.14 (cf. sections 5.1 and 5.2 of [29])**.** Given a representation

$$T_\ell : D \to \overline{\mathbb{Q}}_\ell\text{-vect} .$$

as above, let Σ be the "basic signature" which has:

- one sort for the coefficient field k and for each object d of D,
- function symbols for the ring structure on k and the k-linear structure on each other sort,
- a function symbol for each arrow of D,
- a relation symbol $\neq 0$ on each sort.

Let $\mathbb{I}_\ell$ be the "basic theory" over Σ having as axioms

- the sequents which ensure that k is a field of characteristic 0 and the other sorts are k-vector spaces,
- the sequents which define the relations $\neq 0$,
- the sequents

$$(\varphi \vdash_{\vec{x}} \psi)$$

  which are satisfied in T_ℓ, where the formulae φ, ψ are finite conjunctions of equalities of terms.

Let Σ' be the signature obtained from Σ by adding a relation symbol

$$R_S$$

for each context $\vec{x}$ (modulo renaming of variables) and subset S of the set of terms in the context $\vec{x}$.

Let $\mathbb{S}_\ell$ be the theory over Σ' obtained from $\mathbb{I}_\ell$ by adding

- the axioms which ensure that for each relation R_S, the terms in S are 0 and the terms not in S are $\neq 0$,
- the axiom
$$(\top \vdash_x \bigvee_S R_S(\vec{x}))$$
for each context $\vec{x}$.

Proposition 3.15 (cf. sections 5.2 and 5.3 [29]). *(i) For each index ℓ, the geometric theory $\mathbb{S}_\ell$ associated with the representation $T_\ell : D \to \overline{\mathbb{Q}}_\ell$-vect is of presheaf type and its category C_ℓ of finitely presentable (equivalently, finitely generated) models has an initial object and satisfies the amalgamation and joint embedding properties.*

(ii) The theory $\mathbb{T}_\ell$ is obtained from $\mathbb{S}_\ell$ by adding all the sequents of the form
$$(R_{\vec{w}_*(S)}(\vec{y}) \vdash_{\vec{y}} (\exists \vec{x})(R_S(\vec{x}) \wedge \vec{w}(\vec{x}) = \vec{y}))$$
for any tuple $\vec{w}$ of terms and any subset S such that the sequent $(R_S(\vec{x}) \vdash_{\vec{x}} \perp)$ is not provable in $\mathbb{S}_\ell$, where $\vec{w}_*(S) = \{s \mid s(\vec{w}(\vec{x})) \in S\}$.

We can now reformulate our initial questions in the following more precise way:

(i) For each index ℓ, is the cohomological representation
$$T_\ell : D \to \overline{\mathbb{Q}}_\ell\text{-vect}$$
a set-based model of $\mathbb{T}_\ell$, that is, a C_ℓ-homogeneous model of the theory $\mathbb{S}_\ell$?

(ii) Are the theories $\mathbb{S}_\ell$ independent from ℓ?

A positive answer to the above two questions would imply all the expected independence from ℓ properties. This is rather surprising since the notion of dimension is not expressible in terms of the axioms of the theories $\mathbb{S}_\ell$ which are of 'algebraic' type . Notice also that question (i) concerns any single ℓ-adic cohomological functor, independently from any other one. As proved in [29] (cf. Theorem 6.4), the axioms of $\mathbb{T}_\ell$ imply all the usual exactness properties of cohomological functors. Homogeneity can thus be regarded as a refinement of the exactness conditions.

Finally, we note that if the two above questions have a positive answer, the category of internal reflective vector spaces in the common classifying topos of the theories $\mathbb{T}_\ell$ provides a natural candidate for a category of motives through which all the different ℓ-adic cohomological functors factor via faithful exact functors. Indeed, any point of an atomic two-valued topos is a surjection, i.e. its inverse image functor is (exact and) faithful.

4 Dualities, bi-interpretations and Morita-equivalences

Since Morita-equivalence represents a natural notion of equivalence of theories, it is natural to wonder what are its relationships with the notions of duality, categorical equivalence, and bi-interpretation.

As we already remarked in section 2.1, categorical dualities or equivalences between 'concrete' categories can often be seen as arising from the process of 'functorializing', by means of geometric morphisms of toposes, Morita-equivalences which express structural relationships between each pair of objects corresponding to each other under the given duality or equivalence. We shall illustrate this general point below, by showing that the classical Stone-type dualities can be naturally obtained by means of this methodology, which also generates many other new dualities or equivalences between partially ordered structures, locales and topological spaces. Starting from geometric morphisms between classifying toposes in place of Morita-equivalences, one obtains reflections extending the corresponding dualities or equivalences.

Another link between the notion of categorical equivalence between 'concrete' categories and that of Morita-equivalence is provided by the observation, made in section 2.1, that categorical equivalences between the categories of set-based models of two geometric theories can often be 'lifted' to Morita-equivalences provided that they are established by only using constructive logic and geometric constructions. As illustrations of this remark, we discuss in section 4.3.1 two examples of Morita-equivalences obtained by 'lifting' classical equivalences in the context of MV-algebras and lattice-ordered abelian groups. As it can be naturally expected, there are many advantages in lifting such a categorical equivalence to a Morita-equivalence; indeed, many more properties and results can be established and transferred across the two theories by using the common classifying topos as a 'bridge'; various examples of such transfers are given in section 4.3.

Concerning the relationship between Morita-equivalences and bi-interpretations, we have already remarked in section 2.1 that any bi-interpretation induces a Morita-equivalence. Nonetheless, very importantly, most Morita-equivalences do *not* arise from bi-interpretations; we shall give in section 4.3.1 two specific examples of naturally arising Morita-equivalences which provably do not arise from bi-interpretations. Notice that, in presence of a bi-interpretation, one does not really need to pass to the classifying topos in order to transfer syntactic results across the two theories since the bi-interpetation provides by itself a 'dictionary' for translating formulas written in the language of one theory into formulas written in the language of the other; still, the consideration of the classifying topos can be useful for investigating the relationships between the semantic aspects of the two theories.

4.1 Dualities, equivalences and adjunctions for preordered structures and topological spaces

In the following sections we briefly review the contents of papers [18], [20], [21] and [22], which investigate the subject of preordered structures, topological spaces and locales from a topos-theoretic perspective.

4.1.1 The topos-theoretic construction of Stone-type dualities and adjunctions

The paper [18] introduces a general topos-theoretic machinery for building 'Stone-type' dualities, i.e. dualities or equivalences between categories of preorders and categories of posets, locales or topological spaces. This machinery allows one to unify all the classical Stone-type dualities as instances of just one topos-theoretic phenomenon, and to generate many new such dualities. We recover in particular the following well-known dualities:

- Stone duality for distributive lattices (and in particular Boolean algebras, cf. [71] and [72])
- Lindenbaum-Tarski duality for atomic complete Boolean algebras
- The duality between spatial frames and sober spaces
- M. A. Moshier and P. Jipsen's topological duality for meet-semilattices (cf. [66])
- Alexandrov equivalence between preorders and Alexandrov spaces
- Birkhoff duality for finite distributive lattices
- The duality between algebraic lattices and sup-semilattices
- The duality between completely distributive algebraic lattices and posets

At the same time, our machinery allows one to generate many new dualities, some examples of which we shall give below.

Our methodology essentially consists in 'functorializing' Morita-equivalences of the form

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{D}, K),$$

where $\mathcal{C}$ is a preorder (regarded as a category), J is a (subcanonical) Grothendieck topology J on $\mathcal{C}$, $\mathcal{C}$ is a K-dense full subcategory of $\mathcal{D}$ (i.e. a full subcategory $\mathcal{C}$ of $\mathcal{D}$ such that for any object d of $\mathcal{D}$ the sieve generated by the arrows from objects of $\mathcal{C}$ to d is K-covering) and J is the induced Grothendieck topology $K|_{\mathcal{C}}$ on $\mathcal{C}$; such equivalences are all instances of Grothendieck's Comparison Lemma. More specifically, we show that if the above Morita-equivalences hold for a class of structures $\mathcal{C}$ (resp. $\mathcal{D}$) and that each of these structures is equipped with a

Grothendieck topology $J = J_C$ (resp. $K = K_{\mathcal{D}}$) intrinsically definable in terms of it, then one can construct, under some hypotheses which are satisfied in a large number of cases, dualities or equivalences between a category of structures C (whose morphisms are maps which induce geometric morphisms between the associated toposes $\mathbf{Sh}(C, J_C)$, either covariantly or contravariantly) and a category of structures $\mathcal{D}$ (whose morphisms are maps which induce geometric morphisms between the associated toposes $\mathbf{Sh}(\mathcal{D}, K_{\mathcal{D}})$, either covariantly or contravariantly). The key point is the possibility, under those hypotheses, of recovering the structures C (resp. $\mathcal{D}$) from the corresponding toposes $\mathbf{Sh}(C, J_C)$ (resp. $\mathbf{Sh}(\mathcal{D}, K_{\mathcal{D}})$) by means of topos-theoretic invariants:

$$\begin{tikzcd}
 & \mathbf{Sh}(C, J_C) \arrow[d] & \simeq & \mathbf{Sh}(\mathcal{D}, K_{\mathcal{D}}) \arrow[d] \arrow[ddr, dashed, no head, bend left] & \\
 & \mathbf{Sh}(C', J_{C'}) & \simeq & \mathbf{Sh}(\mathcal{D}', K_{\mathcal{D}'}) & \\
C \arrow[uur, dashed, no head, bend left] & & & & \mathcal{D} \arrow[d] \\
C' \arrow[u] & & & & \mathcal{D}'
\end{tikzcd}$$

(in this bridge the first arch is contravariant and the second is covariant, but all the variance possibilities are equally feasible).

Notice that a preorder category is simply a preordered set, and a functor between preorder categories is just an order-preserving map between the corresponding preordered sets. A preorder with finite limits is precisely a meet-semilattice. As shown in [18], for any preorder category C and any Grothendieck topology J on C, we have an equivalence

$$\mathbf{Sh}(C, J) \simeq \mathbf{Sh}(Id_J(C)),$$

where $Id_J(C)$ is the frame of J-ideals on C whose elements are the J-ideals on C and where the order relation is the subset-inclusion one (recall that a J-ideal is a subset I of the set of objects of C such that for any arrow $f : a \to b$ in C, if $b \in I$ then $a \in I$ and for any J-covering sieve S on an object c of C, if $\mathrm{dom}(f) \in I$ for all $f \in S$ then $c \in I$). Notice that localic toposes (i.e., toposes of the form $\mathbf{Sh}(L)$ for a locale L) can be identified with the corresponding locales, since a localic topos $\mathcal{E}$ is equivalent to the topos of sheaves on the associated locale L with respect to the canonical topology J_L^{can} on it, and conversely any locale can be recovered from the associated topos as the frame of its subterminal objects (recall that a subterminal object of a topos is an object such that the unique arrow from it to the terminal object is monic).

A Grothendieck topology J on a preorder category C is subcanonical if and only if for every J-covering sieve $\{a_i \to a \mid i \in I\}$, we have that $a = \bigvee_{i \in I} a_i$ in C.

To functorialize the above-mentioned Morita-equivalences, we apply the theory of geometric morphisms between Grothendieck toposes induced by functors satisfying appropriate conditions. Recall that:

(a) Any functor $F : \mathcal{C} \to \mathcal{C}'$ between categories $\mathcal{C}$ and $\mathcal{C}'$ with finite limits which preserves finite limits and is cover-preserving (i.e., sends J-covering sieves to families which generate a J'-covering sieve) induces a geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{C}', J') \to \mathbf{Sh}(\mathcal{C}, J)$. If the topologies J and J' are subcanonical then F can be identified with the restriction of the inverse image $\mathbf{Sh}(F)^* : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{C}', J')$ of $\mathbf{Sh}(F)$ to the representables. (The notion of morphism of sites can be given for arbitrary, i.e. not necessarily cartesian sites, but we do not report it here since it is more complicated and we shall not use it.)

(b) Any functor $f : \mathcal{C} \to \mathcal{C}'$ induces a geometric morphism $E(f) : [\mathcal{C}, \mathbf{Set}] \to [\mathcal{C}', \mathbf{Set}]$. If $\mathcal{C}$ and $\mathcal{C}'$ are Cauchy-complete then f can be identified with the restriction to the representables of the left adjoint $E(f)_! : [\mathcal{C}, \mathbf{Set}] \to [\mathcal{C}', \mathbf{Set}]$ to the inverse image of $E(f)$. If $\mathcal{C}$ and $\mathcal{C}'$ are Cauchy-complete, the geometric morphisms $[\mathcal{C}, \mathbf{Set}] \to [\mathcal{C}', \mathbf{Set}]$ of the form $E(f)$ for some functor $f : \mathcal{C} \to \mathcal{C}'$ can be intrinsically characterized as the *essential* ones (i.e., those whose inverse image admits a left adjoint).

In presence of Morita-equivalences of the above form holding for all preorders $\mathcal{C}$ belonging to a certain class, one can thus try to construct a Stone-type duality for a category $\mathcal{K}$ whose objects are the objects in that class and whose morphisms are the maps which induce (either via (a) or via (b)) geometric morphisms between the associated toposes; indeed, under these hypotheses we automatically have a functor (contravariant or covariant depending on whether one chooses to functorialize the given Morita-equivalences via (a) or (b)) from $\mathcal{K}$ to the category of localic Grothendieck toposes (equivalently, to the opposite of the category of frames). In this way we get the first arches of our 'bridges'. Since our aim is to eventually obtain a categorical equivalence, rather than just a functor, we need to be able to 'exit' the bridges as well; in other words, we need to be able to recover the structures $\mathcal{D}$ (resp. $\mathcal{C}$) appearing in the Morita-equivalences from the associated toposes $\mathbf{Sh}(\mathcal{D}, K)$ (resp. $\mathbf{Sh}(\mathcal{C}, J)$) by means of suitable topos-theoretic invariants. Clearly, for this to be the case, a necessary condition is that the topologies J and K be subcanonical, equivalently that $\mathcal{D}$ (resp. $\mathcal{C}$) embeds into $\mathbf{Sh}(\mathcal{D}, K)$ (resp. $\mathbf{Sh}(\mathcal{C}, J)$) via the Yoneda embedding. Notice that under this hypothesis, the Yoneda embedding $\mathcal{D} \to \mathbf{Sh}(\mathcal{D}, K)$ (resp. $\mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$) sends each element of $\mathcal{D}$ (resp. of $\mathcal{C}$) to the associated principal ideal. Since $\mathbf{Sh}(\mathcal{D}, K) \simeq \mathbf{Sh}(Id_K(\mathcal{D}))$ and $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(Id_J(\mathcal{C}))$, and any frame L can be recovered from the associated topos $\mathbf{Sh}(L)$ of sheaves on it as the frame of its subterminal objects, we are reduced to the problem of characterizing the principal ideals among all the ideals in invariant terms.

Very interestingly, it turns out that if the topologies K (resp. J) can be 'uniformly described through an invariant C of families of subterminals in a topos (equivalently, of families of elements of frames)' (in a sense to be made precise below in Definition 4.1 and Theorem 4.2) then the principal ideals on $\mathcal{C}$ can be characterized among the elements of the frame $Id_J(C)$ precisely as the ones which satisfy a key condition of C-compactness (in the sense of Definition 4.1).

The idea behind the following notion of C-induced topology J on a preorder category $\mathcal{C}$ is that J should be generated by sieves obtained by taking the joins, in appropriate frames extending $\mathcal{C}$ (satisfying enough invariant properties P to make them behave sufficiently like the canonical embedding $\mathcal{C} \hookrightarrow Id_J(\mathcal{C})$), of families of elements satisfying C.

Definition 4.1. Let C be a frame-theoretic invariant property of families of elements of a frame (for example, 'to be finite', 'to be a singleton', 'to be of cardinality at most k (for a given cardinal k)', 'to be formed by elements which are pairwise disjoint','to be directed' etc.).

- Given a preordered structure $\mathcal{C}$, a Grothendieck topology J on $\mathcal{C}$ is said to be *C-induced* if for any J_F^{can}-dense monotone embedding $i : \mathcal{C} \hookrightarrow F$ into a frame F (where J_F^{can} is the canonical topology on F) possibly satisfying some invariant property P which is known to hold for the canonical embedding $\mathcal{C} \hookrightarrow Id_{J_C}(\mathcal{C})$ and such that the J_C-covers on $\mathcal{C}$ are sent by i to covers in F, for any family $\mathcal{A}$ of elements in $\mathcal{C}$ there exists a J_C-cover S on an element $c \in \mathcal{C}$ such that the elements $a \in \mathcal{A}$ such that $a \leq c$ generate S if and only if the image $i(\mathcal{A})$ of the family $\mathcal{A}$ in F has a refinement satisfying C made of elements of the form $i(c')$ (for $c' \in \mathcal{C}$).

- An element u of a frame F is said to be *C-compact* if every covering of u in F has a refinement satisfying C.

Here are some examples of Grothendieck topologies on preordered structures.

- If P is a preorder, the *trivial topology* on P is the one in which the only covering sieves are the maximal ones.

- If D is a distributive lattice, the *coherent topology* J_D^{coh} on D is the one in which the covering sieves are exactly those which contain finite families whose join is the given element.

- If F is a frame, the *canonical topology* J_F^{can} on F is the one in which the covering sieves are exactly the families whose join is the given element.

- If D is a disjunctively distributive lattice (i.e. a meet-semilattice in which finite joins of pairwise disjoint elements - that is, of elements whose meet is the zero element - exist and distribute over finite meets), the *disjunctive topology* J_D^{dj} on D is the one in which the covering sieves are exactly those which contain finite families of pairwise disjoint elements whose join is the given element.

- If U is a k-frame (i.e. a meet-lattice in which joins of families of less than k elements exist and distribute over finite meets), the *k-covering topology* J_U^k on U is the one in which the covering sieves are those which contain families of less than k elements whose join is the given element.

- If V is a preframe (i.e. a meet-semilattice in which joins of directed families of elements exist and distribute over finite meets), the *directed topology* J_V^{dir} on V is the one in which the covering sieves are precisely those which contain directed families of elements whose join is the given element.

Notice that the distributivity property of the joins in the structures considered above is crucial for the associated Grothendieck topologies to be well-defined; indeed, it corresponds precisely to the pullback stability property for covering sieves. In the absence of such distributive laws, one can consider the Grothendieck topology *generated* by the sieves corresponding to joins of the appropriate kind in the given structure, but such a topology no longer admits a description in invariant terms as required by our general theory.

It is shown in [18] that the above topologies are all C-induced for an invariant C as in Definition 4.1; more precisely, the trivial (resp. coherent, canonical, disjunctive, k-covering, directed) topology is C-induced where C is the invariant 'to be a singleton' (resp. 'to be finite', 'to be any family', 'to be of cardinality at most k','to be formed by elements which are pairwise disjoint', 'to be directed').

On the other hand, consider the canonical topology J_P^{can} on a preorder P which is not a frame; this is defined by stipulating that for any sieve $\{a_i \to a \mid i \in I\}$, $S \in J_P^{\mathrm{can}}(a)$ if and only if $a = \bigvee_{i \in I} a_i$ and this join is *distributive* in the sense that for any $b \leq a$, b is equal to the sup of all the elements $\leq b$ which are $\leq a_i$ for some $i \in I$. The fact that the form of the distributive joins in P can vary with the elements of P in a way which cannot be described in general invariant (whence independent from P) terms is responsible for the fact that this topology, unlike those considered above, is not C-induced for an invariant C.

The following theorem shows the importance of the notion of C-induced topology for building Stone-type dualities.

Theorem 4.2 ([18]). *(a) If a Grothendieck topology J on a poset $\mathcal{C}$ is C-induced for some invariant C of families of elements of a frame and the invariant C satisfies the property that for any structure $\mathcal{C}$ in $\mathcal{K}$ and for any family $\mathcal{F}$ of principal J-ideals on $\mathcal{C}$, $\mathcal{F}$ has a refinement satisfying C (if and) only if it has a refinement satisfying C made of principal J-ideals on $\mathcal{C}$, then the poset $\mathcal{C}$ can be recovered from the frame $Id_J(\mathcal{C})$ (resp. from the topos $\mathbf{Sh}(\mathcal{C}, J)$) as the poset of its C-compact elements (resp. subterminals).*

(b) If all the Grothendieck topologies $J_{\mathcal{C}}$ associated with structures $\mathcal{C}$ in a certain class $\mathcal{K}$ are C-induced for a given invariant C (relatively to a property P as in Definition 4.1) then the frames of the form $Id_{J_{\mathcal{C}}}(\mathcal{C})$ can be intrinsically characterized as the frames F with a basis B_F of C-compact elements which, regarded

as a poset with the induced order, belongs to $\mathcal{K}$, and such that the embedding $B_F \hookrightarrow F$ satisfies property P, the property that every covering in F of an element of B_F is refined by a covering made of elements of B_F which satisfies the invariant C, and the property that the J_{B_F}-covering sieves are sent by the embedding $B_F \hookrightarrow F$ into covering families in F (where J_{B_F} is the Grothendieck topology with which B_F comes equipped as a structure in $\mathcal{K}$).

Applying Theorem 4.2 in connection with the general theorems on geometric morphisms of toposes recalled above, one can easily generate a great number of Stone-type dualities following the methodology outlined above.

For obtaining dualities with categories of topological spaces rather than locales/frames, one can use the following construction, which provides a canonical way for endowing a given set of points of a topos with a natural topology. Recall that, given a point p of a topos $\mathcal{E}$, the image under p^* of a subterminal object of $\mathcal{E}$ is a subterminal object of $1_{\mathbf{Set}} = \{*\}$, in other words it is either the singleton or the emptyset.

Definition 4.3. Let $\xi : X \to P$ be an indexing of a set P of points of a Grothendieck topos $\mathcal{E}$ by a set X. The *subterminal topology* $\tau_\xi^{\mathcal{E}}$ is the image of the function $\phi_{\mathcal{E}} : \mathrm{Sub}_{\mathcal{E}}(1) \to \mathscr{P}(X)$ (where $\mathrm{Sub}_{\mathcal{E}}(1)$ is the frame of subterminal objects in $\mathcal{E}$) given by

$$\phi_{\mathcal{E}}(u) = \{x \in X \mid \xi(x)^*(u) \cong 1_{\mathbf{Set}}\}\ .$$

We denote the space X endowed with the topology $\tau_\xi^{\mathcal{E}}$ by $X_{\tau_\xi^{\mathcal{E}}}$.

If $\mathcal{E}$ is a localic topos and ξ indexes *all* the points of $\mathcal{E}$ (up to isomorphism) then the space $X_{\tau_\xi^{\mathcal{E}}}$ is called the *space of points* of $\mathcal{E}$.

Indeed, the following proposition shows that in order to obtain a topological duality or equivalence from a localic one, it suffices to make a functorial choice of a separating set of points of the toposes $\mathbf{Sh}(\mathcal{C}, J)$ and $\mathbf{Sh}(\mathcal{D}, K)$ (in a sense which is made precise in [18]). Recall that a set of points of a topos is said to be *separating* if their inverse image functors are jointly conservative.

Proposition 4.4. *If P is a separating set of points for $\mathcal{E}$ (for example, the set of points of a localic topos having enough points) then the frame $\mathcal{O}(X_{\tau_\xi^{\mathcal{E}}})$ of open sets of $X_{\tau_\xi^{\mathcal{E}}}$ is isomorphic to* $\mathrm{Sub}_{\mathcal{E}}(1)$ *(via $\phi_{\mathcal{E}}$).*

Remark 4.5. *Any* topological space X is homeomorphic to the space $X_{\tau_\xi^{\mathbf{Sh}(X)}}$, where ξ is the canonical indexing of the points of $\mathbf{Sh}(X)$ given by the points of X.

A topological space X is said to be *sober* if every irreducible closed subset is the closure of a unique point. Sober spaces can also be characterized as the topological spaces X such that the canonical map $X \to X_{\tau_\chi^{\mathbf{Sh}(X)}}$, where χ is the indexing of the set of *all* points of $\mathbf{Sh}(X)$ is a homeomorphism (cf. Theorem 2.10 [18]). Affine algebraic varieties with the Zariski topology are sober spaces (cf. the topos-theoretic interpretation of the Zariski topology below).

The interest of the notion of subterminal topology lies in its level of generality and its formulation as a topos-theoretic invariant admitting natural site characterizations, as shown by the following theorem.

Definition 4.6. Let $(C, \leq)$ be a preorder and J a Grothendieck topology on it. A *J-prime filter* on C is a subset $F \subseteq C$ such that F is non-empty, $a \in F$ implies $b \in F$ whenever $a \leq b$, for any $a, b \in F$ there exists $c \in F$ such that $c \leq a$ and $c \leq b$, and for any J-covering sieve $\{a_i \to a \mid i \in I\}$ in C if $a \in F$ then there exists $i \in I$ such that $a_i \in F$.

Theorem 4.7 ([18]). *Let C be a preorder and J be a Grothendieck topology on it. Then the space $X_{\tau_{\mathbf{Sh}(C,J)}}$ has as set of points the collection $\mathcal{F}^J_C$ of the J-prime filters on C and as open sets the sets the form*

$$\mathcal{F}_I = \{F \in \mathcal{F}^J_C \mid F \cap I \neq \emptyset\},$$

where I ranges among the J-ideals on C. In particular, a sub-basis for this topology is given by the sets

$$\mathcal{F}_c = \{F \in \mathcal{F}^J_C \mid c \in F\},$$

where c varies among the elements of C.

Here are some examples of 'subterminal topologies':

- The *Alexandrov topology*: taking $\mathcal{E} = [\mathcal{P}, \mathbf{Set}]$, where $\mathcal{P}$ is a preorder and ξ is the indexing of the set of points of $\mathcal{E}$ corresponding to the elements of $\mathcal{P}$, we obtain the Alexandrov topological space $\mathcal{A}_{\mathcal{P}}$ associated with $\mathcal{P}$.

- The *Stone topology for distributive lattices*: taking $\mathcal{E} = \mathbf{Sh}(\mathcal{D}, J^{\mathrm{coh}}_{\mathcal{D}})$ where $\mathcal{D}$ is a distributive lattice and ξ is an indexing of the set of all the points of $\mathcal{E}$, we obtain the space of prime filters on D.

- A *topology for meet-semilattices*: taking $\mathcal{E} = [\mathcal{M}^{\mathrm{op}}, \mathbf{Set}]$ for a meet-semilattice $\mathcal{M}$ and ξ is an indexing of the set of all the points of $\mathcal{E}$, we obtain the space of filters on $\mathcal{M}$.

- The *space of points of a locale L* is obtained by taking $\mathcal{E} = \mathbf{Sh}(L)$ and ξ to be an indexing of the set of all the points of $\mathcal{E}$.

- A *logical topology* on a set $\mathcal{S}$ of set-based models of a geometric theory: taking $\mathcal{E}$ equal to the classifying topos $\mathbf{Sh}(C_{\mathbb{T}}, J_{\mathbb{T}})$ of a geometric theory $\mathbb{T}$ and ξ any indexing of the points in $\mathcal{S}$, we obtain a space whose points are the models of $\mathbb{T}$ in $\mathcal{S}$ and whose open sets are given by the subsets $\{M \in \mathcal{S} \mid M \vDash \phi\}$ for a geometric sentence ϕ over the signature of $\mathbb{T}$.

- The *Zariski topology* on the set of prime filters of a commutative ring with unit A is recovered by taking in the previous point $\mathbb{T}$ equal to the theory $\mathbb{P}_A$ of prime filters on A defined as follows. The signature of $\mathbb{P}_A$ consists of a propositional symbol P_a for each element $a \in A$, and the axioms of $\mathbb{P}_A$ are the following:

$$(\top \vdash P_{1_A});$$

$$(P_{0_A} \vdash \bot);$$

$$(P_{a \cdot b} \dashv\vdash P_a \wedge P_b)$$

for any a, b in A;

$$(P_{a+b} \vdash P_a \vee P_b)$$

for any $a, b \in A$.

The models of $\mathbb{P}_A$ in **Set** are precisely the prime filters on A, that is the subsets S of A such that the complement $A \setminus S$ is a prime ideal. If, instead of taking the theory $\mathbb{P}_A$ of prime filters, we had considered the propositional theory of prime ideals (axiomatized over the same signature in the obvious way), we would have obtained a classifying topos inequivalent to the small Zariski topos of A, in spite of the fact that the two theories have the same models in **Set**.

The classifying topos of $\mathbb{P}_A$ is precisely the small Zariski topos of A.

The fact that many different 'concrete' dualities or equivalences can be generated by applying our machinery is due to the generality of the method (which in fact goes well beyond the setting of preordered categories) and to the degrees of freedom implicit in it: the choice of the structures $\mathcal{C}$, that of the structures $\mathcal{D}$, that of the topologies J and K, that of the points of the toposes $\mathbf{Sh}(\mathcal{C}, J)$ and $\mathbf{Sh}(\mathcal{D}, K)$ and even that of the way for functorializing the given Morita-equivalences.

Let us discuss, by way of illustration, how two classical dualities can be naturally recovered by applying our machinery, and a few examples of new dualities obtained through it.

Stone duality between the category of distributive lattices and that of coherent spaces is obtained by functorializing the Morita-equivalences of the form

$$\mathbf{Sh}(D, J_D^{\mathrm{coh}}) \simeq \mathbf{Sh}(X_D),$$

where D is any distributive lattice and X_D is the Stone space associated with D. Indeed, the morphisms $D \to D'$ of distributive lattices are precisely the morphisms of sites $(D, J_D^{\mathrm{coh}}) \to (D', J_{D'}^{\mathrm{coh}})$, and any distributive lattice D can be recovered from $\mathbf{Sh}(D, J_D^{\mathrm{coh}})$ as the lattice of its compact subterminals; accordingly, the arrows in the target category are the continuous maps between coherent spaces whose inverse image send compact open sets to compact open sets. The space X_D is the space of points of the locale $Id_{J_D^{\mathrm{coh}}}(D)$ of ideals of D. As predicted by Theorem 4.2, the coherent spaces are precisely the sober topological spaces with a basis of compact open sets which forms a distributive lattice (equivalently, with a basis of compact open sets which is closed under finite intersections).

Lindenbaum-Tarski duality between the category of sets and the category of complete atomic Boolean algebras and frame homomorphisms between them which preserve arbitrary infima can be obtained by functorializing the Morita-equivalences of the form

$$[A, \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{P}(A)),$$

where A is any set and $\mathscr{P}(A)$ is the powerset of A, or of the form

$$\mathbf{Sh}(B) \simeq \mathbf{Sh}(\mathrm{At}(B)),$$

where B is any complete atomic Boolean algebra and $\mathrm{At}(B)$ is the set of its atoms. Here B is viewed as a frame and equipped with the canonical topology, with respect to which the full subcategory $\mathrm{At}(B)$ of B is dense (by definition of atomic frame).

Notice that any map of sets $f : A \to B$ induces a frame homomorphism $\mathscr{P}(f) : \mathscr{P}(B) \to \mathscr{P}(A)$ which preserves arbitrary infima. A geometric morphism $[A, \mathbf{Set}] \to [B, \mathbf{Set}]$ is of the form $E(f)$ for some map $f : A \to B$ (resp. a frame homomorphism $\mathscr{P}(B) \to \mathscr{P}(A)$ is of the form $\mathscr{P}(f)$ for some map $f : A \to B$) if and only if it is essential (resp. it admits a left adjoint or, equivalently by the Adjoint Functor Theorem, it preserves arbitrary infima). The invariant notion of essential geometric morphism thus provides a most abstract explanation for the fact that one has to take frame homomorphisms which preserve arbitrary infima as arrows between complete atomic Boolean algebras in order to obtain a duality with the category of sets.

Among the new dualities or equivalences that we obtain by means of our machinery we mention the following:

- A duality between the category of meet-semilattices and meet-semilattices homomorphisms betweeen them and the category of locales whose objects are the locales with a basis of supercompact elements which is closed under finite meets and whose arrows are the locale maps whose associated frame homomorphisms send supercompact elements to supercompact elements.

- A duality between the category of frames with a basis of supercompact elements and complete homomorphisms between them and the category of posets (endowed with the Alexandrov topology).

  This duality restricts to Lindenbaum-Tarski duality.

- A duality between the category of disjunctively distributive lattices and the category whose objects are the sober topological spaces which have a basis of disjunctively compact open sets which is closed under finite intersection and satisfies the property that any covering of a basic open set has a disjunctively compact refinement by basic open sets and whose arrows are the continuous maps between such spaces such that the inverse image of any disjunctively compact open set is a disjunctively compact open set.

- For any regular cardinal k, a duality between the category of k-frames and the category whose objects are the frames which have a basis of k-compact elements which is closed under finite meets and whose arrows are the frame homomorphisms between them which send k-compact elements to k-compact elements.

- A duality between the category of disjunctive frames and the category $\mathbf{Pos}_{dis}$ which has as objects the posets $\mathcal{P}$ such that for any $a, b \in \mathcal{P}$ there exists a family $\{c_i \mid i \in I\}$ of elements of $\mathcal{P}$ such that for any $p \in \mathcal{P}$, $p \leq a$ and $p \leq b$ if and only if $p \leq c_i$ for a unique $i \in I$ and as arrows $\mathcal{P} \to \mathcal{P}'$ the monotone maps $g : \mathcal{P} \to \mathcal{P}'$ such that for any $b \in \mathcal{P}'$ there exists a family $\{c_i \mid i \in I\}$ of elements of $\mathcal{P}$ such that for any $p \in \mathcal{P}$, $g(p) \leq b$ if and only if $p \leq c_i$ for a unique $i \in I$.

- A duality between the category **DirIrrPFrm** of directedly generated preframes whose objects are the directedly generated preframes and whose arrows $\mathcal{D} \to \mathcal{D}'$ are the preframe homomorphisms $f : \mathcal{D} \to \mathcal{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathcal{D}}}(\mathcal{D}) \to Id_{J'_{\mathcal{D}}}(\mathcal{D}')$ which sends an ideal I of $\mathcal{D}$ to the ideal of $\mathcal{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category $\mathbf{Pos}_{dir}$ having as objects the posets $\mathcal{P}$ such that for any $a, b \in \mathcal{P}$ there is $c \in \mathcal{P}$ such that $c \leq a$ and $c \leq b$ and for any elements $d, e \in \mathcal{P}$ such that $d, e \leq a$ and $d, e \leq b$ there exists $z \in \mathcal{P}$ such that $z \leq a$, $z \leq b$, $d, e \leq z$, and as arrows $\mathcal{P} \to \mathcal{P}'$ the monotone maps $g : \mathcal{P} \to \mathcal{P}'$ with the property that for any $b \in \mathcal{P}'$ there exists $a \in \mathcal{P}$ such that $g(a) \leq p$ and for any two $u, v \in \mathcal{P}$ such that $g(u) \leq b$ and $g(v) \leq b$ there exists $z \in \mathcal{P}$ such that $u, v \leq z$ and $g(z) \leq b$.

  This duality restricts to the well-known duality between algebraic lattices sup-semilattices.

- An equivalence between the category of meet-semilattices and the category whose objects are the meet-semilattices F with a bottom element 0_F which have the property that for any $a, b \in F$ with $a, b \neq 0$, $a \wedge b \neq 0$ and whose arrows are the meet-semilattice homomorphisms $F \to F'$ which send 0_F to $0_{F'}$ and any non-zero element of F to a non-zero element of F'.

- A duality between the category **IrrDLat** whose objects are the irreducibly generated distributive lattices and whose arrows $\mathcal{D} \to \mathcal{D}'$ are the distributive lattices homomorphisms $f : \mathcal{D} \to \mathcal{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathcal{D}}}(\mathcal{D}) \to Id_{J'_{\mathcal{D}}}(\mathcal{D}')$ which sends an ideal I of $\mathcal{D}$ to the ideal of $\mathcal{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category $\mathbf{Pos}_{comp}$ whose objects are the posets and whose arrows $\mathcal{P} \to \mathcal{P}'$ are the monotone maps $g : \mathcal{P} \to \mathcal{P}'$ such that for any $q \in \mathcal{P}'$, there exists a finite family $\{a_k \mid k \in K\}$ of elements of $\mathcal{P}$ such that for any $p \in \mathcal{P}$, $g(p) \leq q$ if and only if $p \leq a_k$ for some $k \in K$.

  This duality restricts to *Birkhoff duality*.

- A duality between the category **AtDLat** whose objects are the atomic distributive lattices and whose arrows $\mathcal{D} \to \mathcal{D}'$ are the distributive lattices homomorphisms $f : \mathcal{D} \to \mathcal{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathcal{D}}}(\mathcal{D}) \to Id_{J'_{\mathcal{D}}}(\mathcal{D}')$ which sends an ideal I of $\mathcal{D}$ to the ideal of $\mathcal{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category $\mathbf{Set}_f$ whose objects are the sets and whose arrows $A \to B$ are the functions $f : A \to B$ such that the inverse image under f of any finite subset of B is a finite subset of A.

Let us explain, by way of illustration, how the duality between **DirIrrPFrm** and $\mathbf{Pos}_{dir}$ is obtained. Given a preframe $\mathcal{D}$, we say that an element $d \in \mathcal{D}$ is *directedly irreducible* if any directed sieve on d is maximal, i.e. if for any directed family $\{d_i \mid i \in I\}$ of elements of $\mathcal{D}$ such that $d = \bigvee\limits_{i \in I} d_i$ there exists $i \in I$ such that $d = d_i$; given a preframe $\mathcal{D}$, we denote by $\mathrm{DirIrr}(\mathcal{D})$ the poset of directedly irreducible elements of $\mathcal{D}$. We shall call the preframes in which every element is a directed join of directedly irreducible elements the *directedly generated* preframes. For any directedly generated preframe $\mathcal{D}$, the Comparison Lemma yields an equivalence of toposes

$$\mathbf{Sh}(\mathcal{D}, J_{\mathcal{D}}^{\mathrm{dir}}) \simeq [\mathrm{DirIrr}(\mathcal{D})^{\mathrm{op}}, \mathbf{Set}] \ .$$

The invariant $C =$‘to be a directed family’ satisfies the hypotheses of Theorem 4.2 for each directedly generated preframe $\mathcal{D}$, each of which equipped with the directed topology $J_{\mathcal{D}}$. We functorialize the above equivalences contravariantly on the side of preframes using morphisms of sites and covariantly on the side of their sets of directedly irreducible elements using essential geometric morphisms. In this way, we obtain an equivalence between the category **DirIrrPFrm** defined above and the category whose objects are the posets $\mathcal{P}$ such that the subset $Id_{dir}(\mathcal{P})$ of C-compact elements of $Id(\mathcal{P})$ is closed in $Id(\mathcal{P})$ under finite meets and whose arrows $\mathcal{P} \to \mathcal{P}'$ are the monotone maps $g : \mathcal{P} \to \mathcal{P}'$ between them such that the inverse image $g^{-1} : Id(\mathcal{P}') \to Id(\mathcal{P})$ sends ideals in $Id_{dir}(\mathcal{P}')$ to ideals in $Id_{dir}(\mathcal{P})$. It is not hard to see that the latter category coincides with the category $\mathbf{Pos}_{dir}$ defined above.

It should be noted that all the Stone-type dualities or equivalences generated through our machinery have essentially the same level of ‘mathematical depth’ as the classical Stone duality. Still, the technical sophistication of the different dualities can vary substantially from one to the other; compare for instance the duality for directedly irreducible preframes described above with the classical Stone duality. Whilst the latter was discovered by Stone without the help of toposes, it can be argued that without the insight provided by toposes one could have hardly discovered the former duality, just because of its higher technical sophistication; interestingly, its restriction to the categories of algebraic lattices and of sup-semilattices had been found before (without using toposes) but the description of the categories and functors involved in this restricted duality is much simpler than in the general case.

Quite remarkably, even when we recover classical dualities by applying our machinery, the description of the categories or the functors involved can be different from (and even simpler than) the classical one, reflecting the peculiar nature of our methodology. As an example, take Moshier and Jipsen's topological duality for meet-semilattices established in [66]. Our duality for meet-semilattices is based on the equivalence

$$[\mathcal{M}^{\mathrm{op}}, \mathbf{Set}] \simeq \mathbf{Sh}(X_{\mathcal{M}})$$

holding for each meet-semilattice $\mathcal{M}$ naturally in $\mathcal{M}$, where $X_{\mathcal{M}}$ is the topological space obtained by putting the subterminal topology on the set of points of the topos $[\mathcal{M}^{\mathrm{op}}, \mathbf{Set}]$, namely the space of *filters* on $\mathcal{M}$. Since $[\mathcal{M}^{\mathrm{op}}, \mathbf{Set}] \simeq \mathbf{Sh}(Id_{\mathcal{M}})$, the meet-semilattice $\mathcal{M}$ can be recovered from the locale $Id_{\mathcal{M}}$ (resp. from the space $X_{\mathcal{M}}$) as the subset of supercompact (i.e., not admitting any non-trivial covering in the frame) elements in it (resp. as the subset of supercompact open sets of $X_{\mathcal{M}}$). In [66] instead, the meet-semilattice $\mathcal{M}$ is recovered from $X_{\mathcal{M}}$ as the set of its compact open sets which satisfy a certain filtering property with respect to the specialization order.

In [18], we also establih a number of other results on the general theme of Stone-type dualities, including adjunctions between categories of preorders and categories of frames or locales, for instance between meet-semilattices (resp. distributive lattices, preframes, Boolean algebras) and frames. We also apply the methodology of 'toposes as bridges' for translating properties of preordered structures into properties of the corresponding posets, locales or topological spaces; an example of such transfers, involving the Alexandrov space $\mathcal{A}_{\mathcal{P}}$ associated with a preorder $\mathcal{P}$, was discussed in section 3.1 above.

The key feature of toposes that we have exploited in the context of this investigation is the fact that they allow one to embody in a single object the abstract (i.e., categorically invariant) relationships existing between different pairs of structures, in this case between structures related by a density condition (in the sense of the Comparison Lemma). Let us illustrate this point by discussing the case of a topological space X and a basis $\mathcal{B}$ for it. Concretely, the set $O(X)$ of open sets of a topological space X (and hence the space X itself, if it is sober) can be reconstructed from any basis $\mathcal{B}$, regarded concretely as a subset of $\mathscr{P}(X)$, by taking the subsets of X which are unions of subsets in $\mathcal{B}$. Abstractly, that is regarding $\mathcal{B}$ as a preorder category, a further ingredient is necessary for recovering $O(X)$ from $\mathcal{B}$: this ingredient is precisely the Grothendieck topology $J^{\mathrm{can}}_{O(X)}|_{\mathcal{B}}$ induced by the canonical topology $J^{\mathrm{can}}_{O(X)}$ on $O(X)$. Indeed, by the Comparison Lemma we have an equivalence of toposes

$$\mathbf{Sh}(X) \simeq \mathbf{Sh}(\mathcal{B}, J^{\mathrm{can}}_{O(X)}|_{\mathcal{B}})$$

or, equivalently, an isomorphism of frames

$$O(X) \cong Id_{J^{\mathrm{can}}_{O(X)}|_{\mathcal{B}}}(\mathcal{B}) \ .$$

The most interesting situations, that is those which give rise to genuine representation theorems, are those in which the induced topology $J^{\mathrm{can}}_{O(X)}|_{\mathcal{B}}$ admits an intrinsic description in terms of the categorical structure on $\mathcal{B}$. Indeed, in such a situation one can recontruct the space X from $\mathcal{B}$ alone, without the need of any additional *datum*. Notice that this is for instance the case of Stone duality: given a coherent space X, the induced topology on the basis for X formed by the compact open sets, which is a distributive lattice, is equal to the coherent topology on it. We shall encounter other situations of this kind in section 4.1.4 below.

The possibility of embodying abstract relationships by means of equivalences between two different representations of the same topos has great conceptual and technical power. Indeed, it allows a structural, and hence most canonical, investigation of the relationships existing between the different structures related by the equivalence founded on the duality between toposes and their sites of definition.

Our general topos-theoretic framework for Stone-type dualities can also be used for generating dualities for more complex algebraic or topological structures through the identification of appropriate topos-theoretic invariants; by way of illustration, we discuss the topos-theoretic generation of analogues of Priestley duality for distributive lattices for other kinds of partially ordered structures in section 4.1.3 below. It can also be used for establishing completeness theorems for propositional logics whose syntactic sites are preorded sites to which our theory applies.

The last part of [18] is devoted to the problem of constructing partially ordered structures presented by generators and relations. As we have already explained in section 2.1.3, the theory of syntactic categories can be systematically used for addressing this kind of problems; in fact, we show in [18] that a large class of structures (called 'ordered infinitary Horn theories') presented by generators and relations can be realized as syntactic categories of suitable theories. The interest of the technique of toposes as 'bridges' in this context lies in the fact that for a theory belonging to a fragment of geometric logic any different representation of its classifying topos will provide a different description of its syntactic category as a full subcategory of it.

For instance, given a commutative ring with unit $(A, +, \cdot, 0_A, 1_A)$, the distributive lattice $L(A)$ generated by symbols $D(a)$, $a \in A$, subject to the relations $D(1_A) = 1_{L(A)}$, $D(a \cdot b) = D(a) \wedge D(b)$, $D(0_A) = 0_{L(A)}$, and $D(a + b) \leq D(a) \vee D(b)$ can be characterized (up to isomorphism) as the lattice of compact elements of the frame of open sets of the prime spectrum of A endowed with the Zariski topology since it coincides with the coherent syntactic category of the theory $\mathbb{P}_A$ of prime filters on A introduced above. Indeed, we have an equivalence

$$\mathbf{Sh}(L(A), J^{\mathrm{coh}}_{L(A)}) \simeq \mathbf{Sh}(\mathrm{Spec}(A)) .$$

The topos $\mathbf{Sh}(\mathrm{Spec}(A))$ also admits alternative representations reflecting the different points of view that one can have on the Zariski spectrum; for instance, we have an equivalence

$$\mathbf{Sh}(\mathrm{Spec}(A)) \simeq \mathbf{Sh}(\mathrm{Rad}(A), J_A),$$

where $\mathrm{Rad}(A)$ is the frame of radical ideals of A (with the subset-inclusion ordering) and J_A is the canonical topology on it, and an equivalence

$$\mathbf{Sh}(\mathrm{Spec}(A)) \simeq \mathbf{Sh}(S(A), C),$$

where $S(A)$ is the meet-semilattice given by the quotient of the underlying monoid of A by the smallest congruence which identifies a and a^2 for all elements a of A, with the order given by $[a] \leq [b]$ if and only if $a \cdot b = a$ (where $[a]$ denotes the equivalence class of a in $S(A)$), while C_A is the Grothendieck topology on $S(A)$ generated by the covering families of the form $\emptyset \in C_A([0_A])$ and of the form $[a_i] \to [a]$ (for $i = 1, \dots, n$), where $[a_i] \leq [a]$ and $[a_1 + \cdots + a_n] = [a] = x$ (cf. section 8.7 of [18] and section V3.1 of [50]).

Another example is provided by the construction of the free frame $F(A)$ on a complete join-semilattice A. This can be built as the geometric syntactic category of the theory $\mathbb{L}_A$ defined as follows: the signature Σ_A of $\mathbb{L}_A$ consists of one 0-ary relation symbol F_a for each element $a \in A$, and the axioms of $\mathbb{L}_A$ are, besides those of geometric logic, all the sequents over Σ_A of the form

$$(\bigvee_{i \in I} F_{a_i} \dashv\vdash F_a)$$

for any family of elements $\{a_i \mid i \in I\}$ in A such that $a = \bigvee_{i \in I} a_i$ in A, and of the form

$$(F_a \vdash F_b)$$

for any elements $a, b \in A$ such that $a \leq b$.

This preorder category is the frame of subterminal objects of the classifying topos of $\mathbb{L}_A$, so any representation $\mathbf{Sh}(\mathcal{C}, J)$ of this classifying topos will yield a representation of $F(A)$ as the frame of J-ideals on $\mathcal{C}$. One can take for instance $\mathcal{C}$ to be the opposite of the category of finitely presentable models of the empty cartesian theory over the signature of $\mathbb{L}_A$ (which can be identified with the set $\mathscr{P}_{\mathrm{fin}}(A)$ of finite subsets of A) and J to be the Grothendieck topology associated with $\mathbb{L}_A$ as a quotient of this theory, or choose any other representation to obtain an alternative description of the frame $F(A)$.

4.1.2 A general method for building reflections

The notion of adjunction between two categories represents a natural weakening of that of categorical equivalence; indeed, any adjunction restricts to a categorical equivalence between the subcategories of fixed points of the two adjoint functors. An adjunction is said to be a *reflection* (resp. a coreflection) is the counit (resp. the unit) is an isomorphism, equivalently if the right (resp. left) adjoint functor is full and faithful. Reflections and coreflections are dual to each other, and they suffice to generate all adjunctions since, as it is shown in [57], any adjunction can be canonically obtained by composing a coreflection with a reflection by means of the comma category construction.

We have seen in section 4.1.1, reviewing the contents of paper [18], that many Stone-type dualities or equivalences between categories of preorders and categories of posets, locales and topological spaces can be obtained from the process of appropriately 'functorializing' families of categorical equivalences between toposes. Since Stone-type dualities or equivalences normally extend to adjunctions between larger categories, it is natural to wonder whether such adjunctions can also be obtained starting from relationships between the toposes associated with the structures as in the theory of [18] which are looser than that of categorical equivalence. Paper [20] provides a positive answer to this question, by showing that families of geometric morphisms between those toposes satisfying a number of natural conditions are liable to generate reflections extending the given dualities or equivalences.

For example, Stone's adjunction between the category of Boolean algebras and that of topological spaces can be obtained starting from the canonical geometric morphisms

$$\mathbf{Sh}(X) \to \mathbf{Sh}(B_X, J^{\mathrm{coh}}_{B_X}),$$

where B_X is the Boolean algebra of clopen sets of X, while Alexandrov's equivalence between the category of preorders and that of topological spaces can be obtained starting from the canonical geometric morphisms

$$[X_{\leq}, \mathbf{Set}] \to \mathbf{Sh}(X),$$

where $X_{\leq}$ is the preorder obtained by equipping the set of points of a topological space X with the specialization pre-ordering.

We develop our technique for generating reflections in full generality, implementing the idea that adjunctions between a given pair of categories can be naturally generated starting from a pair of functors from each of the two categories into a third one together with some relationships between them. Our method is complete, in the sense that *any* reflection between categories can be obtained as an application of it, but its main interest lies in its inherent technical flexibility; indeed, it happens very often in practice that two different categories are best understood in relation with each other from the point of view of a third category to which both are related (cf. the philosophy of toposes as 'bridges' or the concept of comma category [57]).

Our method proceeds in a 'top-down' way starting from a family of arrows (satisfying appropriate conditions) which express relations between the 'realizations' of objects and arrows of two given categories in a 'bridge category' to which both of them map: the unit and counit are directly derived from the arrows in the family, and the categories and functors yielding the reflection are equally built from the given family through a general formal procedure.

4.1.3 Priestley-type dualities for partially ordered structures

In this section, which is based on [21], we briefly review a general method, based on the 'bridge' technique, for building natural analogues of the classical Priestley duality for distributive lattices for other kinds of partially ordered structures.

Recall that Priestley duality for distributive lattices (cf. [69] and [70]) is a duality between the category of distributive lattices and the category of *Priestley spaces*. Via this duality, a distributive lattice D corresponds to the ordered topological space P_D obtained by equipping the set $\mathcal{F}_D$ of prime filters on D with the *patch topology* (i.e., the topology having as a sub-basis the collection of the sets of the form $\{P \in \mathcal{F}_D \mid d \in P\}$ for $d \in D$ and their complements) and the specialization order $\leq$ on $\mathcal{F}_D$ with respect to the coherent topology on $\mathcal{F}_D$ (i.e., the topology having as a basis the collection of sets of the form $\{P \in \mathcal{F}_D \mid d \in P\}$): notice that this order is precisely the subset-inclusion one. The assignment $D \to P_D$ can be made functorial as follows: any morphism $D \to D'$ of distributive lattices induces an order-preserving continuous map $P_{D'} \to P_D$. We thus have a functor $P : \mathbf{DLat} \to \mathbf{PTop}$, to which we shall refer as the *Priestley functor*, where **DLat** is the category of distributive lattices and **PTop** is the category of ordered topological spaces. Any distributive lattice D can be recovered from the associated Priestley space as the poset of clopen $\leq$-upper sets; in fact, this assignment defines a functor from the category of Priestley spaces to the opposite of the category of distributive lattices which yields the other half of Priestley duality. The ordered topological spaces $(X, \tau, \leq)$ which are, up to isomorphism, in the image of the Priestley functor are called *Priestley spaces*; notably, these spaces admit natural intrinsic topological characterizations as the compact ordered topological spaces satisfying the *Priestley separation axiom*: if for any $x, y \in X$ such that $x \nleq y$, there is a clopen $\leq$-upper set U of τ such that $x \in U$ and $y \notin U$.

Looking at Priestley duality from an algebraic viewpoint, we see that we can characterize the algebra of clopen subsets of the Priestley space associated to a distributive lattice D via Priestley duality as the free Boolean algebra on D.

In order to build natural 'Priestley-type' for partially ordered structures other than distributive lattices, we generalize both the topological and the algebraic viewpoint on the classical duality. As we have just remarked, topologically Priestley duality is based on the patch topology construction, while algebraically the Boolean algebra of clopen sets of the Priestley space associated to a distributive lattice can be characterized as the free Boolean algebra on it. The unification between the algebraic and the topological formulations of the duality is conveniently provided by the notion of topos; in fact, the toposes involved in Priestley-type dualities admit, on the one hand, an algebraic representation (as categories of sheaves on a preordered structure with respect to an appropriate Grothendieck topology on it) and on the other hand a topological one (as categories of sheaves on suitable spectra of these structures, as provided by the techniques of [18]).

Topologically, our 'Priestley-type' dualities are built by considering natural spectra for the given partially ordered structures, generating patch-type topologies from them and equipping the resulting spaces with the specialization ordering on the original spectra; algebraically, the dualities are obtained by assigning to any given ordered structure a Boolean algebra which is free on it (in an appropriate sense), equipped with a natural ordering on the points of its spectrum.

We discuss in [21] specific examples of dualities generated through this method, including 'Priestley-type' dualities for coherent posets, meet-semilattices and disjunctively distributive lattices.

For instance, our Priestley-type duality for meet-semilattices reads as follows. We have a categorical equivalence

$$B : \mathbf{MsLat}^{\mathrm{op}} \to \mathbf{PTop}_M$$

between the category **MsLat** of meet-semilattices and a subcategory $\mathbf{PTop}_M$ of the category of Priestley spaces.

Given a meet-semilattice $\mathcal{M}$, the Priestley space $B(\mathcal{M})$ is the ordered topological space whose underlying set is the collection $X_{\mathcal{M}}$ of all the filters on $\mathcal{M}$, endowed with the topology generated by the sets of the form $\{F \in X_{\mathcal{M}} \mid m \in F\}$ and their complements in $\mathscr{P}(X_{\mathcal{M}})$, and with the order $\leq_{\mathcal{M}}$ defined as follows: for any $F, F' \in X_{\mathcal{M}}$, $F \leq_{\mathcal{M}} F'$ if and only if $F \subseteq F'$. Given a meet-semilattice homomorphism $f : \mathcal{M} \to \mathcal{N}$, $B(f) : X_{\mathcal{N}} \to X_{\mathcal{M}}$ is the map sending any filter F in $X_{\mathcal{N}}$ to the filter in $\mathcal{M}$ given by the inverse image $f^{-1}(F)$.

Given a Priestley space, we call a clopen upper set which cannot be decomposed as a proper union of clopen upper sets *weakly indecomposable*.

The subcategory $\mathbf{PTop}_M$ can be characterized as the category of ordered topological spaces whose objects are the Priestley spaces $(X, \tau, \leq)$ such that for any $x, y \in X$ with $x \nleq y$, there is a weakly indecomposable clopen $\leq$-upper set U of τ with the property that $x \in U$ and $y \notin U$, and such that the intersection of any two weakly indecomposable clopen $\leq$-upper set is weakly indecomposable, and whose arrows are the continuous order-preserving maps between them such that the inverse image of any weakly indecomposable upper clopen set is weakly indecomposable.

Any meet-semilattice $\mathcal{M}$ can be recovered from the associated Priestley space $B(\mathcal{M})$ as the set of weakly indecomposable clopen $\leq$-upper sets.

Algebraically, the algebra of clopen subsets of the Priestley space $B(\mathcal{M})$ can be identified with the free Boolean algebra on the meet-semilattice $\mathcal{M}$ (for any $\mathcal{M} \in$ **MsLat**).

In [21] we also establish a link between the construction of free structures and that of Stone-type or Priestley-type dualities. Note that in the classical Priestley duality one recovers a distributive lattice from the free Boolean algebra on it equipped with a partial order on its spectrum (as the set of its upper elements with respect to this order), while in the classical Stone duality one recovers a distributive lattice from the free frame on it (as the set of its compact elements). In fact, all the Priestley-type dualities established by means of the method of [21] arise, if viewed algebraically, from the construction of free Boolean algebras on particular kinds of posets, while the Stone-type dualities established in [18] arise from the construction of free frames or posets on preordered structures.

A key result in this respect which is proved in [21] concerns the spatial realization of these free structures as substructures of powerset structures. Notice that the free Boolean algebra on a distributive lattice D can be realized as the sub-Boolean algebra of the powerset $\mathscr{P}(X_D)$, where X_D is the Stone spectrum of D, generated by $D \rightarrowtail O(X_D) \hookrightarrow \mathscr{P}(X_D)$. To obtain an analogue of this result for general Priestley-type dualities, we treat the matter in full generality by introducing the notion of 'the free $(\mathcal{L}, \mathcal{M})$-structure on a preordered structure C' (where $\mathcal{L}$ is a category of first-order structures over a one-sorted signature Σ and $\mathcal{M}$ is a class of functions from C to structures in $\mathcal{L}$) as a structure equipped with an embedding i of C in it such that any function f in $\mathcal{M}$ can be uniquely extended via i to an arrow in $\mathcal{L}$.

We then prove a general theorem allowing us to identify, under appropriate assumptions including in particular the requirement that any powerset $\mathscr{P}(X)$ be naturally identified wtih a Σ-structure in $\mathcal{L}$ and for any function $f : X \to Y$, the function $\mathscr{P}(f) : \mathscr{P}(Y) \to \mathscr{P}(X)$ be an arrow in $\mathcal{L}$, the free $(\mathcal{L}, \mathcal{M})$-structure on C' as the Σ-substructure of $\mathscr{P}(X_C)$ generated by C (where X_C is a space associated with C as in Theorem 2.7 [21]). Notice that the problem of choosing a space X_C such that a free structure on C can be realized as a substructure of $\mathscr{P}(X_C)$ generated by C is quite delicate since in general the two constructions have nothing to do with each other. Take for example $C = \mathscr{P}(X)$; C, regarded as a meet-semilattice, coincides with the Boolean algebra generated by itself, but it is *not* the free Boolean algebra on it (as a meet-semilattice), since Boolean algebra homomorphisms of Boolean algebras do not in general coincide with meet-semilattice homomorphisms of their underlying meet-semilattices. Anyway, as shown by Theorem 2.7 [21], for a great variety of structures C there is a natural choice of a set X such that C can be embedded as a substructure of the powerset $\mathscr{P}(X)$ and the free $(\mathcal{L}, \mathcal{M})$-structure on C can be identified with the substructure of $\mathscr{P}(X)$ generated by C. The way X_C is constructed in [21] is by taking the space of points of a topos $\mathbf{Sh}(C, J_C)$, where J_C is a Grothendieck topology on C satisfying some natural assumptions.

As an illustration of this result, consider the problem of building the free distributive lattice $D_{\mathcal{M}}$ on a meet-semilattice $\mathcal{M}$. We show that we have an equivalence of toposes

$$\mathbf{Sh}(D_{\mathcal{M}}, J^{\mathrm{coh}}_{D_{\mathcal{M}}}) \simeq [\mathcal{M}^{\mathrm{op}}, \mathbf{Set}]$$

and that $D_{\mathcal{M}}$ can be realized as the sublattice of $\mathscr{P}(X_{\mathcal{M}})$ generated by $\mathcal{M} \hookrightarrow O(X_{\mathcal{M}}) \subseteq \mathscr{P}(X_{\mathcal{M}})$ (where $X_{\mathcal{M}}$ is the space of filters on $\mathcal{M}$ as defined in [18], cf. section 4.1.1 above).

The meet-semilattice $\mathcal{M}$ can be recovered from $D_{\mathcal{M}}$ as the set of elements of $D_{\mathcal{M}}$ which are indecomposable, in the sense that they cannot be written as a proper finite join of elements of $D_{\mathcal{M}}$. The free functor $\mathbf{MSLat} \to \mathbf{DLat}$ thus has an inverse defined on its essential image, which sends a distributive lattice D in it to the set of its indecomposable elements. This equivalence can be extended to a coreflection from $\mathbf{MSLat}$ to the subcategory of $\mathbf{DLat}$ whose objects are the distributive lattices whose set of indecomposable elements forms, with the induced order, a meet-semilattice, and whose arrows are the distributive lattice homomorphisms which send indecomposable elements to indecomposable elements; indeed, for any such distributive lattice D, denoting by $M_{\mathcal{D}}$ the collection of its indecomposable elements, we have a canonical geometric morphism

$$\mathbf{Sh}(\mathcal{D}, J_{\mathcal{D}}^{\mathrm{coh}}) \to [M_{\mathcal{D}}{}^{\mathrm{op}}, \mathbf{Set}] \; .$$

The essential image of the free functor $\mathbf{MSLat} \to \mathbf{DLat}$ can be characterized as the subcategory of $\mathbf{DLat}$ whose objects are the distributive lattices D such that any element can be written as a finite join of indecomposable elements and the meet of any two indecomposable elements is indecomposable and whose arrows are the distributive lattice homomorphisms between such lattices which send indecomposable elements to indecomposable elements. We thus have the following criterion for a meet-semilattice homomorphic inclusion $i : \mathcal{M} \hookrightarrow D$ of a meet-semilattice $\mathcal{M}$ into a distributive lattice: i realizes D as the free distributive lattice on $\mathcal{M}$ if and only if i realizes $\mathcal{M}$ as the set of indecomposable elements of D and every element of D can be written as a finite join of indecomposable elements.

4.1.4 Gelfand spectra and Wallman compactifications

In this section we briefly review the contents of paper [22]. In this work we carry out a systematic, topos-theoretically inspired, investigation of the notion of Wallman compactification with a particular emphasis on its relationships with Gelfand spectra and Stone-Čech compactifications. We show that the notion of Wallman base can serve in many contexts as a convenient tool for representing topological spaces, to the point of leading to useful dualities between notable categories of topological spaces, such as the category of T_1 compact spaces or that of compact Hausdorff spaces, and natural categories of distributive lattices. In fact, in addition to proving several specific results about Wallman bases and maximal spectra of distributive lattices, we establish a general framework for functorializing the representation of a topological space as the maximal spectrum of a Wallman base for it, which allows to generate different dualities between categories of topological spaces and subcategories of the category of distributive lattices; in particular, this leads to a categorical equivalence between the category of commutative C^*-algebras and a natural category of distributive lattices. We also establish a general theorem concerning the representation of the Stone-Čech compactification of a locale as a Wallman compactification, which subsumes the previous results obtained on this problem.

Let us start by describing our general method for building dualities for topological spaces by using Wallman bases.

Recall that a topological space X is said to be T_1 if for every pair of distinct points, each has a neighborhood not containing the other. A *Wallman base* B for a topological space X is a sublattice of the frame $\mathcal{O}(X)$ of open sets of X which is a base for X and such that for any $x \in X$ and $U \in B$ such that $x \in U$, there exists a $V \in B$ such that $x \notin V$ and $U \cup V = X$.

We have already observed in section 4.1.1 that by Grothendieck's Comparison Lemma for any topological space X and base B for it we have an equivalence of toposes

$$\mathbf{Sh}(X) \simeq \mathbf{Sh}(B, J^{\text{can}}_{O(X)}|_B),$$

and that if the topology $J^{\text{can}}_{O(X)}|_B$ can be characterized 'intrinsically' in terms of the partially ordered structure of B induced by the inclusion $B \subseteq O(X)$ then the space X admits an 'intrinsic' representation in terms of B which, if appropriately functorialized, can lead to a duality between a category of such spaces X and a category of such posets B. The notion of Wallman basis is particularly relevant in this respect since for many Wallman bases B of topological spaces X, the Grothendieck topology $J^{\text{can}}_{O(X)}|_B$ admits an intrinsic description in terms of the categorical structure on B. For example:

- If X is compact and B is a Wallman base for it then, under a form of the axiom of choice, the topology $J^{\text{can}}_{O(X)}|_B$ can be identified with a Grothendieck topology J^B_m intrinsically defined in terms of the lattice structure on B: its covering sieves $\{c_i \to c \mid i \in I\}$ are precisely those such that whenever $c \vee b = 1$ in B there exists a finite subset $J \subseteq I$ such that $\bigvee_{i \in J} c_i \vee b = 1$;

- In the particular case of the above point when X is Hausdorff and B is equal to the lattice $Coz(X)$ of cozero sets on X (recall that a cozero set is an open set of the form $f^{-1}(\mathbb{R} \setminus \{0\})$ for a continuous function $f : X \to \mathbb{R}$) the topology J^B_m even admits a further representation, as the Grothen-dieck topology on B whose covering sieves are those which contain countable covering families (in the usual set-theoretic sense);

- If X is a T_1 compact space and B is equal to $O(X)$ then the induced Grothendieck topology $J^{\text{can}}_{O(X)}|_B$ obviously coincides with the canonical one.

A well-known result states that if X is a compact space and B is a Wallman basis for it then X is homeomorphic to the maximal spectrum $\text{Max}(B)$ of B, whose points are the maximal ideals of B (regarded as a distributive lattice - recall that an ideal of a distributive lattice is a lower set which is closed under finite joins) and whose topology has as basis of open sets of the form $\{M \in \text{Max}(B) \mid b \notin M\}$ for an element $b \in B$.

This motivates the consideration of spaces of the form $\text{Max}(D)$ where D is a distributive lattice. One can see that these spaces are always T_1 and compact and that we have an equivalence

$$\mathbf{Sh}(\text{Max}(D)) \simeq \mathbf{Sh}(D, J^D_m),$$

equivalently, $O(\text{Max}(D)) \cong Id_{J^D_m}(D)$.

If one wants to drop the compactness condition, one can consider the spaces of maximal ideals $\text{Max}(F)$ of frames F (an ideal of a frame is a lowerset which is closed under arbitrary joins); the spaces of these form are always T_1 and we have an equivalence

$$\mathbf{Sh}(\text{Max}(F)) \simeq \mathbf{Sh}(F, J^F_f),$$

where the Grothendieck topology J_f^F is defined by saying that its covering sieves $\{c_i \to c \mid i \in I\}$ are precisely those such that whenever $c \vee b = 1$ in F, $\bigvee_{i \in I} c_i \vee b = 1$.

Note that if F is a compact frame then any maximal ideal of F considered as a distributive lattice is also a maximal ideal of F considered as a frame; in particular, we have an equality of Grothendieck topologies $J_m^F = J_f^F$.

More generally, one can define the notion of *maximal spectrum* $\mathrm{Max}(\mathcal{E})$ of a localic topos $\mathcal{E}$ as the subspace of the space of points of $\mathcal{E}$ (cf. Definition 4.3) which are minimal with respect to the specialization ordering, equivalently which are closed in the topology. Notice that this notion is a topos-theoretic invariant of $\mathcal{E}$. This invariant admits a natural behaviour with respect to sites; indeed, for any preorder site $(\mathcal{C}, J)$, one can give an explicit description of the maximal spectrum $\mathrm{Max}(\mathcal{C}, J)$ of the topos $\mathbf{Sh}(\mathcal{C}, J)$ as the space whose points are the J-ideals on $\mathcal{C}$ which are maximal with respect to the subset-inclusion ordering, and whose open sets are the subsets of the form $\{H \in \mathrm{Max}(\mathcal{C}, J) \mid I \nsubseteq H\}$ for a J-ideal I on $\mathcal{C}$. Since $\mathrm{Max}(\mathcal{C}, J)$ is a subspace of the space of points $\mathrm{Spec}(\mathcal{C}, J))$ of the topos $\mathbf{Sh}(\mathcal{C}, J)$, there exists a unique Grothendieck $M_J^{\mathcal{C}}$ on $\mathcal{C}$ refining J which makes the diagram

$$\begin{tikzcd}
\mathbf{Sh}(\mathrm{Spec}(\mathcal{C}, J)) \arrow[r, phantom, "\simeq"] & \mathbf{Sh}(\mathcal{C}, J) \\
\mathbf{Sh}(\mathrm{Max}(\mathcal{C}, J)) \arrow[u, hook] \arrow[r, phantom, "\simeq"] & \mathbf{Sh}(\mathcal{C}, M_J^{\mathcal{C}}) \arrow[u, hook]
\end{tikzcd}$$

commute. This topology admits a natural description in many cases of interest (cf. for instance the topologies J_m^D $(= M^D_{J^{\mathrm{coh}}_D})$ on distributive lattices D or the topologies J_f^F $(= M^F_{J^{\mathrm{can}}_F})$ on frames F considered above). One can therefore try to generate dualities involving maximal spectra by constructing and functorializing 'bridges' of the form

$$\begin{tikzcd}
& \mathbf{Sh}(\mathcal{C}, M_J^{\mathcal{C}}) \simeq \mathbf{Sh}(\mathrm{Max}(\mathcal{C}, J)) \arrow[dl, dashed, no head] \arrow[dr, dashed, no head] & \\
\mathcal{C} & & \mathrm{Max}(\mathcal{C}, J)
\end{tikzcd}$$

This is done in section 3 of [22] in the particular setting of distributive lattices; we prove a general duality theorem between an appropriate category of topological spaces each of which equipped with a Wallman base on it and a subcategory of the category of distributive lattices. This duality theorem is then applied to establish a duality for T_1 compact spaces and a duality between the category of compact Hausdorff spaces and a particular category of Alexandrov algebras.

The first duality is obtained by considering $B = \mathcal{O}(X)$ as a Wallman base for any T_1 compact space X and is based on a characterization of the frames D such that the canonical homomorphism $D \to \mathcal{O}(\mathrm{Max}(D)) \cong Id_{J_m^D}(D)$ is an isomorphism. This condition amounts precisely to requiring that $J_D^{\mathrm{can}} = J_m^D$. We call these frames *conjunctive*. The assignments $X \rightsquigarrow \mathcal{O}(X)$ and $D \rightsquigarrow \mathrm{Max}(D)$ thus define a duality between the category of compact T_1 spaces and continuous maps between them and the category whose objects are the compact conjunctive frames and whose arrows are the distributive lattice homomorphisms between them whose inverse image sends maximal ideals to maximal ideals (cf. Theorem 3.8 [22]). This duality actually extends to a duality between the category of T_1 spaces and continuous maps between them and the category of conjunctive frames and frame homomorphisms between them whose inverse image sends maximal ideals (of frames) to maximal ideals. Modulo the identification of maximal ideals with coatoms and of conjunctive frames with coatomistic ones (cf. Proposition 3.9 [22] and the discussion preceding it), this is the duality between T_1 spaces and coatomistic frames established in [63].

The second duality is obtained by considering the lattice of cozero sets $B = Coz(X)$ as a Wallman base for any compact Hausdorff space X and is based on a characterization of the distributive lattices D of the form $Coz(X)$ such that the space $\mathrm{Max}(D)$ is (compact and) Hausdorff. Recall from section [50] A *Alexandrov algebra* A is a distributive lattice (with bottom and top element) satisfying the following properties:

- A is *normal*, i.e. for every pair of elements $a_1, a_2 \in A$ such that $a_1 \vee a_2 = 1$, there exist elements $b_1, b_2 \in A$ such that $b_1 \wedge b_2 = 0$, $b_1 \vee a_2 = 1$ and $b_2 \vee a_1 = 1$;
- Countable joins exist in A and they distribute over finite meets;
- The following 'approximation property' holds: for any element $a \in A$ there exist countable sequences $\{b_n \mid n \in \mathbb{N}\}$ and $\{c_n \mid n \in \mathbb{N}\}$ such that $\bigvee\limits_{n\in\mathbb{N}} c_n = a$, $b_n \vee a = 1$ for all $n \in \mathbb{N}$ and $b_n \wedge c_n = 0$ for all $n \in \mathbb{N}$.

We show, by building on previously known results for completely regular spaces, that the lattices of cozero sets of compact Hausdorff spaces are precisely the Alexandrov algebras which are *countably compact*, in the sense that whenever $\bigvee\limits_{n\in\mathbb{N}} c_n = 1$, there exists a finite subset $I \subseteq \mathbb{N}$ such that $\bigvee\limits_{n\in I} c_n = 1$. Moreover, we prove that any compact Hausdorff space can reconstructed from the Alexandrov algebra $A = Coz(X)$ not only by taking the maximal spectrum $\mathrm{Max}(Coz(X))$ on it, but also as the space of points of the topos $\mathbf{Sh}(A, C)$, where C is the Grothendieck topology on A given by countable joins.

This duality is also analyzed, in view of Gelfand duality [45] between commutative C^*-algebras and compact Hausdorff spaces, from the point of view of C^*-algebras leading to an explicit categorical equivalence between the category of C^*-algebras and this category of lattices; in particular, any C^*-algebra is shown to be recoverable from the associated Alexandrov algebra through a construction of essentially order-theoretic and arithmetic nature. In passing, we observe that in order to construct the Gelfand spectrum of a real C^*-algebra A one does not need to invoke the full structure of A; indeed, the *reticulation* of A, namely the distributive lattice given by the coherent syntactic category of the theory $\mathbb{P}_A$ of prime filters on A considered in section 4.1.1, suffices to construct its Gelfand spectrum since the equivalence of toposes

$$\mathbf{Sh}(\mathrm{Spec(A)}) \simeq \mathbf{Sh}(\mathcal{C}^{\mathrm{coh}}_{\mathbb{P}_A}, J^{\mathrm{coh}}_{\mathbb{P}_A})$$

yields a homeomorphism

$$\mathrm{Max}(A) \cong \mathrm{Max}(\mathcal{C}^{\mathrm{coh}}_{\mathbb{P}_A}, J^{\mathrm{coh}}_{\mathbb{P}_A})$$

(recall that the maximal spectrum of a topos is an invariant). The results are presented for *real* C^*-algebras (that is, for rings of real-valued continuous functions on a compact Hausdorff space) but they can be straightforwardly extended to the context of complex C^*-algebras.

The duality between compact Hausdorff spaces and countably compact Alexandrov algebras can actually be extended to a duality between the whole category of Alexandrov algebras (and lattice homomorphisms between them preserving countable joins) and the category of completely regular spaces with the Lindelöf property (that every open cover of the space has a countable subcover) and continuous maps between them. This duality has a number of interesting consequences; for instance, it follows immediately from the representation of X as the space of points of the topos $\mathbf{Sh}(Coz(X), C)$ that any open cover of a cozero set in such a space (for instance, in every compact Hausdorff space) has a countable subcover, and that every prime ideal of $Coz(X)$ which is closed under taking countable joins in $Coz(X)$ is maximal.

Another topic which is treated in [22], always by adopting our viewpoint of toposes as 'bridges' is that of the relationships between the Stone-Čech compactification of a topological space X and its Wallman compactifications. The study of such relations had been initiated by Wallman himself, who proved that for any normal completely regular space X its Stone-Čech compactification can be identified with the Wallman compactification $\mathrm{Max}(\mathcal{O}(X))$, and was continued by several authors, including Gillman and Jerison [46], Frink [42] and Johnstone [50] and [51]. We introduce the concept of A-conjunctive sublattice of a frame A as the natural lattice-theoretic counterpart of the concept of Wallman base: given a frame A and a sublattice B of A, we say that B is *A-conjunctive* if for any $a \in A$ and $b \in B$, if $\{c \in B \mid c \vee b = 1 \text{ in } A\} \subseteq \{c \in B \mid c \vee d = 1 \text{ in } A \text{ for some } d \in B \text{ such that } d \leq a\}$ then $b \leq a$. This concept admits a natural characterization involving the toposes $\mathbf{Sh}(A, J^{\mathrm{can}}_A)$ and $\mathbf{Sh}(B, J^B_m)$. We then prove a general theorem based on this notion which subsumes all the previous results obtained on this problem and allows one to establish (iso)morphisms between the Stone-Čech compactification of a locale and its Wallman compactifications in many new cases which were not covered by the past treatments. Our result reads as follows.

Theorem 4.8 (Theorem 2.11 [22])**.** *Assume the axiom of (countable) dependent choices. Let A be a locale, and $\eta_A : A \to \beta(A)$ its Stone-Čech compactification.*

Let B be a normal conjunctive sublattice of A. Then the canonical map $h : \beta(A) \to Id_{J_m^B}(B)$ is an isomorphism if and only if the direct image of η_A sends B injectively to $\beta(A)$ and $\eta_A(B) \subseteq \beta(A)$ is a $\beta(A)$-conjunctive base for $\beta(A)$.

In particular, if A is a completely regular locale and B is a conjunctive normal base for it (for example, if B is a normal A-conjunctive base for A) then h is an isomorphism if and only if $\eta_A(B) \subseteq \beta(A)$ is a $\beta(A)$-conjunctive base for $\beta(A)$.

Moreover, we discuss and interpret several different representations for Gelfand spectra and Stone-Čech compactifications as Morita-equivalences between different geometric (propositional) theories having the same classifying topos.

Finally, in section 4 of [22] we investigate the notion of maximal spectrum of a commutative ring with unit from the point of view of the distributive lattice consisting of the compact open sets of its Zariski spectrum. This leads to a logical characterization of the topos of sheaves on such a spectrum as the classifying topos of a certain propositional geometric theory which, if the spectrum is sober, axiomatizes precisely the maximal ideals of the ring. Next, we explicitly characterize the class of rings with the property that the corresponding distributive lattice is conjunctive, and remark that any finite-dimensional commutative C^*-algebra enjoys this property; this leads in particular to an explicit algebraic characterization of the lattice of cozero sets on its Gelfand spectrum as a distributive lattice presented by generators and relations.

4.2 Duality between equations and solutions

The theme of the duality between equations and solutions spans many different fields of Mathematics. From a logical point of view, it can be viewed as a manifestation of the fundamental duality between syntax and semantics, the equations giving the syntax and the solutions giving the interpretation of the syntax in a suitable structure: the syntactic side normally consists of *presentations*, and the solutions are obtained by mapping the structures presented by them to a fixed structure.

This theme is addressed in full generality in [30], where we develop an abstract categorical framework generalizing the classical "system-solution" Galois connection in affine algebraic geometry. We show that such adjunctions take place in a multitude of contexts and study them at different levels of generality, from syntactic categories to equational classes of algebras. Notably, classical dualities like Stone duality for Boolean algebras, Gelfand duality for commutative C^*-algebras, Pontryagin duality for Abelian groups, turn out to be special instances of this framework.

To determine how such general adjunctions restrict to dualities we prove abstract analogues of Hilbert's Nullstellensatz and Gelfand-Kolmogorov-Stone's lemma, completely characterising the fixed points on one side of the adjunction. We also investigate the relationship between our framework for generating affine adjunctions and the theory of dualities generated by a "schizophrenic" object, showing that under some natural assumptions the dualities generated through our machinery are induced by a "schizophrenic" object. Our framework is actually *complete* in that, as we show in the paper, *any* duality between categories in which one has equalizers and arbitrary intersections of subobjects arises as an appliation of our machinery.

A natural framework in which the results of [30] can be applied is that of theories of presheaf type. In fact, as we prove in [15], for any theory of presheaf type the semantic (i.e., Gabriel-Ulmer) and the syntactic notions of finite presentability coincide and yield an equivalence between a full subcategory of the geometric syntactic category of the theory and the opposite of the category of finitely presentable models of the theory.

In the following two sections we succinctly review the contents of [30] and discuss the above-mentioned duality theorem for theories of presheaf type.

4.2.1 General affine adjunctions

The general framework of [30] is based on a categorical abstraction of the classical notions of affine subset and of ideal in a polynomial ring.

In classical affine algebraic geometry, one studies solutions to systems of polynomial equations with coefficients in an algebraically closed field k. For any subset R of the polynomial ring over finitely many variables $k[X] := k[X_1, \ldots, X_n]$, one can consider the affine subset $\mathbb{V}(R) \subseteq k^n$ of solutions of the equations

$$p(X_1, \ldots, X_n) = 0, \; p \in R \, .$$

over k^n, where k^n is the affine n-space over k.

Conversely, for any subset $S \subseteq k^n$, one can consider the ideal $\mathbb{C}(S) \subseteq k[X]$ of polynomials that vanish over S, and the quotient k-algebra $k[X]/\mathbb{C}(S)$, called the *coordinate ring* of the affine set S.

This yields a Galois connection. The fixed points of the closure operator $\mathbb{V} \circ \mathbb{C}$ are then precisely the affine sets in k^n. Since $\mathbb{V} \circ \mathbb{C}$ is a *topological* closure operator, i.e. it commutes with finite unions, affine algebraic sets are the closed sets of a topology on k^n, namely, the *Zariski topology*. The fixed points of the dual closure operator $\mathbb{C} \circ \mathbb{V}$, on the other hand, may be identified thanks to Hilbert's *Nullstellensatz*: they are precisely the *radical ideals* of $k[X]$.

The Galois connection given by the pair $(\mathbb{C}, \mathbb{V})$ can be made functorial by considering, on the algebraic side, the category of finitely presented k-algebras with their homomorphisms and, on the geometric side, a category whose objects are subset embeddings $S \subseteq k^n$, for each finite n and whose arrows $(S, S \subseteq k^n) \to (T, T \subseteq K^m)$ are the regular maps $S \to T$ (i.e., the equivalence classes of polynomial functions $f \colon k^n \to k^m$ which restrict to functions $S \to T$, two such functions being equivalent if and only if they agree on S). There is a functor that associates to each regular map $S \to T$ a contravariant homomorphism of the coordinate rings of $\mathbb{V} \circ \mathbb{C}(T)$ and $\mathbb{V} \circ \mathbb{C}(S)$, and a functor that associates to each homomorphism of presented k-algebras $k[X]/I \to k[Y]/J$, with $Y = \{Y_1, \ldots, Y_m\}$ and J an ideal of $k[Y]$, a contravariant regular map $\mathbb{V}(J) \to \mathbb{V}(I)$. These two functors yield a contravariant adjunction; upon restricting each functor to the fixed points in each domain, one obtains the classical duality between affine algebraic varieties and their coordinate rings. It is important to note that in this adjunction, coordinate rings are *presented*, that is, they are not merely isomorphic to a ring of the form $k[X]/I$: they come with a specific defining ideal I.

Let us first generalize this setup to the setting of universal algebra.

The main observation is that in any variety of algebras, the *free algebras* play the same role as the ring of polynomials. Ideals of the ring of polynomials become then, in full generality, congruences on some free algebra, while the ground field k is replaced by any algebra A in the variety.

In Corollary 6.13 [30] we obtain an adjunction between $\mathsf{V}_{\mathsf{p}}^{\mathrm{op}}$, the opposite of the category V_{p} of presented V-algebras (morphisms being the ring homomorphisms between them), and the category of subsets of (the underlying set of) A^{μ}, as μ ranges over all cardinals, with definable maps as morphisms. Notice that, assuming the axiom of choice, the category V_{p} of presented V-algebras is equivalent to the category of V-algebras.

The functors defining the adjunction act on objects by taking a subset $R \subseteq \mathscr{F}(\mu)\times\mathscr{F}(\mu)$ (where $\mathscr{F}(\mu)$ is the free V-algebra on μ generators), that is, a "system of equations in the language of V", to its solution set $\mathbb{V}(R) \subseteq A^{\mu}$, where $\mathbb{V}(R)$ is the set of elements of A^{μ} such that each pair of terms in R evaluate identically over it, and a subset $S \hookrightarrow A^{\mu}$ to its "coordinate V-algebra", namely, $\mathscr{F}(\mu)/\,\mathbb{C}(S)$, where $\mathbb{C}(S)$ is the congruence on $\mathscr{F}(\mu)$ consisting of all pairs of terms that evaluate identically at each element of S.

To identify the fixed points of this general affine adjunction on the algebraic side, we prove an appropriate generalisation of the *Nullstellensatz* based on the identification of a suitable notion of radical congruence. The identification of an appropriate type of representation for those V-algebras that are fixed under the adjunction (in part (iii) of the theorem) also leads to a result reminiscent of Birkhoff's Subdirect Representation Theorem.

As far as it concerns the *affine* side, we observe that in several cases the composition $\mathbb{V}\circ\mathbb{C}$ gives a topological closure operator. The resulting topology is a generalisation of the Zariski topology. Under the hypothesis that this topology is Hausdorff and that all the definable functions are continuous with respect to the product topology, we characterize in Lemma 8.3 [30] the fixed points on this side as the closed subsets with respect to the product topology. This characterization allows one to naturally recover Gelfand duality and Stone duality as particular instances of our framework.

In [30] we also lift the general affine adjunction for varieties of algebras of Corollary 6.13 to a more general categorical context. Conceptually, the key ingredient in the algebraic construction sketched above is the functor $\mathscr{I}_A\colon \mathsf{T} \to \mathbf{Set}$ sending the free algebra $\mathscr{F}(\mu)$ on μ generators to the set A^{μ}. In the categorical abstraction, the basic *datum* is *any* functor $\mathscr{I}\colon \mathsf{T} \to \mathsf{S}$, which can be conceived as the *interpretation* of the "syntax" T into the "semantics" S, along with a distinguished object $\vartriangle$ of T (in the algebraic specialisation, $\vartriangle$ is $\mathscr{F}(1)$, the free singly generated V-algebra). Here T and S are simply arbitrary locally small categories. Out of these data, we construct two categories D and R respectively of subobjects and of relations.

The category D abstracts that of sets affinely embedded into A^μ; here, sets are replaced by objects of S, the powers A^μ are replaced by objects $\mathscr{I}(t)$ as t ranges over objects of T, and the morphisms of S that are "definable" are declared to be those in the range of $\mathscr{I}$. The category R abstracts the category of relations (not necessarily congruences) on the free V-algebras $\mathscr{F}(\mu)$; that is, its objects are relations on the hom-set $\hom_{\mathsf{T}}(t, \triangle)$, as t ranges over objects of T. Arrows are T-arrows that preserve the given relations.

It is possible, in this setting, to define the operator $\mathbb{C}$ in full generality. In order to define an appropriate abstraction of the operator $\mathbb{V}$, we need to require that S has enough limits, as "solutions" to "systems of equations" are computed by intersecting solutions to "single equations". The pair $(\mathbb{C}, \mathbb{V})$ yields a Galois connection which functorially lifts to an adjunction between D and R. This represents a weak form of the algebraic adjunction, since in the algebraic setting one is involved with *quotients* of the categories D and R rather than with D and R themselves. To obtain an appropriate categorical generalization of the affine algebraic adjunction, we consider quotient categories D^q of D and R^q of R respectively obtained by identifying pairs of definable morphisms in D which agree on the given "affine subobject" and pairs of morphisms that agree on the same "presented object" (in a suitable abstract sense). Interestingly, in order for the adjunction between D and R to descend to the quotients, it is necessary to impose a condition on the object $\triangle$, namely that it be an *$\mathscr{I}$-coseparator* (meaning that for any T-object t, the family of arrows $\mathscr{I}(\varphi)\colon \mathscr{I}(t) \to \mathscr{I}(\triangle)$, as φ ranges over all T-arrows $\varphi\colon t \to \triangle$, is jointly monic in S); in the algebraic specialisation, we show that this assumption on $\triangle = \mathscr{F}(1)$ is automatically satisfied. Under this additional assumption, which is satisfied in many cases of interest beyond the algebraic setting, we obtain our general affine adjunction between D^q and R^q (see Theorem 3.8 [30]).

The following table, taken from [30], illustrates the correspondences between the main concepts involved in the affine adjunctions at differne levels of generality.

Theorem 3.8 can notably be applied in the context of theories of presheaf type, yielding adjunctions between categories of congruences on finitely presentable models and categories of definable sets and definable homomorphisms between them. In fact, the generality of our categorical adjunction theorem is such that, as we prove in Theorem 4.1 [30], *any* duality between categories such that one of them has equalizers and arbitrary intersections of subobjects is a restriction of an adjunction of the form specified by Theorem 3.8 [30].

Finally, we show that there is a natural connection between our general categorical framework and the classical theory of dualities generated by a "schizofrenic object". Recall that a contravariant adjunction $\mathscr{F} : \mathsf{A} \to \mathsf{B}^{\mathrm{op}}$ and $\mathscr{G} : \mathsf{B}^{\mathrm{op}} \to \mathsf{A}$ between two categories A and B equipped with functors $\mathscr{U}_{\mathsf{A}} : \mathsf{A} \to \mathbf{Set}$ and $\mathscr{U}_{\mathsf{B}} : \mathsf{B} \to \mathbf{Set}$ is said to be induced by a "schizofrenic" or "dualising" object if there exist objects $a \in \mathsf{A}$ and $b \in \mathsf{B}$ such that

Algebraic geometry	Universal algebra	Categories
Ground field k	Any algebra A in V	Functor $\mathscr{I} : \mathsf{T} \to \mathsf{S}$
Class of k-algebras	Any variety V	Category R
$k[X_1, \ldots, X_n]$	Free algebras	Objects in T
Ideals	Congruences	Subsets of $\hom^2_{\mathsf{T}}(t, \vartriangle)$ with t in T
Assignment $k[X_1, \ldots, X_n] \to k$	Assignment $\mathcal{F}(\mu) \to A$	Object $\vartriangle$ in T
Regular map	Definable map	Restriction of $\mathscr{I}(f)$
Coordinate algebra of S	Algebra presented by $\mathbb{C}(S)$	Pair $(t, \mathbb{C}(S))$ in R
Affine variety	$\mathbb{V} \circ \mathbb{C}$-closed set	Pair $(t, \mathbb{V}(R))$ in S

Table 1: Corresponding concepts in the geometric, algebraic, and categorical setting.

- $\mathscr{U}_{\mathsf{A}}$ is representable by a,
- $\mathscr{U}_{\mathsf{B}}$ is representable by b,
- the composite functor $\mathscr{U}_{\mathsf{B}} \circ \mathscr{F}$ is represented by the object $\mathscr{G}(b)$,
- the composite functor $\mathscr{U}_{\mathsf{A}} \circ \mathscr{G}$ is represented by the object $\mathscr{F}(a)$, and
- $\mathscr{U}_{\mathsf{B}}(\mathscr{F}(a)) \cong \mathscr{U}_{\mathsf{A}}(\mathscr{G}(b))$.

We prove in Theorem 4.8 [30] that if $\mathsf{S} = \mathbf{Set}$ and the functor $\mathscr{I} : \mathsf{T} \to \mathsf{S}$ is representable then a suitable restriction of the adjunction of Theorem 3.8 is induced by a "schizophrenic" object (in a sense which is made precise in the paper). This is for instance the case of the ring-theoretic affine adjunction of classical algebraic geometry, in which we have S equal to **Set**, T equal to the opposite of the category of free k-algebras (i.e. the polynomial rings in a finite number of variables) and $\mathscr{I}$ equal to the representable functor $\mathrm{Hom}_{\mathsf{T}}(k, -)$; for any affine variety $V \subseteq k^n$, the corresponding ideal I_V can be realized as the set of morphisms $(V, k^n) \to (\{0, k\})$ in the category R^q, while for any ideal I of a polynomial ring $k[X_1, \ldots, X_n]$, the corresponding variety V_I can be realized as the set of morphisms $(I, k[X_1, \ldots, X_n]) \to (1_k, k)$ in the category D^q (where 1_k is the equality relation on k).

4.2.2 Finite presentability in the setting of theories of presheaf type

Let us recall the following standard notions of finite presentability for models of geometric theories.

Definition 4.9. Let $\mathbb{T}$ be a geometric theory over a signature Σ and M a set-based $\mathbb{T}$-model. Then

(a) The model M is said to be *finitely presentable* if the representable functor $\mathrm{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(M, -) : \mathbb{T}\text{-mod}(\mathbf{Set}) \to \mathbf{Set}$ preserves filtered colimits;

(b) The model M is said to be *finitely presented* if there is a geometric formula $\{\vec{x} . \phi\}$ over Σ and a string of elements $(\xi_1, \ldots, \xi_n) \in MA_1 \times \cdots \times MA_n$ (where $A_1, \ldots, A_n$ are the sorts of the variables in $\vec{x}$), called the *generators* of M, such that for any $\mathbb{T}$-model N in **Set** and string of elements $(b_1, \ldots, b_n) \in MA_1 \times \cdots \times MA_n$ such that $(b_1, \ldots, b_n) \in [[\vec{x} . \phi]]_N$, there exists a unique arrow $f : M \to N$ in $\mathbb{T}$-mod(**Set**) such that $(f_{A_1} \times \cdots \times f_{A_n})((\xi_1, \ldots, \xi_n)) = (b_1, \ldots, b_n)$.

Notice that the syntactic notion of finite presentability is signature-dependent, while the semantic one is an invariant of the category $\mathbb{T}$-mod(**Set**).

As we have already remarked in section 3.1, irreducible formulae play a fundamental role in the study of theories of presheaf type, since they correspond to the irreducible objects of their classifying toposes. Their role is further clarified by the following theorem, which generalizes the classical duality between the cartesian formulae and the models presenting them holding for all cartesian theories.

Theorem 4.10 (Theorem 4.3 [15]). *Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ. Then*

(i) Any finitely presentable $\mathbb{T}$-model in **Set** *is presented by a $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ.*

(ii) Conversely, any $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ presents a $\mathbb{T}$-model.

*In fact, the category f.p.$\mathbb{T}$-mod(**Set**)^{op} is equivalent to the full subcategory $\mathcal{C}_{\mathbb{T}}^{irr}$ of $\mathcal{C}_{\mathbb{T}}$ on the $\mathbb{T}$-irreducible formulae.*

In particular, the syntactic and the semantic notions of finite presentability coincide for $\mathbb{T}$.

This result arises from the following 'bridge':

$$\begin{array}{ccc} & \overset{\textit{Irreducible object}}{[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})} & \\ \underset{\textit{Any object}}{\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\mathrm{op}}} & & \underset{\mathbb{T}\textit{-irreducible formula}}{(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})} \end{array}$$

Indeed, since the category f.p.$\mathbb{T}$-mod(**Set**) is Cauchy-complete, the irreducible objects of the topos [f.p.$\mathbb{T}$-mod(**Set**), **Set**] are precisely (the objects which are isomorphic to) the representable functors, while an object A of the topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ which is irreducible is necessarily, up to isomorphism, of the form $y_{\mathbb{T}}(\{\vec{x} . \phi\})$ for some object $\{\vec{x} . \phi\}$ of $\mathcal{C}_{\mathbb{T}}$ (where $y_{\mathbb{T}}$ is the Yoneda embedding $\mathcal{C}_{\mathbb{T}} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$), since it can be covered by objects coming from $\mathcal{C}_{\mathbb{T}}$ (via $y_{\mathbb{T}}$) and the image of $\mathcal{C}_{\mathbb{T}}$ in $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is closed under subobjects in $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ (notice that the irreducibility of A implies that A is a retract in $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of one of the objects from $\mathcal{C}_{\mathbb{T}}$ which cover it).

By Theorem 6.1.17 [8], the model presented by a $\mathbb{T}$-irreducible formula $\{\vec{x} . \phi\}$ can be built as the image of the 'tautological' universal model of $\mathbb{T}$ inside $\mathcal{C}_{\mathbb{T}}$ under the flat $J_{\mathbb{T}}$-continuous functor $\mathrm{Hom}_{\mathcal{C}_{\mathbb{T}}}(\{\vec{x} . \phi\}, -) : \mathcal{C}_{\mathbb{T}} \to \mathbf{Set}$.

It is worth to note that, unlike the case of finitary algebraic (or more generally, cartesian) theories, free models, i.e. models presented by formulae of the form $\{x_1^{A_1}, \dots, x_n^{A_n} . \top\}$, do not necessarily exist for general theories of presheaf type; nonetheless, the formulae which present the finitely presentable models of such a theory $\mathbb{T}$ 'cover' all the others in the syntactic category of the theory, in the sense that every geometric formula-in-context over the signature of $\mathbb{T}$, regarded as an object of its syntactic category $\mathcal{C}_{\mathbb{T}}$, admits a $J_{\mathbb{T}}$-covering sieve generated by arrows whose domains are $\mathbb{T}$-irreducible formulae (cf. Theorem 3.3 above). In fact, whilst theories of presheaf type may syntactically not look at all like finitary algebraic theories, they actually share with them many fundamental properties that are revealed by the study of their classifying toposes. The novelty and depth of the topos-theoretic viewpoint is witnessed by the fact that many of the results that we have established for general theories of presheaf type (for instance, Theorems 2.15 and 3.3) were not even known to hold for finitary algebraic theories.

On the other hand, the level of generality of the concept of theory of presheaf type goes well beyond that of universal algebra since it is essentially that of category theory. Indeed, any small category $\mathcal{C}$ can be regarded, up to Cauchy completion, as the category of finitely presentable models of a theory of presheaf type (namely, the geometric theory of flat functors on $\mathcal{C}^{\mathrm{op}}$). This fact - which follows from the fact that every Cauchy complete category $\mathcal{C}$ can be recovered, up to equivalence, from the ind-completion Ind-$\mathcal{C}$ as its full subcategory on the finitely presentable objects (cf. Proposition C4.2.2 [54]) noticing that the ind-completion of a small category is equivalent to the ind-completion of its Cauchy completion - is of fundamental importance since it allows one to prove categorical results by logical means. In fact, realizing a Grothendieck topos $\mathbf{Sh}(\mathcal{C}, J)$ as the classifying topos of a geometric theory $\mathbb{T}$ (such as the theory of J-continuous flat functors on $\mathcal{C}$) can be a poweful tool since syntactic sites $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of geometric theories $\mathbb{T}$ are categorically better-behaved than most sites; for instance, they are always subcanonical and $\mathcal{C}_{\mathbb{T}}$ is closed under subobjects in $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$. As an illustration of this remark, consider our topos-theoretic interpretation of topological Galois theory given by Theorem 3.8. This categorical theorem was actually proved in [28] by logical means thanks to the equivalence

$$\mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \simeq \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\mathrm{at}}, J_{\mathrm{at}}),$$

where $\mathbb{T}'$ is the theory of homogeneous $\mathbb{T}$-models, using the syntactic site of $\mathbb{T}'$ and the identification of the atoms of the classifying topos of $\mathbb{T}'$ as the $\mathbb{T}'$-complete formulae; it seems impossible, or at least much more technically complicated, to give a direct categorical proof holding in full generality. This is due to the fact that the atoms of the topos admit an easy description in terms of the site $(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$ but a much more complicated one in terms of the site (f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$, J_{at}).

4.3 Lattice-ordered groups and MV-algebras

In section 4.1 we have illustrated, by means of our topos-theoretic interpretation and generation of Stone-type, Priestley-type and Gelfand-type dualities, the general point of section 2.1.1 according to which dualities or equivalences between 'concrete' categories can be naturally obtained by 'functorializing' Morita-equivalences. Another way in which Morita-equivalences can lead to categorical equivalences between 'concrete' categories, is simply by restriction to a given fixed topos, in particular to the topos **Set**, that is by considering the induced categorical equivalence between the categories of models of the two theories in the given topos. In particular, any double representation of a given topos yields a categorical equivalence between the two *avatars* of the category of points of the topos obtained by calculating this invariant on the one hand by using the first representation and on the other by using the second representation. This principle is illustrated in section 4.3.2, where we derive a whole class of new Morita and categorical equivalences between MV-algebras and ℓ-groups from a representation result for a class of classifying toposes as presheaf toposes.

In the converse direction, as remarked in section 2.1.1, if two geometric theories $\mathbb{T}$ and $\mathbb{T}'$ have equivalent categories of models in **Set** then, provided that the given categorical equivalence is established by only using constructive logic and geometric constructions, one can naturally expect it to 'lift' to a Morita-equivalence between $\mathbb{T}$ and $\mathbb{T}'$. In section 4.3.1 we present two examples of Morita-equivalences (from [25] and [26]) obtained by 'lifting' known categorical equivalences between categories of set-based models of geometric theories, namely Mundici's equivalence between MV-algebras and lattice-ordered abelian groups with strong unit and Di Nola-Lettieri's equivalence betweeen the category of perfect MV-algebras and lattice-ordered abelian groups. In both cases, the 'bridge' technique, applied to the resulting Morita-equivalence, allows one to extract new information about the pair of theories related by it which is not visible by adopting the classical techniques. Indeed, as we show in those papers, both pairs of theories are not bi-interpretable, whence the use of their common classifying topos as a 'bridge' becomes essential for transferring notions and results from one theory to the other.

Remarkably, all the geometric theories involved in the above-mentioned Morita-equivalences are of presheaf type. In section 4.3.3 we encounter yet other notable examples of theories of presheaf type, namely the epicyclic theory (classified by Connes-Consani's epicylic topos), and two related theories respectively classified by the cyclic topos and by the topos $[\mathbb{N}^*, \textbf{Set}]$. Interestingly, as shown in section 4.3.2, the latter theory is strictly related to the geometric theory of finite MV-chains considered in [27].

4.3.1 Mundici's and Di Nola-Lettieri's equivalences from a topos-theoretic viewpoint

In this section we briefly review the contents of papers [25] and [26]. These papers concern the subject of MV-algebras and lattice-ordered abelian groups.

The class of structures known as MV-algebras was introduced in 1958 by C. C. Chang (cf. [31] and [32]) in order to provide an algebraic proof of the completeness of Łukasiewicz's multi-valued propositional logic. As this logic generalizes classical logic, MV-algebras are a generalization of the concept of Boolean algebra (indeed, Boolean algebras can be characterized as the idempotent MV-algebras). After their introduction in the context of algebraic logic, MV-algebras became objects of independent interest and found applications in different areas of Mathematics, the most notable ones being in functional analysis (cf. [68]), in the theory of lattice-ordered abelian groups (cf. [68] and [38]) and in the field of generalized probability theory (cf. Chapters 1 and 10 of [67] for a general overview). Several equivalences between categories of MV-algebras and categories of lattice-ordered abelian groups (ℓ-groups, for short) can be found in the literature, the most important ones being the following:

- Mundici's equivalence [68] between the whole category of MV-algebras and the category of ℓ-groups with strong unit;
- Di Nola-Lettieri's equivalence [38] between the category of perfect MV-algebras (i.e., MV-algebras generated by their radical) and the whole category of ℓ-groups.

In [25] we interpret Mundici's equivalence as an equivalence between the categories of set-based models of two geometric theories, namely the algebraic theory $\mathbb{MV}$ of MV-algebras and the theory $\mathbb{L}_u$ of abelian ℓ-groups with strong unit, and prove that this equivalence generalizes over an arbitrary Grothendieck topos yielding a Morita-equivalence between the two theories.

The signature of the theory $\mathbb{MV}$ of MV-algebras consists of one sort, two function symbols ($\oplus, \neg$) and one constant (0). The axioms of $\mathbb{MV}$ are:

$(\top \vdash_{x,y,z} x \oplus (y \oplus z) = (x \oplus y) \oplus z)$

$(\top \vdash_{x,y} x \oplus y = y \oplus x)$

$(\top \vdash_{x} x \oplus 0 = x)$

$(\top \vdash_{x} \neg\neg x = x)$

$(\top \vdash_{x} x \oplus \neg 0 = \neg 0)$

$(\top \vdash_{x,y} \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x)$

In any MV-algebra there is a natural order $\leq$ defined by: $x \leq y$ if and only if $\neg x \oplus y = 1$; this is a partial order relation which induces a lattice structure on the underlying set of the MV-algebra. One can also define $1 := \neg 0$ and a product operation $\odot$ by setting $x \odot y := \neg(\neg x \oplus \neg y)$. The notation x^n will be used as an abbreviation for $x \odot \cdots \odot x$ n times (for any natural number $n \geq 1$).

Recall that an abelian ℓ-group with strong unit is a structure $\mathcal{G} = (G, +, -, \leq, \inf, \sup, 0, u)$, where $(G, +, -, 0)$ is an abelian group, $\leq$ is a partial order relation that induces a lattice structure and is compatible with addition, i.e., it has the translation invariance property

$$\forall x, y, t \in G \;\; x \leq y \Rightarrow t + x \leq t + y$$

and $u \in G$ is a *strong unit*, i.e., $u \geq 0$ and for any $x \in G$ there is a natural number n such that $x \leq nu$.

The class of abelian ℓ-groups with strong unit can be axiomatized by a geometric theory $\mathbb{L}_u$ over a signature consisting of one sort, four function symbols $+$, $-$, sup and inf, one relation symbol $\leq$ and two constants 0 and u; its axioms are those of the theory of abelian ℓ-groups plus those which define the concept of strong unit:

$(\top \vdash u \geq 0)$

$(x \geq 0 \vdash_x \bigvee_{n \in \mathbb{N}} x \leq nu)$

Our proof of the Morita-equivalence between $\mathbb{MV}$ and $\mathbb{L}_u$ is based on a geometric and constructive reinterpretation of the classical functors defining Mundici's equivalence. Recall that one half of Mundici's equivalence is provided by the unit interval functor Γ, which assigns to each ℓ-group with strong unit $(\mathcal{G}, u)$ the MV-algebra given by the unit interval $[0, u]$ in $\mathcal{G}$ (where $x \oplus y = \min(u, x + y)$ and $\neg x = u - x$). The fact that this functor only involves geometric constructions is straightforward. More subtle is the definition of the functor going in the other direction from MV-algebras to ℓ-groups with strong unit, which assigns to an MV-algebra $\mathcal{A}$ the Grothendieck group of the monoid of *good sequences* on $\mathcal{A}$ (recall that a sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n, \ldots)$ of elements of $\mathcal{A}$ is said to be *good* if $a_i \oplus a_{i+1} = a_i$ for each $i \in \mathbb{N}$ and there is a natural number n such that $a_r = 0$ for any $r > n$). Whilst the set of *all* sequences does not admit a geometric definition in general, the subset of *good* sequences does since one can realize it as the colimit over $\mathbb{N}$ of the sets of good sequences which are 0 in all places $\geq n$, and the latter are geometric subsets of the nth power of $\mathcal{A}$.

The fact that the theories $\mathbb{MV}$ and $\mathbb{L}_u$ have equivalent classifying toposes - rather than merely equivalent categories of set-based models - has a number of non-trivial consequences. For instance, the established Morita-equivalence provides, for any topological space X, an equivalence between the category of sheaves of MV-algebras over X and the category of sheaves of *pointed ℓ-groups* (i.e. abelian ℓ-groups with a distinguished element) over X whose stalks are ℓ-groups with strong unit, which is natural in X and extends Mundici's equivalence at the level of stalks. Most importantly, one can apply the 'bridge' technique to the given Morita-equivalence to transfer properties and results across the two theories:

$$\begin{tikzcd} & \mathcal{E}_{\mathbb{MV}} \simeq \mathcal{E}_{\mathbb{L}_u} \arrow[dl, dashed, no head] \arrow[dr, dashed, no head] & \\ \mathbb{MV} & & \mathbb{L}_u \end{tikzcd}$$

Of course, different invariants considered on the common classifying topos will give different results concerning the two theories.

Taking for instance as invariant the property of being equivalent to a presheaf topos, we immediately obtain that the theory $\mathbb{L}_u$ is of presheaf type as the algebraic theory $\mathbb{MV}$ is.

Choosing instead as invariant the notion of subtopos one obtains, by Theorem 2.5, the striking result that the quotients of $\mathbb{MV}$ and of $\mathbb{L}_u$ (in their respective languages) correspond to each other bijectively. Note that this result would be trivial if the theories $\mathbb{MV}$ and $\mathbb{L}_u$ were bi-interpretable, but, as we show in the paper, this is not the case. Still, there is an interpretation (geometric) functor $I : \mathcal{C}_{\mathbb{MV}} \to \mathcal{C}_{\mathbb{L}_u}$, which for instance allows one to prove identities for MV-algebras by arguing in the language of ℓ-groups.

If we consider the invariant property of objects of toposes to be irreducible we get a logical characterization of the finitely presentable ℓ-groups with strong unit (equivalently, of the ℓ-groups with strong unit corresponding to the finitely presented MV-algebras under Mundici's equivalence) as the ℓ-groups presented by a formula which is $\mathbb{L}_u$-irreducible. By considering as invariants the properties of the classifying topos to have a compact terminal object and to have enough points, we derive a form of compactness and completeness for the infinitary theory $\mathbb{L}_u$ from the fact that they are satisfied by the finitary theory $\mathbb{MV}$.

It is worth to note that, while the language of ℓ-groups with strong unit is often more convenient for performing calculations than that of MV-algebras, mostly due to the fact that these structures are torsion-free, the theory of MV-algebras is finitary algebraic, has close ties with semiring theory and tropical mathematics (cf. for instance [40]) and supports a number of important notions which are neither natural from the point of view of ℓ-groups nor studied in that context (think for example of the notion of infinitesimal element of an MV-algebra).

Let us now turn to paper [26]. As in [25], we lift a categorical equivalence, in this case Di Nola-Lettieri's equivalence between the category of perfect MV-algebras and the category of lattice-ordered abelian groups, to a Morita-equivalence between two geometric theories, by showing that all the concepts involved in the given equivalence admit a constructive and geometric treatment. Recall that an *ideal* of an MV-algebra is a lowerset (with respect to the natural ordering on the algebra) which is closed with respect to the $\oplus$ operation; the *radical* $\mathrm{Rad}(\mathcal{A})$ of an MV-algebra $\mathcal{A}$ is defined as the intersection of all the maximal ideals of $\mathcal{A}$, equivalently the set of infinitesimal elements of $\mathcal{A}$ plus 0 (recall that an element x

of an MV-algebra is said to be infinitesimal if it is non-zero and $nx \leq \neg x$ for all integers $n \geq 0$).

An MV-algebra $\mathcal{A}$ is said to be *perfect* if it is non-trivial (i.e., $A \neq \{0\}$ or equivalently $1 \neq 0$) and $\mathcal{A} = \mathrm{Rad}(\mathcal{A}) \cup \neg\mathrm{Rad}(\mathcal{A})$, where $\neg\mathrm{Rad}(\mathcal{A}) = \{x \in A \mid \neg x \in \mathrm{Rad}(\mathcal{A})\}$ is the coradical of $\mathcal{A}$.

The theories involved are the theory $\mathbb{L}$ of lattice-ordered abelian groups (which is defined in the obvious way over the signature for lattice-ordered abelian groups with strong unit considered above) and the theory $\mathbb{P}$ of perfect MV-algebras, which we define to be the quotient of the theory $\mathbb{MV}$ obtained by adding the axioms

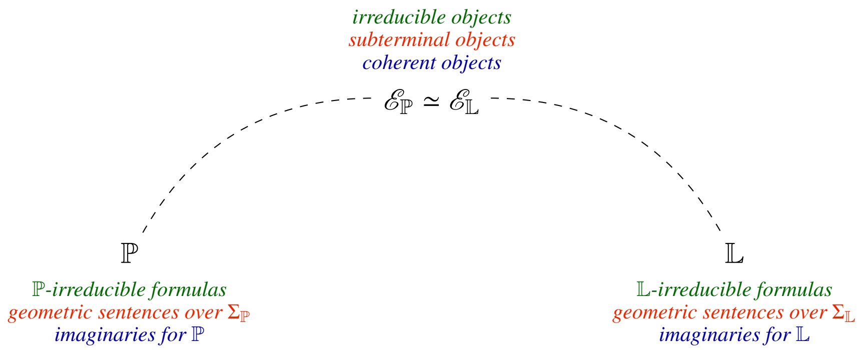
$$(\top \vdash_x x^2 \oplus x^2 = (x \oplus x)^2),$$

$$(x \oplus x = x \vdash_x x = 0 \vee x = 1),$$

$$(x = \neg x \vdash_x \bot).$$

Indeed, the set-based models of the theory $\mathbb{P}$ can be identified precisely with the perfect MV-algebras. As we prove in the paper, the radical of such an algebra $\mathcal{A}$ can be constructively defined as the set $\{x \in \mathcal{A} \mid x \leq \neg x\}$.

The Morita-equivalence between the theories $\mathbb{L}$ and $\mathbb{P}$ has many consequences on the theories which are not visible from different viewpoints. Indeed, as in the case of the theories of [25], we prove that $\mathbb{L}$ and $\mathbb{P}$ are not bi-interpretable in the classical sense; still, by applying the 'bridge' technique in connection with the invariant properties of objects of toposes to be irreducible (resp. subterminals, coherent), we uncover three levels of partial bi-interpretability holding for particular classes of formulas: irreducible formulas, geometric sentences and imaginaries.



The $\mathbb{P}$-irreducible formulae are precisely the formulae that present the finitely presentable perfect MV-algebras, that is the algebras which correspond to the finitely presented ℓ-groups via Di Nola-Lettieri's equivalence. These formulae represent the analogue for the theory $\mathbb{P}$ of the cartesian formulae in the theory of MV-algebras. Indeed, even though the category $\mathbb{P}$-mod(**Set**) is not a variety in the sense of universal algebra (i.e., the category of set-based models of a finitary algebraic theories), it is generated by its finitely presentable objects since the theory $\mathbb{P}$ is of presheaf type classified by the topos [f.p.$\mathbb{P}$-mod(**Set**), **Set**].

In order to explicitly describe the $\mathbb{P}$-irreducible formulae and the above-mentioned partial bi-interpretations, we establish a bi-interpretation between the theory $\mathbb{L}$ of lattice-ordered abelian groups and a cartesian theory $\mathbb{M}$ axiomatizing the positive cones of these groups (which we call the theory of cancellative lattice-ordered abelian monoids with bottom element). In fact, Di Nola-Lettieri's equivalence turns out to admit a simpler formulation in the language of monoids than in that of groups. Interestingly, our bi-interpretation between $\mathbb{M}$ and $\mathbb{L}$ also provides an alternative description of the Grothendieck group of a model $\mathcal{M}$ of $\mathbb{M}$ as a subset, rather than a quotient as in the classical definition, of the product $\mathcal{M} \times \mathcal{M}$; in particular, its underlying set can be identified with the set

$$\{(u, v) \in \mathcal{M} \times \mathcal{M} \mid \inf(u, v) = 0\},$$

where inf is the operation of infimum of a pair of elements in $\mathcal{M}$.

We then investigate the classifying topos of the theory $\mathbb{P}$ of perfect MV-algebras, representing it as a subtopos

$$\mathbf{Sh}(\text{f.p.}\mathbb{C}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\mathbb{P}}) \hookrightarrow [\text{f.p.}\mathbb{C}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$$

of the classifying topos of the algebraic theory $\mathbb{C}$ axiomatizing the variety generated by Chang's MV-algebra, namely the quotient of the theory $\mathbb{MV}$ obtained by adding the sequent $(\top \vdash_x x^2 \oplus x^2 = (x \oplus x)^2)$. We show that $J_{\mathbb{P}}$ is rigid (cf. section 2.2 above for the notion of rigid topology) and subcanonical and derive various results concerning the relationship between these two theories, notably including a representation theorem for the finitely presentable (resp. finitely generated) algebras in Chang's variety $\mathbb{C}$-mod($\mathbf{Set}$) as finite products of finitely presentable (resp. finitely generated) perfect MV-algebras. Indeed, the only non-trivial $J_{\mathbb{P}}$-coverings on a MV-algebra $\mathcal{A} \in$ f.p.$\mathbb{C}$-mod($\mathbf{Set}$) are those which contain families of the form

$$\begin{tikzcd}
 & & & \mathcal{A}/(a_1^1)/\ldots/([\ldots[a_1^n]\ldots]) \\
 & & \mathcal{A}/(a_1^1)/([a_1^2]) \cdots \arrow[ur] \arrow[dr] & \\
 & & & \mathcal{A}/(a_1^1)/\ldots/(\neg[\ldots[a_1^n]\ldots]) \\
 & \mathcal{A}/(a_1^1) \arrow[uur] \arrow[dr] & & \\
 & & \mathcal{A}/(a_1^1)/(\neg[a_1^2]) \cdots & \\
\mathcal{A} \arrow[uur] \arrow[ddr] & & & \\
 & & \mathcal{A}/(\neg a_1^1)/([a_2^2]) \cdots & \\
 & \mathcal{A}/(\neg a_1^1) \arrow[ur] \arrow[ddr] & & \\
 & & & \mathcal{A}/(\neg a_1^1)/\ldots/([\ldots[a_{2^{n-1}}^n]\ldots]) \\
 & & \mathcal{A}/(\neg a_1^1)/(\neg[a_2^2]) \cdots \arrow[ur] \arrow[dr] & \\
 & & & \mathcal{A}/(\neg a_1^1)/\ldots/(\neg[\ldots[a_{2^{n-1}}^n]\ldots])
\end{tikzcd}$$

for any elements a^i_j (where $i = 1, \dots, n$ and $j = 1, \dots, 2^{i-1}$) in the Boolean skeleton $\mathcal{B}(\mathcal{A})$ of $\mathcal{A}$ (i.e., such that $a^i_j \oplus a^i_j = a^i_j$); for any MV-algebra $\mathcal{A}$ in $\mathbb{C}$-mod(**Set**), $\mathcal{A}/\mathrm{Rad}(\mathcal{A}) \cong \mathcal{B}(\mathcal{A})$, so if the elements a^i_j are taken to be a set of generators for $\mathcal{B}(\mathcal{A})$, the algebras appearing as the leaves of the above diagram are perfect MV-algebras, and for each Boolean element x of an MV-algebra $\mathcal{A}$, we have that $\mathcal{A} \cong \mathcal{A}/(x) \times \mathcal{A}/(\neg x)$.

Among the other insights obtained on the relationship between the theories $\mathbb{P}$ and $\mathbb{C}$, we mention a characterization of the perfect MV-algebras which correspond to finitely presented lattice-ordered abelian groups via Di Nola-Lettieri's equivalence as the finitely presented objects of Chang's variety which are perfect MV-algebras, and the property that $\mathbb{C}$ is the cartesianization of $\mathbb{P}$ (i.e., the collection of cartesian sequents which are provable in $\mathbb{P}$).

We then revisit the above-mentioned representation theorem for MV-algebras in Chang's variety from the point of view of subdirect products of perfect MV-algebras. We show that the class of MV-algebras in Chang's variety $\mathbb{C}$-mod(**Set**) constitutes a particularly natural MV-algebraic setting extending the variety of Boolean algebras (recall that every Boolean algebra is an MV-algebra, actually lying in $\mathbb{C}$-mod(**Set**)), with perfect algebras representing the counterpart of the algebra $\{0, 1\}$ and powerset algebras, that is products of the algebra $\{0, 1\}$, corresponding to products of perfect MV-algebras. Theorem 10.2 [26] represents a natural generalization in this setting of the Stone representation of a Boolean algebra as a field of sets, while Theorem 10.6 [26] represents the analogue of Lindenbaum-Tarski's theorem (which characterizes the Boolean algebras isomorphic to powersets as the complete atomic Boolean algebras). As every Boolean algebra with n generators is a product of at most 2^n copies of the algebra $\{0, 1\}$, so every finitely presented algebra in $\mathbb{C}$-mod(**Set**) with n generators is a product of at most 2^n finitely presented perfect MV-algebras (cf. Theorem 9.2 [26]). These relationships are summarized in the table below.

Classical context	**MV-algebraic generalization**
Boolean algebra	MV-algebra in $\mathbb{C}$-mod(**Set**)
$\{0, 1\}$	Perfect MV-algebra
Powerset $\cong$ product of $\{0, 1\}$	Product of perfect MV-algebras
Finite Boolean algebra	Finitely presentable MV-algebra in $\mathbb{C}$-mod(**Set**)
Complete atomic Boolean algebra	MV-algebra in $\mathbb{C}$-mod(**Set**) satisfying the hypotheses of Theorem 10.6 [26]
Representation theorem for finite Boolean algebras	Theorem 9.2 [26]
Stone representation for Boolean algebras	Theorem 10.2 [26]
Lindenbaum-Tarski's theorem	Theorem 10.6 [26]

Finally, we transfer the above-mentioned representation theorems for the MV-algebras in Chang's variety in terms of perfect MV-algebras into the context of ℓ-groups with strong unit and show that a theory of pointed perfect MV-algebras is Morita-equivalent to the theory of lattice-ordered abelian groups with strong unit (whence to that of MV-algebras). Our Morita-equivalence between $\mathbb{P}$ and $\mathbb{L}$ also implies, in light of Theorem 2.5 and our Morita-equivalence lifting Mundici's equivalence, that the theory $\mathbb{L}$ can be identified with a quotient of the theory $\mathbb{L}_u$; we identify this quotient in Remark 11.5 as the theory obtained from $\mathbb{L}_u$ by adding the following axioms:

$$(0 \leq x \wedge x \leq u \vdash_x \sup(0, 2\inf(2x, u) - u) = \inf(u, 2\sup(2x - u, 0))$$
$$(0 \leq x \wedge x \leq u \wedge \inf(2x, u) = x \vdash_x x = 0 \vee x = u)$$

This theory axiomatizes precisely the *antiarchimedean* ℓ-groups with strong unit, i.e. the ℓ-groups corresponding to perfect MV-algebras under Mundici's equivalence.

4.3.2 New Morita-equivalences for local MV-algebras in varieties

In [27] we construct a new class of Morita-equivalences between theories of local MV-algebras and theories of ℓ-groups, which includes the Morita-equivalence obtained in [26] by lifting Di Nola-Lettieri's equivalence. Recall that an MV-algebra $\mathcal{A}$ is said to be *local* if it has exactly one maximal ideal, equivalently if every element x of $\mathcal{A}$ is either in the radical, in the coradical or it is finite (in the sense that $nx = 1$ for some n and $m\neg x = 1$ for some m).

The starting point of our investigation is the observation that the class of perfect MV-algebras is the intersection of the class of local MV-algebras with a specific proper variety of MV-algebras, namely Chang's variety. We wonder what happens if we replace Chang's variety with an arbitrary variety of MV-algebras. We prove that 'globally' the theory of local MV-algebras is *not* of presheaf type, while if we restrict to any proper subvariety V, the theory of local MV-algebras in V is of presheaf type and Morita-equivalent to a theory expanding the theory of ℓ-groups. In order to present this latter theorem, we have to explain a number of preliminary results obtained in [27].

By a result of [56], every proper subvariety of MV-algebras is a so-called *Komori variety*, i.e. it is generated by a finite number of finite simple MV-algebras $S_m = \Gamma(\mathbb{Z}, m)$ and a finite number of Komori chains, i.e. algebras of the form $S_m^\omega = \Gamma(\mathbb{Z} \times_{\text{lex}} \mathbb{Z}, (m, 0))$, where $\mathbb{Z} \times_{\text{lex}} \mathbb{Z}$ is the lexicographic product of the group of integers with itself and Γ is the unit interval functor from the category of ℓ-groups with strong unit to the category of MV-algebras. We show that for any Komori variety $V = V(S_{n_1}, \dots, S_{n_k}, S_{m_1}^\omega, \dots, S_{m_s}^\omega)$, the least common multiple n of the ranks n_i and m_j of the generators is an *invariant* of V, i.e. it does not depend on the choice of the generators for V.

We consider two different axiomatizations for the theory of local MV-algebras in a Komori variety V. The first, which we call $\mathbb{L}oc_V^1$, is obtained from the theory

$\mathbb{T}_V$ of the variety V by adding the sequents

σ_n: $(\top \vdash_x ((n+1)x)^2 = 0 \vee (n+1)x = 1)$;

NT: $(0 = 1 \vdash \bot)$.

The Grothendieck topology J_1 on f.p.$\mathbb{T}_V$-mod$(\mathbf{Set})^{\mathrm{op}}$ associated with $\mathbb{L}oc^1_V$ as a quotient of $\mathbb{T}_V$ is obtained by considering finite multicompositions of diagrams of the form

$$\begin{tikzcd}
 & \mathcal{A}/((n+1)x)^2) \\
\mathcal{A} \arrow[ur] \arrow[dr] & \\
 & \mathcal{A}/(\neg((n+1)x)^2)
\end{tikzcd}$$

These diagrams are product diagrams in $\mathbb{T}_V$-mod$(\mathbf{Set})$, whence J_1 is subcanonical (recall that a Grothendieck topology is subcanonical if and only if its covering sieves are effective-epimorphic, cf. p. 534 [54]). This in turn implies that the algebraic theory $\mathbb{T}_V$ of the variety V is the cartesianization of the theory $\mathbb{L}oc^1_V$; indeed, this result arises from the following 'bridge':

$$\begin{tikzcd}
 & \underset{\text{Validity of a cartesian sequent } \sigma \text{ in the universal model}}{\mathbf{Sh}(\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set})^{\mathrm{op}}, J_1) \hookrightarrow [\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set}), \mathbf{Set}]} \arrow[dl, dashed, no head] \arrow[dr, dashed, no head] & \\
\underset{\text{Provability of } \sigma \text{ in } \mathbb{L}oc^1_V}{\mathbb{L}oc^1_V} & & \underset{\text{Provability of } \sigma \text{ in } \mathbb{T}_V}{\mathbb{T}_V}
\end{tikzcd}$$

We also prove, in a fully constructive way using Di Nola-Lettieri's axiomatization for a Komori variety, that the radical of an MV-algebra in V is definable by the formula $((n+1)x)^2 = 0$ (where n is the invariant of V defined above).

We would like to show that J_1 is rigid in order to generalize the results obtained for perfect MV-algebras in [26]. To this end, we consider another, more refined axiomatization for the theory of local MV-algebras in V. Indeed, it seems impossible to directly establish the rigidity of J_1 since the partitions $\mathcal{A} = \{x \in \mathcal{A} \mid ((n+1)x)^2 = 0\} \cup \{x \in \mathcal{A} \mid (n+1)x = 1\}$ of the algebras $\mathcal{A}$ in $\mathbb{T}_V$-mod$(\mathbf{Set})$ defined by axiom σ_n are not compatible with respect to the MV-algebraic operations. To solve this problem, we observe that for any local MV-algebra in V, the quotient $\mathcal{A}/Rad(\mathcal{A})$ is isomorphic to a finite simple MV-algebra S_m, where m divides the invariant n of the variety V, so we have a canonical MV-algebra homomorphism $\phi_{\mathcal{A}} : \mathcal{A} \to S_n$. Unlike the partitions defined by the topology J_1, the radical classes of $\mathcal{A}$, i.e. the subsets of the form $\mathrm{Fin}^n_d(\mathcal{A}) := \phi^{-1}_{\mathcal{A}}(d)$ for $d \in S_n$, *are* compatible with the MV-algebraic operations. We are able to find geometric formulae defining these subsets in any local MV-algebra in V, which we call $x \in \mathrm{Fin}^n_d$ (for any

$d \in S_n$). Let us consider the geometric sequent

$$\rho_n : (\top \vdash_x \bigvee_{d=0}^{n} x \in \mathrm{Fin}^n_d),$$

and call $\mathbb{L}oc^2_V$ the quotient of $\mathbb{T}_V$ obtained by adding the sequents ρ_n and NT. We show that the theories $\mathbb{L}oc^2_V$ and $\mathbb{L}oc^1_V$ are syntactically equivalent (assuming the axiom of choice), since they are both coherent and their set-based models are the same. Most importantly, we show (in Proposition 6.18 [27]) that the sequents

$$(x \in \mathrm{Fin}^n_d \wedge y \in \mathrm{Fin}^n_b \vdash_{x,y} x \oplus y \in \mathrm{Fin}^n_{d \oplus b}) \tag{1}$$

(for each $d, b \in \{0, \ldots, n\}$ and where with $d \oplus b$ we indicate the sum in $S_n = \{0, 1, \ldots, n\}$) and

$$(x \in \mathrm{Fin}^n_d \vdash_x \neg x \in \mathrm{Fin}^n_{n-d}) \tag{2}$$

(for each $d \in \{0, \ldots, n\}$) are provable in the theory $\mathbb{T}_V$. This fact has important consequences: for instance, it allows us to identify the biggest local subalgebra of an algebra $\mathcal{A}$ in $\mathbb{T}_V$-mod(**Set**) as the algebra

$$\mathcal{A}_{\mathrm{loc}} = \{x \in \mathcal{A} \mid x \in \mathrm{Fin}^n_d(\mathcal{A}) \text{ for some } d \in \{0, \ldots, n\}\}$$

(cf. Proposition 6.19 [27]). This in turn implies that every finitely presentable $\mathbb{L}oc^2_V$-model is finitely presentable also as a $\mathbb{T}_V$-model. So, by Theorem 2.12, to prove that the theory $\mathbb{L}oc^2_V$ is of presheaf type we are reduced to verify that the Grothendieck topology J_2 on f.p.$\mathbb{T}_V$-mod(**Set**)$^{\mathrm{op}}$ associated with $\mathbb{L}oc^2_V$ as a quotient of $\mathbb{T}_V$ is rigid. But this easily follows from the compatibility property of the partition defined by the Fin^n_d with respect to the MV-algebraic operations expressed by the above-mentioned sequents; indeed, the topology J_2 is generated by finite multicompositions of diagrams of the form

$$\begin{array}{ccl} & & \mathcal{A}/(x \in Fin^n_0(\mathcal{A})) \\ & \nearrow & \vdots \\ \mathcal{A} & \longrightarrow & \mathcal{A}/(x \in Fin^n_d(\mathcal{A})) \\ & \searrow & \vdots \\ & & \mathcal{A}/(x \in Fin^n_n(\mathcal{A})) \end{array}$$

where $\mathcal{A}$ is a finitely presentable algebra in V, and if we choose at each step of such a multicomposition one of the generators of the algebra $\mathcal{A}$, the codomain algebras of the resulting diagram will be trivial or local MV-algebras since any MV-algebra which satisfies sequents σ_n and NT is local.

So we have an equivalence of toposes

$$\mathcal{E}_{\mathbb{L}oc^2_V} \simeq \mathbf{Sh}(\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set})^{\mathrm{op}}, J_2) \simeq [\text{f.p.}\mathbb{L}oc^2_V\text{-mod}(\mathbf{Set})]\ .$$

Since the theories $\mathbb{L}oc_V^2$ and $\mathbb{L}oc_V^1$ are syntactically equivalent, it follows from Theorem 2.5 that $J_1 = J_2$; so J_1 is rigid. This in turn implies that every finitely presentable non-trivial algebra in V is a finite direct product of finitely presentable local MV-algebras in V (cf. Theorem 6.23 [27]). This generalizes the representation result obtained in [26] for the finitely presentable MV-algebras in Chang's variety as finite products of perfect MV-algebras. Still, as we show in section 6.3 of [27], there are a number of important structural differences between the particular setting of perfect MV-algebras investigated in [26] and that of local MV-algebras in an arbitrary Komori variety, particularly in connection with the representation of MV-algebras in the variety as Boolean products of local MV-algebras.

Having proved that the theory $\mathbb{L}oc_V^2$ is of presheaf type, we proceed to define an expansion of the theory of ℓ-groups which is Morita-equivalent to it. We shall indicate with the symbol $\delta(I)$ (resp. $\delta(J)$) and $\delta(n)$ the set of divisors of a number in I (resp. in J) and the set of divisors of n.

Recall that the rank of a local MV-algebra $\mathcal{A}$ in a Komori variety is the cardinality of the simple MV-algebra $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$. By results of [39], the local MV-algebras in a Komori variety $V = V(S_{n_1}, \dots, S_{n_k}, S^{\omega}_{m_1}, \dots, S^{\omega}_{m_s})$ are precisely the local MV-algebras A of finite rank $\mathrm{rank}(A) \in \delta(I) \cup \delta(J)$, where $I = \{n_1, \dots, n_k\}$ and $J = \{m_1, \dots, m_s\}$, such that if $\mathrm{rank}(\mathcal{A}) \in \delta(I) \setminus \delta(J)$ then $\mathcal{A}$ is simple, and for any such algebra $\mathcal{A}$ of rank k there exists an ℓ-group G and an element $g \in G$ such that $\mathcal{A} \cong \Gamma(\mathbb{Z} \times_{lex} G, (k, g))$. Building on this, we define the theory $\mathbb{G}_{(I,J)}$ as follows. The signature of $\mathbb{G}_{(I,J)}$ is obtained from that of the theory $\mathbb{L}$ of lattice-ordered abelian groups by adding a 0-ary relation symbol R_k for each $k \in \delta(n)$ (where n is the least common multiple of the numbers in $I \cup J$), and a constant. The predicate R_k has the meaning that the rank of the corresponding MV-algebra is a multiple of k (notice that the property 'to have rank *equal* to k' is not definable by a geometric formula since it is not preserved by homomorphisms of local MV-algebras in V). The axioms of $\mathbb{G}_{(I,J)}$ are:

$(\top \vdash R_1)$;

$(R_k \vdash R_{k'})$, for each k' which divides k;

$(R_k \wedge R_{k'} \vdash R_{\mathrm{l.c.m.}(k,k')})$, for any k, k';

$(R_k \vdash_g g = 0)$, for every $k \in \delta(I) \setminus \delta(J)$;

$(R_k \vdash \bot)$, for any $k \notin \delta(I) \cup \delta(J)$.

Theorem 4.11 (Corollary 7.8 [27]). *For any Komori variety*

$$V = V(S_{n_1}, \dots, S_{n_k}, S^{\omega}_{m_1}, \dots, S^{\omega}_{m_s}),$$

the theories $\mathbb{L}oc_V^2$ and $\mathbb{G}_{(I,J)}$, where $I = \{n_1, \dots, n_k\}$ and $J = \{m_1, \dots, m_s\}$, are Morita-equivalent.

This theorem is proved by showing that $\mathbb{G}_{(I,J)}$ is of presheaf type (by applying a theorem in Chapter 8 of [8] asserting that any quotient of a theory of presheaf type $\mathbb{T}$ by adding axioms of the form $(\phi \vdash_{\vec{x}} \bot)$ is again of presheaf type), and by verifying that the categories of set-based models of the theories $\mathbb{G}_{(I,J)}$ and $\mathbb{L}oc^2_V$ are equivalent (notice that two theories of presheaf type are Morita-equivalent if and only if their categories of set-based models are equivalent).

The categories of set-based models of these theories are not in general algebraic as in the case of perfect MV-algebras; however, we characterize the varieties V for which we have algebraicity as precisely those which can be generated by a single chain. We show that all the Morita-equivalences of Theorem 4.11 are non-trivial, i.e. do not arise from bi-interpretations.

Strictly related to the theory of local MV-algebras is the theory of *simple MV-algebras*, i.e. of local MV-algebras whose radical is $\{0\}$; indeed, an MV-algebra A is local if and only if the quotient $A/\mathrm{Rad}(A)$ is a simple MV-algebra. This theory shares many properties with the theory of local MV-algebras: globally it is not of presheaf type but it has this property if we restrict to an arbitrary proper subvariety. On the other hand, while the theory of simple MV-algebras of finite rank is of presheaf type (as it coincides with the geometric theory of finite chains), the theory of local MV-algebras of finite rank is not, as we prove in section 8 of [27].

By applying Theorem 2.11 to the theory $\mathbb{MV}$ of MV-algebras and the full subcategory $\mathcal{F}$ of f.p.$\mathbb{MV}$-mod(**Set**) on the finite chains, we obtain an explicit axiomatization of the geometric theory $\mathbb{F} = \mathbb{MV}_{\mathcal{F}}$ of finite chains (where the notation is that of Theorem 2.10): $\mathbb{F}$ is the theory obtained from $\mathbb{MV}$ by adding the following axiom:

$$(\top \vdash_x \bigvee_{k,t \in \mathbb{N}} (\exists z)((k-1)z = \neg z \wedge x = tz)) \ .$$

We show that the set-based models of $\mathbb{F}$ are precisely the (simple) MV-algebras that can be embedded as subalgebras of the MV-algebra $\mathbb{Q} \cap [0, 1]$.

4.3.3 Cyclic theories

In [24] we investigate from a logical viewpoint some toposes introduced by A. Connes and C. Consani in connection with their research programme for studying the local factors of L-functions attached to arithmetic varieties through cohomology and non-commutative geometry; more specifically, we describe suitable geometric theories classified by them.

Connes-Consani's epicyclic topos (cf. [35]), Connes' cyclic topos (cf. [34]) and the arithmetic topos (cf. [36]) are all presheaf toposes, so our techniques for generating theories classified by a given presheaf topos developed in [8] and recalled in section 2.2 are most relevant in this context. In fact, we shall describe a geometric theory classified by the epicylic topos and two related theories respectively classified by the cyclic topos and by the arithmetic topos. Recall that the epicyclic topos is defined as the topos $[\tilde{\Lambda}, \mathbf{Set}]$, where $\tilde{\Lambda}$ is Goodwillie's epicyclic category, the cyclic topos as the topos $[\Lambda, \mathbf{Set}]$, where Λ is the cyclic category, and the arithmetic topos as the topos $[\mathbb{N}^*, \mathbf{Set}]$, where $\mathbb{N}^*$ is the multiplicative monoid of non-zero natural numbers.

Our strategy for defining geometric theories classified by these toposes is to identify basic theories of presheaf type $\mathbb{T}$ such that the categories $\tilde{\Lambda}$, Λ and $\mathbb{N}^*$ can be identified as full subcategories of the category f.p.$\mathbb{T}$-mod(**Set**), and then apply Theorems 2.10 and 2.11.

In the case of the epicyclic topos, our basic theory of presheaf type is written over a signature which is essentially the one considered in [35], namely that of oriented groupoids with the addition of a non-triviality predicate. In fact, one of the main results obtained in [35] is a characterization of the points of the epicyclic topos in terms of projective geometry over a semi-field of characteristic 1, using this language for describing the objects of the cyclic as well as of the epicyclic category. We show that the theory $\mathbb{G}_T$ of *partially oriented groupoids with (non-triviality) predicate T* (cf. Definition 1 [24]) is of presheaf type and the epicylic category $\tilde{\Lambda}$ can be identified with the full subcategory of $\mathbb{G}_T$-mod(**Set**) consisting of the oriented groupoids of the form $\mathbb{Z} \rtimes X$ for *transitive* $\mathbb{Z}$-actions on *finite* sets X.

In the case of the cyclic topos, the basic theory of presheaf type is the theory $\mathbb{G}_C$ of *partially oriented groupoids with cycles* (cf. Definition 3 [24]), written over the signature of oriented groupoids by adding a function symbol whose intended interpretation is the assignment to an object of the generator of the cyclic group of endomorphisms on it. This reflects the fact that, while morphisms in the epicyclic category can send minimal loops to non-minimal ones, the morphisms in the cyclic category must send minimal loops to minimal loops. We show the cyclic category Λ can be identified with the full subcategory of the category $\mathbb{G}_C$-mod(**Set**) consisting of partially oriented groupoids with cycles of the form $\mathbb{Z} \rtimes \{0, \ldots, n-1\}$ (for $n \geq 1$).

We should mention that in an unpublished note ([64]) I. Moerdijk suggested a theory of "abstract circles" classified by the cyclic topos, but his argument that this theory is classified by it appears incomplete since it only proves that the category of set-based models of this theory is equivalent to the category of points of the cyclic topos. A complete, fully constructive, proof of this fact is given in Chapter 9 of [8]. While the theory of abstract circles is coherent, the *cyclic theory* defined in Theorem 3 [24] is infinitary but presents the technical advantage that, while the domain of definition of the concatenation operation on segments in an abstract circle is not controlled by a geometric formula, in the setting of the cyclic theory, the operation of composition of arrows in a groupoid is always defined whenever the codomain of the first arrow matches the domain of the second.

To describe a theory classified by the arithmetic topos, we observe that sending the unique object of $\mathbb{N}^*$ to $\mathbb{Z}$ realizes the category $\mathbb{N}^*$ as a full subcategory of the category of finitely presentable models of the theory $\mathbb{O}_{\neq}$ of *partially ordered groups with a non-triviality predicate $\neq$*, namely the theory obtained from the (algebraic) theory of groups (where the constant 1 denotes the neutral element of the group) by adding a positivity predicate P and the following axioms:

$(\top \vdash P(1))$

$(P(a) \wedge P(b) \vdash_{a,b} P(a \cdot b))$

$(P(a) \vdash_{a,c} P(c^{-1} \cdot a \cdot c))$

$(P(a) \wedge P(a^{-1}) \vdash_a a = 1)$

$(x \neq x \vdash_x \bot)$.

This theory is of presheaf type since it is obtained from a cartesian theory by adding an axiom of the form $(\phi \vdash_{\vec{x}} \bot)$ (a theorem in Chapter 8 of [8] ensures that any quotient of a theory of presheaf type by adding axioms of the form $(\phi \vdash_{\vec{x}} \bot)$ is again of presheaf type).

By applying Theorem 2.11, one shows (cf. Theorem 4 [24]) that the geometric theory $\mathbb{T}_N := \mathbb{O}_{\neq \mathbb{Z}}$ of $\mathbb{Z}$ (where the notation is that of Theorem 2.10) is obtained from $\mathbb{O}_{\neq}$ by adding the following axioms:

$(\top \vdash_a P(a) \vee P(a^{-1}))$

$(\top \vdash_{x,y} (x \neq y) \vee (x = y))$

$(\top \vdash \exists x (x \neq 1))$

$(P(x) \wedge P(y) \vdash_{x,y} \bigvee_{n,m \in \mathbb{N}^+} \exists z (P(z) \wedge (x = z^n) \wedge (y = z^m)))$.

By Theorem 6 [24], the set-based models of $\mathbb{T}_N$ can be identified with the (non-trivial) ordered groups which are isomorphic to ordered subgroups of $(\mathbb{Q}, \mathbb{Q}^+)$. This enlightens a relationship with the geometric theory of finite chains $\mathbb{F}$ considered in section 4.3.2, whose models are precisely the MV-algebras which can be embedded as subalgebras of $\mathbb{Q} \cap [0,1]$. Since the finite chains can be identified with the simple MV-algebras S_n and there is a (necessarily unique) MV-algebra homomorphism $S_n \to S_m$ if and only if n divides m, the classifying topos of $\mathbb{F}$ is the topos $[\widetilde{\mathbb{N}^*}, \mathbf{Set}]$, where $\widetilde{\mathbb{N}^*}$ is the category whose objects are the non-zero natural numbers and whose arrows are given by the divisibility relation. We thus have a geometric surjection

$$\mathcal{E}_{\mathbb{F}} = [\widetilde{\mathbb{N}^*}, \mathbf{Set}] \to [\mathbb{N}^*, \mathbf{Set}] = \mathcal{E}_{\mathbb{T}_N} .$$

This morphism induces a functor

$$\mathbb{F}\text{-mod}(\mathbf{Set}) \to \mathbb{T}_N\text{-mod}(\mathbf{Set}),$$

which can be identified, by Mundici's equivalence, with the functor sending an ℓ-group with unit (G, u) which can be embedded in $(\mathbb{Q}, 1)$ to the underlying ℓ-group G. It follows in particular that the set of isomorphism classes of points of the topos $[\mathbb{N}^*, \mathbf{Set}]$, which was identified in [36] with the set $\mathbb{Q}_+^* \backslash \mathbb{A}^f / \hat{\mathbb{Z}}^*$ of adèle classes whose archimedean component vanishes, is a quotient of the collection of isomorphism classes of points of the topos $\mathcal{E}_{\mathbb{F}}$.

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