C. 204

The Bicategory of Topoi, and Spectra.

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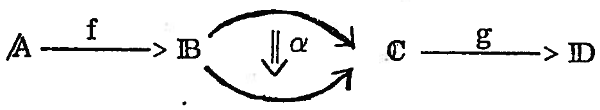
The "spectra" referred to in the title are right adjoints to forgetful functors between categories of topoi-with-structure. Examples are the local-ring spectrum of a ringed topos, the etale spectrum of a local-ringed topos, and many others besides. The general idea is to solve a universal problem which has no solution in the ambient set theory, but does have a solution when we allow a change of topos. The remarkable fact is that the general theorems may be proved abstractly from no more than the fact that Topoi is finitely complete, in a sense appropriate to bicategories.

0. Bicategories.

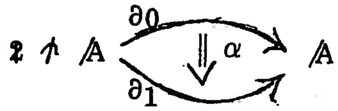
0.1 A 2-category is a Cat-enriched category: it has hom-categories (rather than hom-sets), and composition is functorial, so that the composite of a diagram



denoted $f*\alpha*g$ is unambiguously defined.

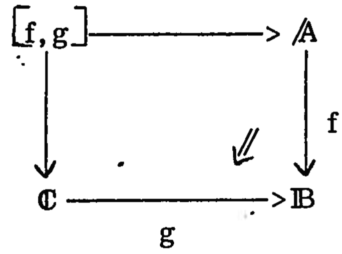
In a 2-category, $\mathcal{A}$, as well as the (ordinary) finite limits obtained from a terminal object and pullbacks, we should consider limits of diagrams having 2-cells.

0.2 For each $\mathbb{A}$, the cotensor with $\mathbf{2}$ of $\mathbb{A}$ is a diagram



for which $\partial_0, \partial_1, \alpha$ induce an isomorphism of hom-categories, $\mathcal{A}(\mathbb{X}, \mathbf{2}\pitchfork\mathbb{A}) = \mathcal{A}(\mathbb{X}, \mathbb{A})^{\mathbf{2}}$, natural in $\mathbb{X}$, where the right-hand category is the (usual) category of morphisms. Thus $\phi : f \Longrightarrow g : \mathbb{X} \longrightarrow \mathbb{A}$ induces a unique $\ulcorner\phi\urcorner : \mathbb{X} \longrightarrow \mathbf{2}\pitchfork\mathbb{A}$ such that $\ulcorner\phi\urcorner * \alpha = \phi$, and with 2-cells $\ulcorner\phi\urcorner \longrightarrow \ulcorner\psi\urcorner$ being induced by commuting squares of 2-cells over $\mathbb{A}$.

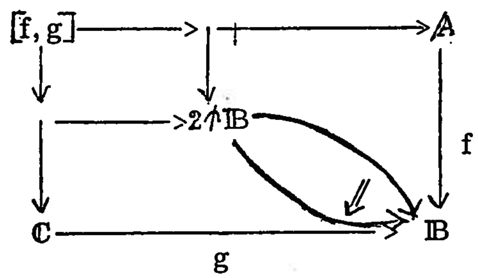
0.3 A <u>comma-object,</u> $[f,g]$ for a pair of 1-cells with common codomain is a square



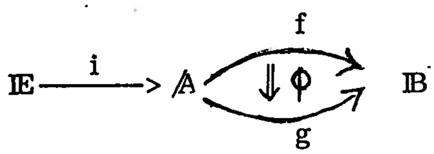
with the universal property, $\mathcal{A}(\mathbb{X}, [f,g]) \cong (\mathcal{A}(\mathbb{X},f), \mathcal{A}(\mathbb{X},g))$, naturally in $\mathbb{X}$, where the right-hand category is the usual comma-category of the composition functors.

In the presence of pullbacks, comma-objects may be constructed from cotensors with $\mathbf{2}$, simply by pulling back ∂_0, ∂_1 along f, g respectively:

0.4



0.5 An <u>identifier</u> is a diagram

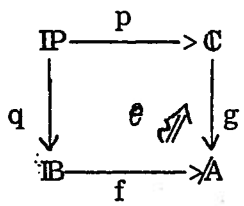


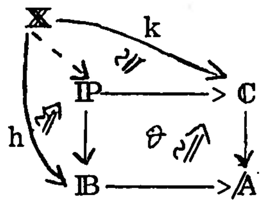
with the universal property that $h\colon \mathbb{X} \longrightarrow \mathbb{A}$ factors (uniquely) through $\mathbb{E} \longrightarrow \mathbb{A}$ iff $h * \phi$ is an identity 2-cell, and with the obvious condition for 2-cells.

Notice that the identity 2-cell $\mathbb{B} \longrightarrow \mathbb{B}$ induces a "diagonal" map, $I\colon \mathbb{B} \longrightarrow \mathbf{2} \pitchfork \mathbb{B}$. It is not hard to see that $I, \partial_0, \partial_1$ make $2 \pitchfork \mathbb{B}$ into a category-object; furthermore, we have adjointness $\partial_1 \dashv I \dashv \partial_0$. The identifier of ϕ may be constructed simply by pulling back $\ulcorner \phi \urcorner\colon \mathbb{A} \longrightarrow \mathbf{2} \pitchfork \mathbb{B}$ along $I\colon \mathbb{B} \longrightarrow \mathbf{2} \pitchfork \mathbb{B}$.

0.6 We say that a 2-category $\mathcal{A}$ is <u>finitely complete</u> if it has a terminal object, pullbacks, and cotensors with $\mathbf{2}$.

0.7 A <u>bicategory</u> (Benabou [2]) has a composition of 1-cells which is associative and unitary only up to a coherent isomorphism (example: a monoidal category is a bicategory with only one object): composition is pseudo-functorial. To translate 2-category notions into the corresponding bicategory Notions, it is hence necessary to replace equality of 1-cells by isomorphisms. In particular, limits defined by an isomorphism of hom-categories must be replaced by Limits, defined by the corresponding equivalence of hom-categories. Unique existence of a 1-cell is replaced by existence, unique up to a unique isomorphism, and so on. We distinguish bicategory Limits from 2-category limits by the use of a capital letter (following Grothendieck [1b]). Thus, 0,8 a <u>Pullback</u> is a square





such that for each h, k, $\lambda : h.f \xrightarrow{\sim} k.g$, there is $\ell : \mathbb{X} \longrightarrow \mathbb{P}$, unique up to unique isomorphism, together with $\kappa : h \xrightarrow{\sim} \ell.q$, $\mu : \ell.p \xrightarrow{\sim} k$ such that $\lambda = (\kappa * f).(\ell * \theta).(\mu * g)$; further, 2-cells, $h \Longrightarrow h'$, $k \Longrightarrow k'$ commuting with λ, λ' induce $\ell \Longrightarrow \ell'$; in short, there is a natural equivalence

$$\mathcal{A}(\mathbb{X},\mathbb{P}) \simeq \mathcal{A}(\mathbb{X},\mathbb{B}) \underset{\mathcal{A}(\mathbb{X},\mathbb{A})}{\times} \mathcal{A}(\mathbb{X},\mathbb{C})$$

where the right-hand category has as objects, triples (h, λ, k) with $\lambda : h.f \xrightarrow{\sim} k.g$. Here, "natural equivalence" in $\mathbb{X}$ means (because of the associativity isomorphisms for composition) that the naturality squares commute up to an isomorphism satisfying the obvious "pasting" condition for composites, $\mathbb{X} \longrightarrow \mathbb{Y} \longrightarrow \mathbb{Z}$.

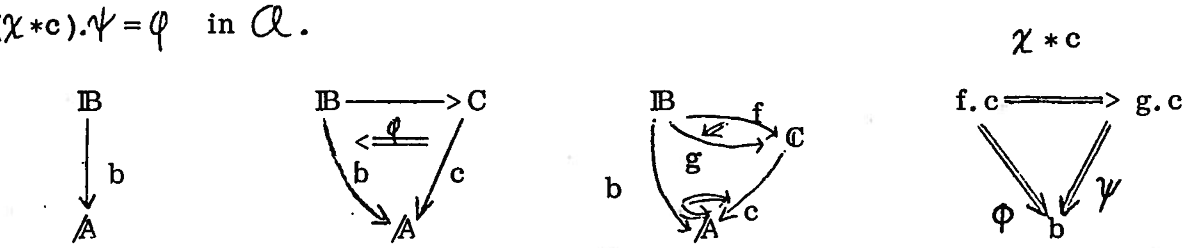
0.9 Other Limits are similarly defined, and we say that a bicategory is <u>finitely Complete</u> if it has a Terminal object, Pullbacks, and coTensors with $\mathbf{2}$, and hence also, Comma-objects, and Inverters (corresponding to identifiers).

A morphism of bicategories can only be a pseudo-functor, and, as we would expect, Limits are pseudo-functorial once they have been chosen (they are, of course, unique up to equivalence).

0.10 A pair of pseudofunctors $U: \mathcal{A} \longrightarrow \mathcal{B}$, $F: \mathcal{B} \longrightarrow \mathcal{A}$ is <u>Adjoint</u>, $F \dashv U$ if there is an equivalence $\mathcal{A}(F(\mathbb{B}), \mathbb{A}) \simeq \mathcal{B}(\mathbb{B}, U(\mathbb{A}))$, natural in $\mathbb{A}$ and $\mathbb{B}$ in the same sense as for 0.8. Equivalently, for each $\mathbb{B}$, there is $\eta: \mathbb{B} \longrightarrow U(F(\mathbb{B}))$ such that for every $h: \mathbb{B} \longrightarrow U(\mathbb{A})$, there is $\bar{h}: F(\mathbb{B}) \longrightarrow \mathbb{A}$, unique-up-to isomorphism, with $\varepsilon: h \overset{\sim}{\Longrightarrow} \eta . U(\bar{h})$; a 2-cell $h \Longrightarrow h'$ induces $\bar{h} \Longrightarrow \bar{h}'$ commuting with ε, ε'.

0.11 We define the <u>comma-bicategory</u>, $\mathcal{A}//\mathbb{A}$ for $\mathbb{A} \in \mathcal{A}$ as follows.

- an object $b \in \mathcal{A}//\mathbb{A}$ is a 1-cell $b: \mathbb{B} \longrightarrow \mathbb{A}$ in $\mathcal{A}$;
- a 1-cell $(f, \varphi): b \longrightarrow c$ in $\mathcal{A}//\mathbb{A}$ is $f: \mathbb{B} \longrightarrow C$ and $\varphi: f.c \Longrightarrow b$ in $\mathcal{A}$;
- a 2-cell $\chi: (f, \varphi) \Longrightarrow (g, \psi)$ in $\mathcal{A}//\mathbb{A}$ is a 2-cell $\chi: f \Longrightarrow g$ such that $(\chi * c).\psi = \varphi$ in $\mathcal{A}$.



object 1-cell 2-cell

0.12 Example. If $u: \mathbb{B} \longrightarrow \mathbb{A}$ induces the obvious $U: \mathcal{A}//\mathbb{B} \longrightarrow \mathcal{A}//\mathbb{A}$, then $[-, u]$ is right Adjoint to U, and this defines the Comma operation.

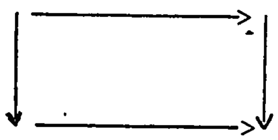
0.13 We recall also that a pair of 1-cells $u: \mathbb{B} \longrightarrow \mathbb{A}$, $f: \mathbb{A} \longrightarrow \mathbb{B}$ is <u>adjoint</u>, $f \dashv u$ if there are 2-cells $\eta: 1_{\mathbb{A}} \Longrightarrow f.u$, $\varepsilon: u.f \Longrightarrow 1_{\mathbb{B}}$ satisfying the usual equations, $(\eta * f).(f * \varepsilon) = 1_f$, $(u * \eta).(\varepsilon * u) = 1_u$. Equivalently, for each $\mathbb{X}$, $\mathcal{A}(\mathbb{X}, f) \dashv \mathcal{A}(\mathbb{X}, u)$, or, again, for each $\mathbb{Y}$, $\mathcal{A}(u, \mathbb{Y}) \dashv \mathcal{A}(f, \mathbb{Y})$, the (ordinary) adjunction transformations being natural-up-to-isomorphism in $\mathbb{X}$ or $\mathbb{Y}$.

A 1-cell is <u>fully-faithful</u>, $f: \mathbb{A} \longrightarrow \mathbb{B}$ if $\mathcal{A}(\mathbb{X}, f)$ is a fully faithful functor for each $\mathbb{X}$; $f \dashv u$ is a <u>reflection,</u> and f the <u>reflector</u>, if u is fully-faithful. Equivalently, the end adjunction, ε, is an isomorphism.

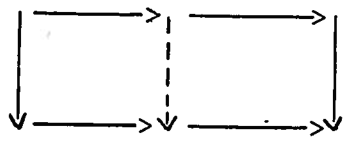
0.14 Lemma. The Pullback of a reflector (coreflector) is a reflector (coreflector).

Proof. Since $\mathcal{E} : u.f \Longrightarrow 1_{\mathbb{B}}$ is an isomorphism, so is its Pullback along $g: \mathbb{C} \longrightarrow \mathbb{A}$, $\bar{\mathcal{E}} : \bar{u}.\bar{f} \Longrightarrow 1_{\bar{\mathbb{B}}}$. But $\eta : 1_{\mathbb{A}} \Longrightarrow f.u$ Pulls back to $\bar{\eta} : 1_{\bar{\mathbb{A}}} \Longrightarrow \bar{f}.\bar{u}$, satisfying the relevant equations. ∎

We consider $\mathcal{E}$ - $\mathcal{M}$ factorisation systems on an ordinary category, $\mathbb{A}$. If $\mathcal{M}$ is a class of maps of $\mathbb{A}$ containing isomorphisms and closed under composition, we say that $\mathcal{M}$ gives best factorisations if every map $A \longrightarrow B$ in A has a factorisation $A \longrightarrow C \longrightarrow B$ with $C \longrightarrow B \in \mathcal{M}$, such that for any other such factorisation, $A \longrightarrow C' \longrightarrow B$ with $C' \longrightarrow B \in \mathcal{M}$, there is a unique $C \longrightarrow C' \in \mathcal{M}$, making both triangles commute. We say that a factorisation is functorial if a commuting square factors into commuting squares:



factors with



commuting

(for an $\mathcal{E}$ - $\mathcal{M}$ factorisation, this is equivalent to the usual diagonal property).

0.15 Proposition. For a category, $\mathbb{A}$, the following data are equivalent:

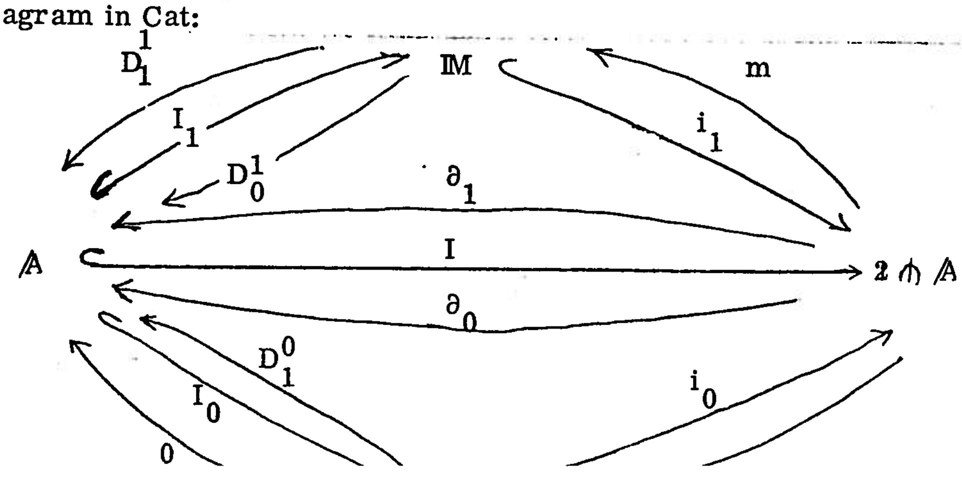
(a) a class $\mathcal{M}$ of maps giving functorial best factorisations;

(b) a class $\mathcal{E}$ of maps giving functorial co-best factorisations;

(c) a functorial $\mathcal{E}$ - $\mathcal{M}$ factorisation;

(d) for each category $\mathbb{X}$, a factorisation of type (a), (b) or (c) on $Cat(\mathbb{X}, \mathbb{A})$ such that for each $f: \mathbb{Y} \longrightarrow \mathbb{X}$, if $\alpha.\beta$ is a factored map in $Cat(\mathbb{X}, \mathbb{A})$, then $(f*\alpha).(f*\beta)$ is the factorisation in $Cat(\mathbb{Y}, \mathbb{A})$ of $f_*(\alpha.\beta)$;

(e) the diagram in Cat:



in which $I_1.i_1 = I = I_0.i_0$, each functor is left adjoint to the one immediately below it, $(\mathbb{A}, I_0, I_1)$ is the Pullback of (i_0, i_1), and $(2 \pitchfork \mathbb{A}, m, e)$ is the Pullback of (D_0^1, D_1^0).

Proof-sketch. Given $\mathcal{M}$, define $\mathcal{E}$ to be the class of maps whose "best $\mathcal{M}$-factor" is an isomorphism, and conversely. This establishes the equivalence of the first three. For (d), take those natural transformations whose components lie in $\mathcal{M}$ or $\mathcal{E}$, and, conversely, take $\mathbb{X} = \mathbb{1}$. Finally, for (e), let $\mathbb{M}$ and $\mathbb{E}$ be the full subcategories of $\mathbb{A}^2$ consisting of those maps which are in $\mathcal{M}$ or $\mathcal{E}$. The functors m, e are "best $\mathcal{M}$-factor" (resp. $\mathcal{E}$-factor). ∎

We take (d) to be the definition in an arbitrary bicategory of <u>$\mathcal{E}$-$\mathcal{M}$ factorisation on $\mathbb{A}$,</u> and say that it is <u>representable</u> if the diagram (e) exists.

0.16 Proposition. If $\mathcal{Q}$ is finitely Complete, then any $\mathcal{E}$-$\mathcal{M}$ factorisation on an object is representable.

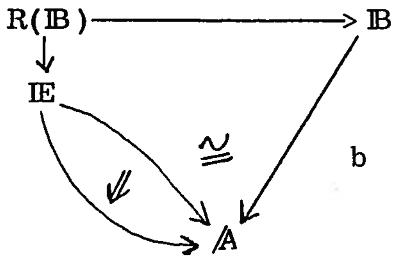
Proof. Factorise the universal 2-cell $\alpha: \partial_0 \Longrightarrow \partial_1: 2 \pitchfork \mathbb{A} \longrightarrow \mathbb{A}$ into $\eta: \partial_0 \Longrightarrow d$, $\mu: d \Longrightarrow \partial_1$. Define $\mathbb{M}$ to be the Inverter of η, and $\mathbb{E}$ the Inverter of μ. Since a map is in $\mathcal{M}$ (resp. $\mathcal{E}$) iff its best $\mathcal{E}$-(resp. $\mathcal{M}$-) factor is an isomorphism, it is clear that $\varphi: f \Longrightarrow g: \mathbb{X} \longrightarrow \mathbb{A}$ is in $\mathcal{M}_{\mathbb{X}}$ (resp. $\mathcal{E}_{\mathbb{X}}$) iff $\ulcorner\varphi\urcorner: \mathbb{X} \longrightarrow 2 \pitchfork \mathbb{A}$ factors through $\mathbb{M}$ (resp. $\mathbb{E}$). The rest of the diagram follows immediately. ∎

Given a class $\mathcal{M}$ of 2-cells over $\mathbb{A}$, containing isomorphisms, closed under composition, and satisfying $\alpha \in \mathcal{M}$ implies $f * \alpha \in \mathcal{M}$, it is clear how we may modify the definition of the comma-bicategory by allowing only 2-cells of $\mathcal{M}$ to appear, giving a bicategory $\mathcal{M}-\mathcal{Q}//\mathbb{A}$.

0.17 Proposition If $\mathcal{Q}$ is finitely Complete, the obvious $\mathcal{M}-\mathcal{Q}//\mathbb{A} \longrightarrow \mathcal{Q}//\mathbb{A}$ has a right Adjoint iff $\mathcal{M}$ forms an $\mathcal{E}$-$\mathcal{M}$ factorisation on $\mathbb{A}$.

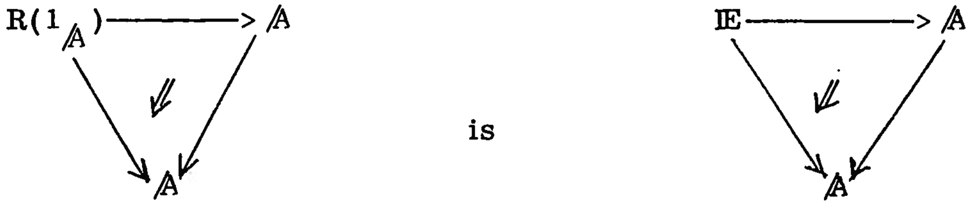
Proof. Given an $\mathcal{E}$-$\mathcal{M}$ factorisation, it is representable, by 0.16. We define the required right Adjoint by taking $R(b: \mathbb{B} \longrightarrow \mathbb{A})$ to be the Pullback of $D_0^0: \mathbb{E} \longrightarrow \mathbb{A}$

along b, with structure-map $R(\mathbb{B}) \longrightarrow \mathbb{E} \xrightarrow{D^0_1} \mathbb{A}$. The end adjunction is the map (in $\mathcal{A}//\mathbb{A}$



where the 2-cell is the universal $\mathcal{E}$-map, whence we see that the end adjunction is the universal $\mathcal{E}$-map with domain b. The universal property emerges immediately from the definition (d) of a factorisation.

Conversely, given the right Adjoint, R, define $\mathbb{E} \Downarrow \mathbb{A}$ by taking $D^0_1 : \mathbb{E} \longrightarrow \mathbb{A}$ to be $R(\mathbb{A} \longrightarrow \mathbb{A})$, and the end adjunction to be the universal $\mathcal{E}$-map:



The universal property of the end adjunction leads directly to the best factorisation property of the class of maps represented by $\mathbb{E}$, whose orthogonal $\mathcal{M}$-class is that originally given. ∎

Notice in particular that since $\mathbb{A}$ is a coreflective subobject of $\mathbb{E}$, the construction of R shows that $R(\mathbb{B})$ contains $\mathbb{B}$ as a coreflective subobject, by 0.14. The inclusion "classifies" the identity-map of b as an $\mathcal{E}$-map.

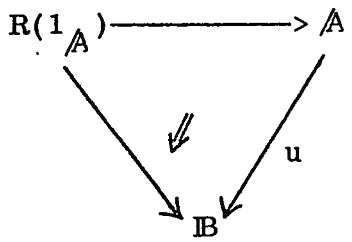
Finally, and not very elegantly, we combine 0.12, 0.17 and a restricted class of 2-cells. Suppose $\mathcal{M}$ is a class of 2-cells, closed under compositions, etc., so that $\mathcal{M}-\mathcal{A}//\mathbb{A}$ is defined, similarly, $\mathcal{N}-\mathcal{A}//\mathbb{B}$, and suppose we are given $u: \mathbb{B} \longrightarrow \mathbb{A}$, such that if $\alpha \in \mathcal{N}$ then $\alpha * u \in \mathcal{M}$. Then we obtain a pseudo-functor, $\mathcal{N}-\mathcal{A}//\mathbb{B} \longrightarrow \mathcal{M}-\mathcal{A}//\mathbb{A}$. In this situation, we say that $\mathcal{N}$ forms a u-$\mathcal{M}$-factorisation if every 2-cell $\alpha: f \Longrightarrow g.u$ has a best factorisation, $f \Longrightarrow h.u, h \Longrightarrow g$, with $h \Longrightarrow g \in \mathcal{N}$ such that for any other such factorisation, $f \longrightarrow h'.u \longrightarrow g.u$, there is a unique $h \Longrightarrow h'$, making both triangles commute, and that such best factorisations are stable under compositions with 1-cells, as for 0.15(d).

0.18 Proposition. If $\mathcal{A}$ is finitely Complete and $\mathcal{M}$ is representable, then

$$\mathcal{N}-\mathcal{A}//\mathbb{B} \dashrightarrow \mathcal{M}-\mathcal{A}//\mathbb{A}$$

has a right Adjoint iff $\mathcal{N}$ forms a u-$\mathcal{M}$-factorisation.

Proof. If $\mathcal{N}$ forms a factorisation, define $\mathbb{E} \longrightarrow [\mathbb{A}, u]$ to represent the $\mathcal{N}$-extremal $\mathcal{M}$-maps (those whose best $\mathcal{N}$-factor is an isomorphism), by Inverting the best $\mathcal{N}$-factor of the universal 2-cell obtained by Pulling back $\mathbb{M} \longrightarrow \mathbf{2}\pitchfork\mathbb{A} \xrightarrow{\partial_1} \mathbb{A}$ along $u\colon \mathbb{B} \longrightarrow \mathbb{A}$. Pull the "domain" map $\mathbb{E} \longrightarrow \mathbb{A}$ along $c\colon \mathbb{C} \longrightarrow \mathbb{A}$ to define $R(c\colon \mathbb{C} \longrightarrow \mathbb{A})$, right Adjoint to the given forgetful functor. Conversely, given the right Adjoint, R, define the universal $\mathcal{N}$-extremal $\mathcal{M}$-map to be the end adjunction for $R(\mathbb{A} \longrightarrow \mathbb{A})$, and proceed as in 0.17:



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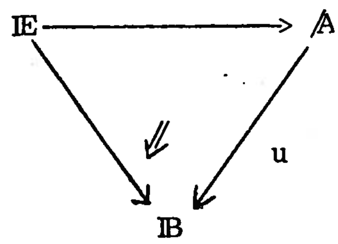


1. Limits in Topoi.

We consider two 2-categories and a bicategory. Lex is the 2-category of finitely complete (small) categories, left-exact functors, and natural transformations. LexSite is the 2-category of finitely complete (small) categories equipped with a Grothendieck topology, left-exact cover-preserving functors, and natural transformations. Topoi is the bicategory of cocomplete topoi (i.e. Sets-topoi), geometric morphisms, and natural transformations between the inverse-image functors (with composition defined whichever way you prefer). While it is true that Topoi may be made into a 2-category, we choose not to. Each way of defining compositions associative up to equality has its disadvantages, and none seems canonical. The real point is that Topoi has Limits, rather than limits. "Straightening out" all the canonical isomorphisms seems an insuperable task, and is probably not worth it: it seems that the cheapest way of handling the difficulties is to put them in at the start.

Since Lex is "monadic" over Cat (in a sense we leave to the experts to make precise), it is clear that (strict) limits may be constructed at the underlying category level. What is a little mysterious is the fact that many of these limits in Lex turn out to be coLimits (in the "underlying" bicategory).

1.1 Lemma. $\mathbb{1}$ is coTerminal in Lex.

Proof. The canonical unique $\mathbb{A} \longrightarrow \mathbb{1}$ has a right adjoint, $\mathbb{1} \longrightarrow \mathbb{A}$ (the terminal object of $\mathbb{A}$) which is unique among left-exact functors. ∎

1.2 Lemma. $\mathbb{A} \times \mathbb{B}$ is the coProduct in Lex.

Proof. The projections have right adjoints, $A \longmapsto (A,1)$, $B \longmapsto (1,B)$ which give injections. Given $h: \mathbb{A} \longrightarrow \mathbb{X}$, $k: \mathbb{B} \longrightarrow \mathbb{X}$, define $\ell : \mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{X}$ by $\ell(A,B) = h(A) \times k(B)$, the product in $\mathbb{X}$. ∎

1.3 Lemma. The cotensor $\mathbf{2} \pitchfork \mathbb{A}$ (the category $\mathbb{A}^{\mathbf{2}}$) is also the Tensor $\mathbf{2} \otimes \mathbb{A}$ in Lex.

Proof. Again, the projections have right adjoints, $\delta_0 : A \longmapsto (A \longrightarrow A)$, $\delta_1 : A \longmapsto (A \longrightarrow 1)$, for injections, with the obvious 2-cell. Given $\alpha : f \Longrightarrow g : \mathbb{A} \longrightarrow \mathbb{X}$, define $\ulcorner \alpha \urcorner : \mathbb{A}^{\mathbf{2}} \longrightarrow \mathbb{X}$ by taking $\ulcorner \alpha \urcorner (a : A_1 \longrightarrow A_2)$ to be the pullback in $\mathbb{X}$ of $g(a)$ along α_{A_2}. ∎

1.4 Lemma. The comma-category $(\mathbb{B}, f)$ for $f : \mathbb{A} \longrightarrow \mathbb{B}$ in Lex is also the coComma object $\langle \mathbb{A}, f \rangle$.

Proof. Just as in 1.3, but with more letters. ∎

Thus Lex is almost finitely coComplete - we lack coEqualisers, which may perhaps be provided by "monadicity" over Cat. Since an intersection of topologies is a topology, we may always find the "least topology such that ...". In an appropriate sense, the forgetful LexSite $\longrightarrow$ Lex is an initial structure functor, which we use to lift coLimits from Lex to LexSite.

1.5 Lemma. LexSite $\longrightarrow$ Lex creates coLimits.

Proof. Given a diagram $\mathcal{D}$ in LexSite having a coLimit in Lex, we simply provide the coLimit with the least topology for which the injections preserve coverings: the smallest containing the images under injection of coverings in the diagram. Then a map out of the coLimit preserves coverings iff its composites with the injections all do, so we are finished. ∎

1.6 Remark. Suppose that $f: \mathbb{A} \longrightarrow \mathbb{B}$ is a topos-map. Then the comma-category $(\mathbb{B}, f_*) \cong (f^*, \mathbb{A})$ since $f^* \dashv f_*$, and it satisfies the coComma property for left-exact functors. In fact, it is the coComma object in the bicategory Topoi, the inverse-image functors being given by the comma-property, the direct-images being provided by the coComma property. Thus Topoi has coComma objects of the form $\langle f, \mathbb{A} \rangle$. With an arbitrary left-exact functor in place of f_*, this construction is the well-known Artin glueing, [1a], [9] .

We turn now to Topoi , and recall that we have pseudo-functors,

$$\widehat{(\ )}: \underline{\underline{\mathrm{Lex}}}^{op} \longrightarrow \underline{\underline{\mathrm{Topoi}}} \qquad \widetilde{(\ )}: \underline{\underline{\mathrm{LexSite}}}^{op} \longrightarrow \underline{\underline{\mathrm{Topoi}}}$$

called "presheaves (resp. sheaves) on (—)"; the "op" indicates that 1-cells are reversed, but 2-cells retain their direction. On 1-cells, the direct-image functors are induced by composition, and the inverse-image functor is the left Kan extension, left-exact because the original 1-cell is.

We state without proof the classification theorem ([1a] , [3]).

1.7 Theorem. (a) $\underline{\underline{\mathrm{Topoi}}}(\underline{E}, \widehat{\mathbb{A}}) \simeq \underline{\underline{\mathrm{Lex}}}(\mathbb{A}, \underline{E})$, (b) $\mathrm{Topoi}(\underline{E}, \widetilde{\mathbb{A}}) \simeq \mathrm{LexSite}(\mathbb{A}, \underline{E})$, naturally in $\underline{E}$ and $\mathbb{A}$. ∎

Note the abuse of language whereby we have treated the (large) underlying category of a topos as an object of Lex, or, with its canonical topology, of LexSite.

1.8 Corollary. $\widehat{(\ )}$, $\widetilde{(\ )}$ take coLimits to Limits.

Proof. The usual argument for adjoint functors also works for this partial Adjointness of pseudo-functors: if $\mathbb{D}$ is a diagram in Lex, having a coLimit, $\mathbb{L}$, and $\widehat{\mathbb{D}}^{op}$ is the corresponding diagram of topoi, then

$$\text{Topoi}(E, \hat{\mathbb{O}}^{op}) \simeq \text{Lex}(\mathbb{O}, E) \qquad (1.7(a))$$

$$\simeq \text{Lex}(\mathbb{L}, E) \quad \text{by definition of coLimit,}$$

$$\simeq \text{Topoi}(E, \hat{\mathbb{L}}) \qquad (1.7(a)),$$

whence $\hat{\mathbb{L}}$ is the Limit in Topoi. A similar argument works for sites. ∎

Recall that a topos is a Grothendieck topos if (it is cocomplete and) it has a (small) set of generators. We denote by GrTopoi the full subcategory of Grothendieck topoi. By relativising these notions to an arbitrary elementary topos playing the role of Sets, Diaconescu arrives at the notion of a bounded topos-map $E \longrightarrow F$, one for which E has an F-object of generators, and by relativising the classification theorem, obtains [3]:

1.9 The Pullback of a topos-map along a bounded topos-map exists. ∎

It is easy to show that any map $\tilde{\mathbb{A}} \longrightarrow E$ is bounded, and the Giraud theorem characterises Grothendieck topoi as those of the form $\tilde{\mathbb{A}}$ for some (not unique) site $\mathbb{A}$.

Combining 1.9 with the results above, we obtain:

1.10 Proposition. GrTopoi is finitely Complete. Topoi has a Terminal object, the Pullback $A \times_C B$ exists if one of A, B is bounded over C, the Comma-topos $[f, g]$ exists if one of f, g has Grothendieck domain and codomain, and the Inverter of $\alpha: f \Longrightarrow g: A \longrightarrow B$ exists if B is Grothendieck.

Proof. Sets is Terminal, Pullbacks along maps between Grothendieck topoi exist, and coTensors with $\mathbf{2}$ are obtained from Tensor-sites. The rest are constructed from these. ∎

We are thus in a situation where the results of § 0 apply.

Needless to say, Inverters may also be constructed as the largest sheaf subtopos for which the components of α are bidense. Conversely, it is not hard to show that every sheaf embedding is an Inverter, by using the relativised version of 1.7(a). $Sh_j(E) \longrightarrow E$ is the Inverter of "1" $\Longrightarrow$ "J": $E \longrightarrow E^{\Omega^{op}}$, where "1", "J" are induced by the flat discrete fibrations $1 \longrightarrow \Omega^{op}$, $J^{op} \longrightarrow \Omega^{op}$.

(This is essentially due to Johnstone [6]).

Adjoint 1-cells in Topoi are just what one would expect.

1.11 If $f: \underline{E} \longrightarrow \underline{F}$, $g: \underline{F} \longrightarrow \underline{E}$ are 1-cells in Topoi, then

$$f \dashv g \text{ iff } f^* \cong g_* \text{ iff } g^* \dashv f^* \text{ iff } g_* \dashv f_*$$

Proof. The equivalence of the last three is immediate from the uniqueness of adjoints. The equivalence with the first is shown simply by unwinding the equational definition of adjointness for the 1- and 2-cells of Topoi. ∎

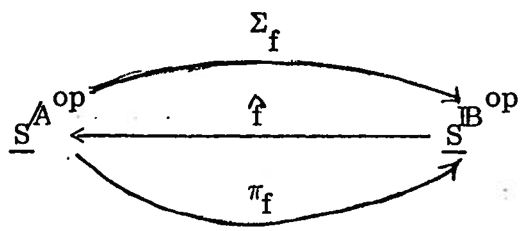
1.12 A 1-cell in Topoi is fully-faithful iff it is equivalent to a sheaf embedding. ∎

2. Examples.

We take the view that "every Grothendieck topos classifies something" (namely, the left-exact, cover-preserving functors — this may be given a first-order syntactic form, albeit with possibly infinitary disjunctions).

If $\underline{T}$ is the classifying topos for the theory $\mathcal{T}$, i.e. Topoi $(\underline{E}, \underline{T}) \simeq \mathcal{T}$-models $(\underline{E})$, naturally in $\underline{E}$, then the inverse-image, $f^*(M)$ of a model by a map of topoi is again a model. Similarly, a map $u: \underline{T}_2 \longrightarrow \underline{T}_1$ induces a "forgetful" functor, $\mathcal{T}_2$-models $(\underline{E}) \longrightarrow \mathcal{T}_1$-models $(\underline{E})$ by composition. A 2-cell $\alpha: f \Longrightarrow g: \underline{E} \longrightarrow \underline{T}$ is interpreted as a $\mathcal{T}$-model homomorphism, whence we see immediately that $2 \pitchfork \underline{T}$ is the $\mathcal{T}$-morphism-classifier. Thus the model theory of topoi is coextensive with the study of the bicategory-structure of Topoi. We shall usually identify a $\mathcal{T}$-model M in $\underline{E}$ with its classifying map $M: \underline{E} \longrightarrow \underline{T}$, hoping that this simplifies life for the reader, rather than confusing him.

In this light, we examine an example of adjoint topos-maps. A left-exact functor $f: \mathbb{A} \longrightarrow \mathbb{B}$ induces three functors,



$$\Sigma_f \dashv \hat{f} \dashv \pi_f$$

Σ_f, π_f being the left and right Kan extensions. We have already identified $\Sigma_f \dashv \hat{f}$ as the topos-map $\hat{f}: \hat{\mathbb{B}} \to \hat{\mathbb{A}}$. But since $\hat{f}$ is left-exact, $\hat{f} \dashv \pi_f$ is also a topos-map $f^{\#}: \hat{\mathbb{A}} \to \hat{\mathbb{B}}$, and from 1.11, we know that $f^{\#} \dashv \hat{f}$.

Now, we may consider $\mathbb{A}$, $\mathbb{B}$ to be the duals of categories of finitely presented algebras, thus thinking of $\mathbb{A}, \mathbb{B}$ themselves as algebraic theories, with f an interpretation. Then $\hat{\mathbb{A}}$, $\hat{\mathbb{B}}$ are the $\mathbb{A}$- and $\mathbb{B}$-algebra classifiers, $\hat{f}$ represents the "forgetful" $\mathbb{B}\text{-alg}(\underline{E}) \to \mathbb{A}\text{-alg}(\underline{E})$ and $f^{\#}$ represents its left adjoint, the relatively free functor. For example, if $f^{op}: \mathbb{A}^{op} \to \mathbb{B}^{op}$ is the abelianisation functor from finitely presented groups to f.p. abelian groups, we obtain the abelian-group classifier as a reflective sub-topos of the group classifier. We mention a further point of interest for this example. If the interpretation is finitary - involves the imposition of finitely many new axioms - as for the example of groups and abelian groups, in the sense that if B is a finitely presented $\mathbb{B}$-algebra then U(B) is finitely-presented as an $\mathbb{A}$-algebra, then the forgetful functor restricts to $\mathbb{B}^{op} \to \mathbb{A}^{op}$, providing a left adjoint g to f. We obtain from this a fourth functor, $\Sigma_g \dashv \Sigma_f$ between the classifying topoi, so that the "forgetful" map $\hat{f}$ is actually an essential topos-map.

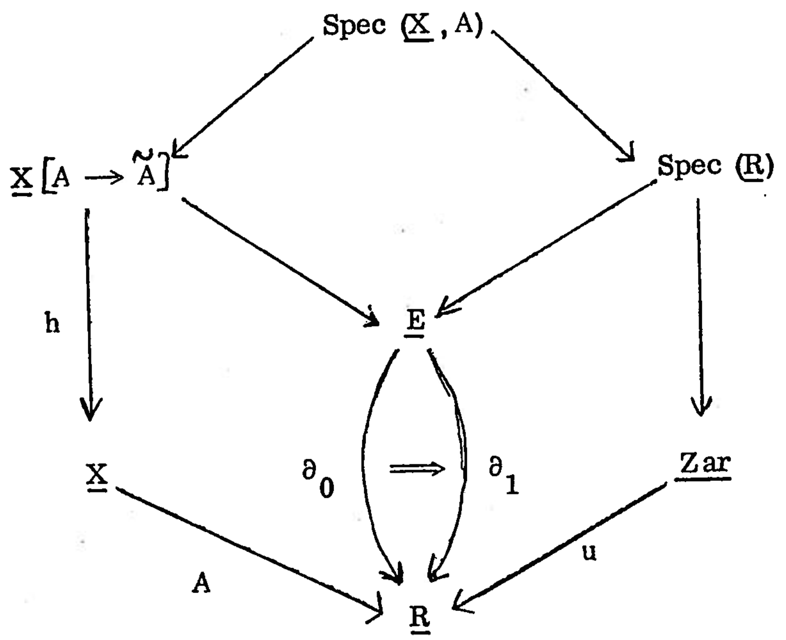
A more geometric example of adjoint topos-maps is furnished by the relationship (given in [1a]) between the "gros" topos of a space and its "ordinary" topos. In fact, since the map of sites, Open (X) $\xrightarrow{i}$ Spaces /X is cover-reflecting, it induces not only the "restriction" map i: TOP(X) $\to$ Sh(X) but also the left adjoint inclusion, Sh(X) $\to$ TOP(X), so that Sh(X) is a coreflective sub-topos of TOP(X). The remark that these topoi are therefore cohomologically equivalent applies equally to other coreflective situations. For example, the Zariski topos, Zar, sheaves on affine schemes of finite type, may (by the Lemme de Comparaison of [1a]) equally be constructed as sheaves on the category of schemes. The Zariski topology is less fine than the canonical, and so, using the Yoneda functor, we may consider a scheme X both as a ringed space and as an object of Zar. Essentially the same argument as for the gros topos shows that Sh(X) is a coreflective subtopos of Zar/X.

Suppose $\mathcal{D}$ is a finite diagram-type, and $\underline{T}$ a classifying topos - the object-classifier, pour fixer les idees. We may form $\mathcal{D} \pitchfork \underline{T}$, the $\mathcal{D}$-diagram classifier (of $\mathcal{T}$-models) by taking Pullbacks and Comma-objects according to the recipe by which $\mathcal{D}$ is made up from nodes, arrows, and commutativity relations. Thus for example, the classifier for diagrams $\cdot \longrightarrow \cdot \longleftarrow \cdot$ is obtained by Pulling back $\partial_1 : 2 \pitchfork \underline{T} \longrightarrow \underline{T}$ along itself; the commuting-square-classifier is got by Pulling ∂_1 along ∂_0 to obtain $3 \pitchfork \underline{T}$, and then Pulling the "composition" map $3 \pitchfork \underline{T} \longrightarrow 2 \pitchfork \underline{T}$ back along itself.

Just as $2 \pitchfork \underline{T}$ is the $\mathcal{T}$-morphism classifier, so, for $u : \underline{T}_2 \longrightarrow \underline{T}_1$, the Comma-topos, $[\underline{T}_1, u]$ classifies $\mathcal{T}_1$-morphisms $A \longrightarrow u(B)$, where A is a $\mathcal{T}_1$-model and B is a $\mathcal{T}_2$-model. Recall that 1.4, 1.5 construct a site of definition of $[\underline{T}_1, u]$ from a site-map defining u. The sites defining the spectra of Hakim [5] are of closely related form, with a finer topology. Given a particular model, $A : \underline{E} \longrightarrow \underline{T}_1$, the Comma-topos $[A, \underline{T}_1]$ classifies "$\mathcal{T}_1$-maps with domain A": given $f : \underline{F} \longrightarrow \underline{E}$, maps $\underline{F} \longrightarrow [A, \underline{T}_1]$ over $\underline{E}$ correspond to $\mathcal{T}_1$-maps $f^*(A) \longrightarrow (-)$ where $(-)$ is any $\mathcal{T}_1$-model in $\underline{F}$. Similarly $[u, A]$ classifies $\mathcal{T}_1$-maps $u(-) \longrightarrow A$, and $[A, u]$ classifies maps $A \longrightarrow u(-)$.

Clearly various "epi-mono" factorisations of $\mathcal{T}$-maps give rise to applications of 0.17. It is easy to see that $\underline{\underline{\text{Topoi}}}//\underline{T}$ is the category of $\mathcal{T}$-modelled topoi, defined in the same way as the usual category of ringed topoi. For the category of local-ringed topoi, however, we must insist that all the ring-homomorphisms be local, i.e. reflect the units (invertible elements), whence, in our previous notation, Loc-$\underline{\underline{\text{Topoi}}}$//Zar is the category of local-ringed topoi. We see from §0 that the existence of a right Adjoint (the spectrum Hakim [5]) to the forgetful Loc-$\underline{\underline{\text{Topoi}}}$//Zar $\longrightarrow$ $\underline{\underline{\text{Topoi}}}//\underline{R}$ ($\underline{R}$ the ring classifier) is equivalent to the fact that a ring-homomorphism $A \longrightarrow L$ with L a local ring has a best factorisation $A \longrightarrow F \longrightarrow L$ with $F \longrightarrow L$ a local map; the associated extremal maps A—>F are the localisations, obtained by pulling back the units of L to A, and forming the ring-of-fractions to invert this "prime co-ideal", giving the local ring F(Tierney [8]).

It is worth unravelling the proof of the relevant version of 0.18 for this case. There is an underlying factorisation of ring-homomorphisms (not just those with local codomain), namely, with $\mathcal{M} = \{$unit-reflecting maps$\}$ and $\mathcal{E} = \{$ring-of-fractions maps$\}$ (of the form $A \rightarrow A[S^{-1}]$ for some multiplicatively closed subobject S of A). We factorise the universal ring-homomorphism, $2 \pitchfork \underline{R} \overset{\Downarrow}{\rightrightarrows} \underline{R}$, and invert its $\mathcal{M}$-part, to obtain the fractions-map-classifier, $\underline{E}$. Now Pull back the "codomain" map along $u: Zar \rightarrow \underline{R}$ to obtain the localisation-classifier (the spectrum of the universal ring), and finally, Pull the "domain" map back along a given ring $A: \underline{X} \rightarrow \underline{R}$ to obtain Spec $(\underline{X}, A)$. Notice that these steps all commute with each other: we may factorise and Pull back in any convenient order.



In particular, $\underline{X}[A \rightarrow \tilde{A}]$ is the classifier for "fractions-maps with domain A". It has a map to the topos $\underline{X}$, so we may imagine $\underline{X}[A \rightarrow \tilde{A}]$ as being a topos of sheaves with values in $\underline{X}$, the direct-image functor, h_*, being thought of as "globalsections", the inverse-image, h^*, being "constant sheaf".

The universal fractions-map, $\underline{X}\,[A \longrightarrow \tilde{A}] \longrightarrow \underline{E} \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \underline{R}$ then looks like a fractions-map $h^*(A) \longrightarrow \tilde{A}$ of rings in $\underline{X}\,[A \longrightarrow \tilde{A}]$, corresponding by adjointness to $A \longrightarrow h_*(\tilde{A})$. Thus A is represented in the "global sections of a sheaf". (When $\underline{X}$ = Sets, this is literally true). But recall that since $\underline{R}$ is a coreflective subtopos of $\underline{E}$, by 0.14, $\underline{X}$ is a coreflective subtopos of $\underline{X}\,[A \longrightarrow \tilde{A}]$, whence by 1.11, we see that the functor h_* is actually the inverse-image functor of the inclusion $\underline{X} \longrightarrow \underline{X}\,[A \longrightarrow \tilde{A}]$. The front adjunction isomorphism then gives immediately that $A \longrightarrow h_*(\tilde{A})$ is an isomorphism, since the inclusion classifies $A \longrightarrow A$ as fractions-map. This argument shows that for "spectra" of the kind given by 0.17, the "representation of A in a sheaf" is always an isomorphism, $A \xrightarrow{\simeq} h_*(A)$. However, for most purposes, this is not enough: we "force" the codomain, $\tilde{A}$, to be a model of a richer theory (local rings in this case), by Pulling back ∂_1, (along $\underline{Zar} \longrightarrow \underline{R}$), which obstructs the argument. This author suspects that further progress will involve considering the Beck condition for Pullbacks of coherent topoi.

Another example is furnished by the étale spectrum of a local-ringed topos (Hakim [5]). Joyal and Wraith have determined that Hakim's strictly local rings are those local rings, A, which are "separably closed" in the following sense. If $f(t) \in A[t]$ is monic (i.e. has leading coefficient 1), consider $D(f)(t) = t^n - \prod_{i=1}^{n}(t - f'(\alpha_i))$, where $\alpha_1, \ldots, \alpha_n$ are the roots of f (in some hypothetical extension of A), and f' is the formal derivative of f. Since $D(f)$ is symmetric in the α_i's, it has coefficients lying in A (Newton's theorem on symmetric polynomials), whence we have a purely combinatorial procedure for defining $D(f)(t)$ without reference to any roots. Classically, $D(f) \equiv 0$ iff all the roots of f are repeated roots. The axiom for a local ring to be strictly local says: $D(f)(t)$ has an invertible coefficient implies $\exists a \in A: f(a) = 0$ and $f'(a)$ is invertible. Hakim considers local homomophisms between strictly local rings, and constructs a "spectrum" to "strictify" a local ring, universally, of which the étale topos of a scheme is an example. Wraith has (tentatively) identified the extremal maps for the best factorisation of a local map A—>S into A—>T—>S, with S, T strictly local, as being those maps φ : A—>T for which ever $t \in T$ satisfies a polynomial equation $(\varphi(f))(t) = 0$ with $(\varphi(f))'(t)$ invertible, f a monic polynomial over A (T is "separably integral" over A) and T is strictly local. Such a factorisation, stable under inverse-image functors, is equivalent to Hakim's construction of a right Adjoint to the forgetful

Loc-Topoi //StrZar ———> Loc-Topoi //Zar, by 0.18.

In similar vein, it is conjectured that the crystalline topos of a scheme will be associated with a universal extremal "extension of A by a nil-ideal with divided power structure" I $\hookrightarrow$>B —>A (plus further structure whose details are here irrelevant).

An unfamiliar application is to ordered sets. An order-preserving map P—>L from an ordered set to a linearly ordered set has a best factorisation whose second factor is order-reflecting $(f(x) < f(y)$ implies $x < y)$ between linear orderings: pull back the ordering of L to P, and quotient by the antisymmetry law. Hence there is a right Adjoint to the forgetful OrdRefl-Topoi// L ——> Topoi//P, where L, P classify linear, resp. partial orderings.

As a final example, we construct a spectrum for ordered rings, for which the "Zariski topology" would better be called the Euclidean topology. An ordered ring in this case means a ring with a predicate P(x) (read "x is positive") satisfying $\neg P(0)$, $P(1)$, $P(x) \wedge P(y)$ implies $P(x+y) \wedge P(xy)$. Say that the ring A is <u>linear</u> if in addition, $P(x+y)$ implies $P(x) \vee P(y)$, and $P(xy)$ implies $P(x) \vee P(-x)$. Call A <u>full</u> if $P(x)$ implies $\exists y(xy=1)$. Since the positive elements are multiplicatively closed, any ring may be made full by taking fractions. A linear full ordered ring is called local (and is local in the usual sense). A map of local ordered rings (a homomorphism preserving positivity) is local iff it reflects positivity, i.e. it reflects the ordering. To factorise a map A—>L from an ordered ring to a local ordered ring, proceed as above to linearise, and then add in fullness: pull back the ordering from L to A, and make A full with respect to this finer ordering. Extremal maps are localisations in the ordinary sense, but thought of primarily as linearisations of the ordering. This leads to a spectrum, right Adjoint to Loc-Topoi //OrdZar—> Topoi//OrdR. Closer analysis (private communication with M. P. Fourman) reveals that a base of "open sets" of this spectrum is of the form $\{\{x: f(x)>0\} : f \in A\}$ whereas the Zariski base is of the form $\{\{x: f(x) \neq 0\} \ f \in A\}$, x ranging over the "points" of the spectrum (it is indeed "spatial" over its domain, in the sense that it is generated by its subobjects of 1; and when Zorn's Lemma holds in the domain topos, it has enough points, so that it is spatial in the strong sense.)

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