Topos Theory

Lectures 5-6: Sheaves on a topological space

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Presheaves on a topological space

Definition

Let X be a topological space. A presheaf $\mathcal{F}$ on X consists of the data:

(i) for every open subset U of X, a set $\mathcal{F}(U)$ and

(ii) for every inclusion $V \subseteq U$ of open subsets of X, a function $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ subject to the conditions

- $\rho_{U,U}$ is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$ and
- if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

The maps $\rho_{U,V}$ are called restriction maps, and we sometimes write $s|_V$ instead of $\rho_{U,V}(s)$, if $s \in \mathcal{F}(U)$.

A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ on a topological space X is a collection of maps $\mathcal{F}(U) \to \mathcal{G}(U)$ which is compatible with respect to restriction maps.

Remark

*Categorically, a presheaf $\mathcal{F}$ on X is a functor $\mathcal{F} : \mathcal{O}(X)^{\text{op}} \to$ **Set**, where $\mathcal{O}(X)$ is the poset category corresponding to the lattice of open sets of the topological space X (with respect to the inclusion relation).*

A morphism of presheaves is then just a natural transformation between the corresponding functors.

So we have a category $[\mathcal{O}(X)^{\text{op}}, \textbf{Set}]$ of presheaves on X.

Sheaves on a topological space

Definition

A sheaf $\mathcal{F}$ on a topological space X is a presheaf on X satisfying the additional conditions

(i) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if $s, t \in \mathcal{F}(U)$ are elements such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$;

(ii) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i, with the property that for each $i, j \in I$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ (necessarily unique by (i)) such that $s|_{V_i} = s_i$ for each i.

A morphism of sheaves is defined as a morphism of the underlying presheaves.

Remark

*Categorically, a sheaf is a functor $\mathcal{O}(X)^{\text{op}} \to$ **Set** which satisfies certain conditions expressible in categorical language entirely in terms of the poset category $\mathcal{O}(X)$ and of the usual notion of covering on it. The category $\mathbf{Sh}(X)$ of sheaves on a topological space X is a full subcategory of the category $[\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$ of presheaves on X.*

This paves the way for a significant categorical generalization of the notion of sheaf, leading to the notion of Grothendieck topos.

Categorical reformulations

- The sheaf condition for a presheaf $\mathcal{F}$ on a topological space X can be categorically reformulated as the requirement that the canonical arrow

  $$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$$

  given by $s \to (s|_{U_i} \mid i \in I)$ should be the equalizer of the two arrows

  $$\prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

  given by $(s_i \to (s_i|_{U_i \cap U_j}))$ and $(s_i \to (s_j|_{U_i \cap U_j}))$.
- For any covering family $F = \{U_i \subseteq U \mid i \in I\}$, giving a family of elements $s_i \in \mathcal{F}(U_i)$ such that for any $i, j \in I$ $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ is equivalent to giving a family of elements $\{s_W \in \mathcal{F}(W) \mid W \in S_F\}$ such that for any open set $W' \subseteq W$, $s_W|_{W'} = s_{W'}$, where S_F is the sieve generated by F.

Examples of sheaves

Examples

- the sheaf of continuous real-valued functions on any topological space
- the sheaf of regular functions on a variety
- the sheaf of differentiable functions on a differentiable manifold
- the sheaf of holomorphic functions on a complex manifold

In each of the above examples, the restriction maps of the sheaf are the usual set-theoretic restrictions of functions to a subset.

Remark

Sheaves arising in Mathematics are often equipped with more structure than the mere set-theoretic one; for example, one may wish to consider sheaves of modules (resp. rings, abelian groups, ...) on a topological space X.
The natural categorical way of looking at these notions is to consider them as models of certain (geometric) theories in a category $\mathbf{Sh}(X)$ of sheaves of sets.

The sheaf of cross-sections of a bundle

Definition

- For any topological space X, a continuous map $p : Y \to X$ is called a bundle over X. In fact, the category of bundles is the slice category $\mathbf{Top}/X$.
- Given an open subset U of X, a cross-section over U of a bundle $p : Y \to X$ is a continuous map $s : U \to Y$ such that the composite $p \circ s$ is the inclusion $i : U \hookrightarrow X$. Let

$$\Gamma_p U = \{s \mid s : U \to Y \text{ and } p \circ s = i : U \to X\}$$

  denote the set of all such cross-sections over U.
- If $V \subseteq U$, one has a restriction operation $\Gamma_p U \to \Gamma_p V$. The functor $\Gamma_p : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ obtained in this way is a sheaf and is called the sheaf of cross-sections of the bundle p.

The bundle of germs of a presheaf

Definition

- Given any presheaf $\mathcal{F} : \mathcal{O}(X)^{\text{op}} \to \mathbf{Set}$ on a space X, a point $x \in X$, two open neighbourhoods U and V of x, and two elements $s \in \mathcal{F}(U), t \in \mathcal{F}(V)$. We say that s and t have the same germ at x when there is some open set $W \subseteq U \cap V$ with $x \in W$ and $s|_W = t|_W$. This relation 'to have the same germ at x' is an equivalence relation, and the equivalence class of any one such s is called the germ of s at x, in symbols $germ_x(s)$ or s_x.
- Let
$$\mathcal{F}_x = \{germ_x(s) \mid s \in \mathcal{F}(U),\ x \in U \text{ open in } X\}$$
be the stalk of $\mathcal{F}$ at x, that is the set of all germs of $\mathcal{F}$ at x.
- Let $\Gamma_{\mathcal{F}}$ be the disjoint union of the $\mathcal{F}_x$
$$\Lambda_{\mathcal{F}} = \{\langle x, r\rangle \mid x \in X, r \in \mathcal{F}_x\}$$
topologized by taking as a base of open sets all the image sets $\tilde{s}(U)$, where $\tilde{s} : U \to \Lambda_{\mathcal{F}}$ is the map induced by an element $s \in \mathcal{F}(U)$ by taking its germs at points in U.
- With respect to this topology, the natural projection map $\Lambda_{\mathcal{F}} \to X$ becomes a continuous map, called the bundle of germs of the presheaf $\mathcal{F}$.

Sheaves as étale bundles I

Definition

- A bundle $p : E \to X$ is said to be étale (over X) when p is a local homeomorphism in the following sense: for each $e \in E$ there is an open set V, with $e \in V$, such that $p(V)$ is open in X and $p|_V$ is a homeomorphism $V \to p(V)$.
- The full subcategory of $\mathbf{Top}/X$ on the étale bundles is denoted by $\mathbf{Etale}(X)$.

Theorem

- *For any topological space X, there is a pair of adjoint functors*

$$\Gamma : \mathbf{Top}/X \to [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}], \quad \Lambda : [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}] \to \mathbf{Top}/X,$$

  where Γ assigns to each bundle $p : Y \to X$ the sheaf of cross-sections of p, while its left adjoint Λ assigns to each presheaf $\mathcal{F}$ the bundle of germs of $\mathcal{F}$.
- *This adjunction restricts to an equivalence of categories*

$$\mathbf{Sh}(X) \simeq \mathbf{Etale}(X) .$$

Sheaves as étale bundles II

This adjunction is naturally presented by speciying its unit and counit:

- The unit $\eta : 1_{[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]} \to \Gamma \circ \Lambda$ acts on a presheaf $\mathcal{F}$ by sending a section $s \in \mathcal{F}(U)$ to the section $\dot{s} \in \Gamma_{\Lambda_{\mathcal{F}}}(U)$;
- The counit $\epsilon : \Lambda \circ \Gamma \to 1_{\mathbf{Top}/X}$ acts on a bundle $p : Y \to X$ by sending any element $(x, germ_x(s))$ of Λ_{Γ_p} to the value $s(x)$.

One then verifies that these natural transformations satisfy the triangular identities:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\Gamma\eta} & \Gamma \circ \Lambda \circ \Gamma \\
 & \searrow^{1_\Gamma} & \downarrow{\scriptstyle \epsilon\Gamma} \\
 & & \Gamma
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\Lambda & \xrightarrow{\eta\Lambda} & \Lambda \circ \Gamma \circ \Lambda \\
 & \searrow^{1_\Lambda} & \downarrow{\scriptstyle \Lambda\epsilon} \\
 & & \Lambda
\end{array}
$$

One further proves that if p is étale then ϵ_p is an isomorphism (and conversely), while if $\mathcal{F}$ is a sheaf then $\eta_{\mathcal{F}}$ is an isomorphism (and conversely). It thus follows from general abstract nonsense that the adjunction restricts to a duality between the full subcategories on sheaves and on étale bundles.

The associated sheaf functor

Theorem

Given a presheaf $\mathcal{F}$, there is a sheaf $a(\mathcal{F})$ and a morphism $\theta : \mathcal{F} \to a(\mathcal{F})$, with the property that for any sheaf $\mathcal{G}$, and any morphism $\phi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : a(\mathcal{F}) \to \mathcal{G}$ such that $\psi \circ \theta = \phi$.

The sheaf $a(\mathcal{F})$ is called the sheaf associated to the presheaf $\mathcal{F}$.

Remark

Categorically, this means that the inclusion functor $i : \mathbf{Sh}(X) \to [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$ has a left adjoint $a : [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(X)$.

The left adjoint $a : [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(X)$ is called the associated sheaf functor.

Theorem

The associated sheaf functor a is given by the composite $\Gamma \circ \Lambda$.

Concretely, $a(\mathcal{F})(U)$ is the collection of functions $s : U \to \Lambda_{\mathcal{F}}$ which satisfy the following properties:

- $s(x) \in \mathcal{F}_x$ for each $x \in U$;
- for each $x \in U$ there exist an open set $Z_x \subseteq U$ containing x and a section $\xi^{Z_x} \in \mathcal{F}(Z_x)$ such that $s(y) = (\xi^{Z_x})_y$ for each $y \in Z_x$.

Limits and colimits in $\mathbf{Sh}(X)$

Theorem

(i) *The category $\mathbf{Sh}(X)$ is closed in $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ under arbitrary (small) limits.*

(ii) *The associated sheaf functor $a : [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(X)$ (having a right adjoint) preserves all (small) colimits.*

- *Part (i) follows from the fact that limits commute with limits, in light of the characterization of sheaves in terms of limits.*
- *From part (ii) it follows that $\mathbf{Sh}(X)$ has all small colimits, which are computed by applying the associated sheaf functor to the colimit of the diagram considered with values in $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$.*

Adjunctions induced by points

Let x be a point of a topological space X.

Definition

Let A be a set. Then the skyscraper sheaf $\mathrm{Sky}_x(A)$ of A at x is the sheaf on X defined as

- $\mathrm{Sky}_x(A)(U) = A$ if $x \in U$
- $\mathrm{Sky}_x(A)(U) = 1 = \{*\}$ if $x \notin U$

and in the obvious way on arrows.

The assignment $A \to \mathrm{Sky}_x(A)$ is clearly functorial.

Theorem

The stalk functor $\mathrm{Stalk}_x : \mathbf{Sh}(X) \to \mathbf{Set}$ *at x is left adjoint to the skyscraper functor* $\mathrm{Sky}_x : \mathbf{Set} \to \mathbf{Sh}(X)$.

In fact, as we shall see later in the course, points in topos theory are defined as suitable kinds of functors (more precisely, colimit and finite-limit preserving ones).

Open sets as subterminal objects

Since limits in a category $\mathbf{Sh}(X)$ are computed as in the category of presheaves $[\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$, a subobject of a sheaf F in $\mathbf{Sh}(X)$ is just a subsheaf, that is a subfunctor which is a sheaf.
Notice that a subfunctor $S \subseteq F$ is a sheaf if and only if for every open covering $\{U_i \subseteq U \mid i \in I\}$ and every element $x \in F(U)$, $x \in S(U)$ if and only if $x|_{U_i} \in S(U_i)$.

Definition

In a category with a terminal object, a subterminal object is an object whose unique arrow to the terminal object is a monomorphism.

Theorem

Let X be a topological space. Then we have a frame isomorphism

$$\mathrm{Sub}_{\mathbf{Sh}(X)}(1) \cong \mathcal{O}(X) .$$

between the subterminal objects of $\mathbf{Sh}(X)$ and the open sets of X.

For further reading

S. Mac Lane and I. Moerdijk.
Sheaves in geometry and logic: a first introduction to topos theory
Springer-Verlag, 1992.