TOPOS THEORY (a.y. 2018/2019) - Exercises

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1. Let X be a topological space. Show that there is a geometric morphism $\mathbf{Set}/X \to \mathbf{Sh}(X)$ whose inverse image sends a sheaf F to the disjoint union of its stalks F_x, $x \in X$, with the obvious projection to X, and whose direct image sends $p \colon E \to X$ to the sheaf F such that $F(U)$ is the set of all sections of p over U (that is, continuous functions $s \colon U \to E$ such that $p \circ s$ is the inclusion $U \to X$).

2. Let $f : A \to B$ be an arrow in a Grothendieck topos $\mathcal{E}$. Show that the pullback functor $f^* : \mathcal{E}/B \to \mathcal{E}/A$ is faithful if and only if f is an epimorphism.

3. Show that

(a) For any preorder $\mathcal{C}$ and Grothendieck topology J on $\mathcal{C}$, the points of the topos $\mathbf{Sh}(\mathcal{C}, J)$ correspond precisely to the J-*prime filters* on $\mathcal{C}$ (by a J-prime filter on $\mathcal{C}$ we mean a subset $F \subseteq \mathcal{C}$ such that F is non-empty, $a \in F$ implies $b \in F$ whenever $a \leq b$, for any $a, b \in F$ there exists $c \in F$ such that $c \leq a$ and $c \leq b$, and for any J-covering sieve $\{a_i \to a \mid i \in I\}$ in $\mathcal{C}$ if $a \in F$ then there exists $i \in I$ such that $a_i \in F$).

(b) For any frame L, the points of the topos $\mathbf{Sh}(L)$ correspond precisely to the frame homomorphisms $L \to \{0, 1\}$, equivalently to the *completely prime filters* on L (i.e., the subsets $S \subseteq L$ such that $1 \in S$, $a \wedge b \in S$ if and only if $a \in S$ and $b \in S$, and for any family of elements $\{a_i \mid i \in I\}$ whose join is a, $a \in S$ implies $a_i \in S$ for some i).

(c) For any small category $\mathcal{C}$ and any object c of $\mathcal{C}$, there is a point $ev_c : \mathbf{Set} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ of the topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ whose inverse image is the evaluation functor at the object c.

4. Show that if $f : \mathcal{C} \to \mathcal{D}$ is a functor between small categories, the inverse image $f^* : [\mathcal{D}^{\text{op}}, \mathbf{Set}] \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ of the essential geometric morphism $[\mathcal{C}^{\text{op}}, \mathbf{Set}] \to [\mathcal{D}^{\text{op}}, \mathbf{Set}]$ induced by f is faithful if every object of $\mathcal{D}$ is a retract of an object in the image of f. [If you want, try also to prove the converse.]

5. The class of *cartesian formulae* relative to a theory $\mathbb{T}$ is defined as follows: atomic formulae and $\top$ are cartesian, $(\phi \wedge \psi)$ is cartesian provided both ϕ and ψ are, and $(\exists x)\phi$ is cartesian provided ϕ is cartesian and the sequent $((\phi \wedge \phi[x'/x]) \vdash_{x,x',\vec{y}} (x = x'))$ is derivable in $\mathbb{T}$. (Here $x, \vec{y}$ is a suitable context for ϕ, and x' is a variable not in it.) A theory $\mathbb{T}$ is said to be *cartesian* if its axioms can be ordered in such a way that each involves formulae which are cartesian relative to the theory formed by the earlier axioms. Write down a presentation of the theory of categories (as a two-sorted theory, with function symbols for domain, codomain and identities, and a ternary relation $T(x, y, z)$ to express "z is the composite of x and y"), and verify that it is cartesian. Show also that if $\mathbb{T}$ is a cartesian theory then $\mathbb{T}$-mod(**Set**) is closed under (finite) limits in Σ-str(**Set**).

6. Let $\mathcal{E}$ be a Grothendieck topos with internal language $\Sigma_{\mathcal{E}}$. We write $\mathcal{E} \models \sigma$, where σ is a sequent over $\Sigma_{\mathcal{E}}$, to mean that σ is satisfied in the canonical $\Sigma_{\mathcal{E}}$-structure in $\mathcal{E}$. Show that

(a) $1_A : A \to A$ is the identity arrow on A if and only if $\mathcal{E} \models (\top \vdash_x (\ulcorner 1_A \urcorner(x) = x))$;

(b) $f : A \to C$ is the composite of $g : A \to B$ and $h : B \to C$ if and only if $\mathcal{E} \models (\top \vdash_x (\ulcorner f \urcorner(x) = \ulcorner h \urcorner(\ulcorner g \urcorner(x))))$;

(c) $f : A \to B$ is monic if and only if $\mathcal{E} \models ((\ulcorner f \urcorner(x) = \ulcorner f \urcorner(x')) \vdash_{x,x'} (x = x'))$;

(d) $f : A \to B$ is an epimorphism if and only if $\mathcal{E} \models (\top \vdash_y (\exists x)\ulcorner f \urcorner(x) = y)$;

(e) A is a terminal object of $\mathcal{E}$ if and only if $\mathcal{E} \models (\top \vdash_{[]} (\exists x)\top)$ and $\mathcal{E} \models (\top \vdash_{x,x'} x = x')$ (here x and x' are of sort $\ulcorner A \urcorner$).

Optional exercise: n. 8 from Chapter 8 of the book *Sheaves in Geometry and Logic* by S. Mac Lane and I. Moerdijk.