Cohomology of toposes

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Chapter VII:

Operations on linear sheaves on sites and Grothendieck's six operations for étale cohomology

Reminder on sheaves on Grothendieck sites

Definition: Let $\mathcal{C}$ = (essentially) small category.

(i) A sieve S on an object X of $\mathcal{C}$ is a subobject

$$S \hookrightarrow \mathrm{Hom}(\bullet, X) \qquad \text{in} \qquad \widehat{\mathcal{C}} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]\,.$$

In other words it is a collection of arrows

$$X' \longrightarrow X$$

such that, for any $X'' \xrightarrow{g} X' \xrightarrow{f} X$,

$$f \in S \Rightarrow f \circ g \in S\,.$$

(ii) For any morphism $X \xrightarrow{f} Y$ of $\mathcal{C}$
and any sieve S on Y, $f^{-1}S$ is the sieve on X

$$S \times_{\mathrm{Hom}(\bullet, Y)} \mathrm{Hom}(\bullet, X) \hookrightarrow \mathrm{Hom}(\bullet, X)\,.$$

In other words, an arrow $X' \xrightarrow{a} X$ is in $f^{-1}S$ if and only if $f \circ a : X' \to Y$ is in S.

Remarks:

- Any intersection of sieves on X is a sieve on X.
- Any family of arrows $X_i \xrightarrow{f_i} X$ generates a sieve on X. It consists in the morphisms $X' \to X$ which factorise through at least one of the f_i's.

Definition:
Let $\mathcal{C}$ = (essentially) small category.
A topology J on $\mathcal{C}$ is a map

$$\begin{array}{ccc} X & \longmapsto & J(X) \\ \| & & \| \\ \text{object of } \mathcal{C} & & \text{set of sieves on } X \end{array}$$

which verifies the following axioms:

(Maximality) Fo any X, the maximal sieve

$$\mathrm{Hom}(\bullet, X) \quad \text{consisting of all arrows} \quad X' \to X$$

is an element of $J(X)$.

(Stability) For any morphism $f : X \to Y$, the map

$$S \longmapsto f^{-1}S$$

sends $J(Y)$ into $J(X)$.

(Transitivity) If X is an object and $S \in J(X)$, a sieve S' on X such that $f^{-1}S' \in J(X')$, $\forall\, (X' \xrightarrow{f} X) \in S$, necessarily belongs to $J(X)$.

Remark: A family of morphisms $X_i \xrightarrow{f} X$ is called "J-covering" if the sieve it generates belongs to $J(X)$.

Definition:

(i) A site is a pair $(\mathcal{C}, J)$ consisting in
$\mathcal{C} =$ (essentially) small category,
$J =$ topology on $\mathcal{C}$.

(ii) A sheaf on a site $(\mathcal{C}, J)$ is a presheaf

$$F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

such that, for any X and $S \in J(X)$, the canonical map

$$F(X) \longrightarrow \varprojlim_{(X' \xrightarrow{a} X) \in S} F(X')$$

is one-to-one.

(iii) The category of sheaves on $(\mathcal{C}, J)$, denoted

$$\widehat{\mathcal{C}}_J = \mathrm{Sh}(\mathcal{C}, J)\,,$$

is the full subcategory of

$$\widehat{\mathcal{C}} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \qquad (= \text{category of presheaves on } \mathcal{C})$$

on presheaves F which are sheaves.
In other words, a morphism of sheaves is a morphism of presheaves.

The sheafification functor

Proposition: Let $(\mathcal{C}, J) =$ site.
Then the canonical embedding functor

$$j_* : \widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

has a left adjoint

$$\begin{array}{cccc} j^* & : & \widehat{\mathcal{C}} & \longrightarrow & \widehat{\mathcal{C}}_J \\ & & P & \longmapsto & j^* P \end{array}$$

characterized by the property that any morphism

$$P \longrightarrow F$$

from a presheaf P to a sheaf F uniquely factorises as

$$P \longrightarrow j^* P \longrightarrow F\,.$$

Remarks:

(i) The sheafification $j^* P$ of P can be constructed by the formula

$$j^* P = (P^+)^+$$

where $(P^+)(X) = \varinjlim_{S \in J(X)} \varprojlim_{(X' \xrightarrow{a} X) \in S} P(X')$.

(ii) There is a canonical composed functor

$$\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J\,.$$

Exactness properties

Proposition:

(i) The category $\widehat{\mathcal{C}}$ has arbitrary limits and colimits and they are computed component-wise, i.e.

$$\Big(\varprojlim_D P_d\Big)(X) = \varprojlim_D P_d(X)\,,$$
$$\Big(\varinjlim_D P_d\Big)(X) = \varinjlim_D P_d(X)\,.$$

(ii) The category $\widehat{\mathcal{C}}_J$ has arbitrary limits and colimits with

$$\Big(\varprojlim_D F_d\Big)(X) = \varprojlim_D F_d(X)\,,$$
$$\varinjlim_D F_d = j^*\Big(\varinjlim_D j_* F_d\Big)\,.$$

(iii) The functor

$$j_* : \widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

respects arbitrary limits, while its left adjoint

$$j^* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}_J$$

respects arbitrary colimits and finite limits.

Corollary:

(i) A group object [resp. ring object, resp. module object over a ring object] of $\widehat{\mathcal{C}}_J$ is a sheaf of sets

$$X \longmapsto \mathcal{G}(X) \qquad [\text{resp.} \quad \mathcal{O}(X), \text{ resp. } \mathcal{M}(X)\ ]$$

endowed with a structure of group [resp. ring, resp. module over the ring $\mathcal{O}(X)$] on each

$$\mathcal{G}(X) \qquad [\text{resp.} \quad \mathcal{O}(X), \text{ resp. } \mathcal{M}(X)\ ]$$

such that all restriction maps induced by morphisms $X \xrightarrow{f} Y$ of $\mathcal{C}$

$$\mathcal{G}(Y) \to \mathcal{G}(X) \qquad [\text{resp.} \quad \mathcal{O}(Y) \to \mathcal{O}(X), \text{ resp. } \mathcal{M}(Y) \to \mathcal{M}(X)\ ]$$

are group [resp. ring, resp. module] morphisms.

(ii) A morphism of group objects [resp. ring objects, resp. module objects over some ring object $\mathcal{O}$] is a morphism of sheaves

$$\mathcal{G}_1 \to \mathcal{G}_2 \qquad [\text{resp.} \quad \mathcal{O}_1 \to \mathcal{O}_2, \text{ resp. } \mathcal{M}_1 \to \mathcal{M}_2\ ]$$

such that all maps

$$\mathcal{G}_1(X) \to \mathcal{G}_2(X) \qquad [\text{resp.} \quad \mathcal{O}_1(X) \to \mathcal{O}_2(X), \text{ resp. } \mathcal{M}_1(X) \to \mathcal{M}_2(X)\ ]$$

are group [resp. ring, resp. module] morphisms.

The abelian categories of Modules

Definition:

Let $(\mathcal{C}, J, \mathcal{O})$ = ringed site
= site $(\mathcal{C}, J)$
+ ring object $\mathcal{O}$ of $\widehat{\mathcal{C}}_J$.
Then module objects over $\mathcal{O}$ in $\widehat{\mathcal{C}}_J$
are called $\mathcal{O}$-Modules and their category is denoted

$$\mathcal{M}od_{\mathcal{O}} \, .$$

Proposition:

For any ringed site $(\mathcal{C}, J, \mathcal{O})$,

$$\mathcal{M}od_{\mathcal{O}}$$

is an abelian category
with arbitrary limits and colimits.

Change of structure ring-sheaf

Proposition:
Let $(\mathcal{C}, J) =$ site,
$(\mathcal{O}_1 \to \mathcal{O}_2) =$ morphism of ring objects in $\widehat{\mathcal{C}}_J$.
Then the forgetful functor

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}_2} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_1}\,, \\ \mathcal{M} & \longmapsto & \mathcal{M}\,, \end{array}$$

has a left adjoint denoted

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}_1} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_2}\,, \\ \mathcal{M} & \longmapsto & \mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{M}\,. \end{array}$$

Remarks:

(i) For any object $\mathcal{M}$ of $\mathcal{M}od_{\mathcal{O}_1}$,

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{M}$$

is constructed as the sheafification of the presheaf

$$X \longmapsto \mathcal{O}_2(X) \otimes_{\mathcal{O}_1(X)} \mathcal{M}(X)\,.$$

(ii) The forgetful functor respects arbitrary limits and colimits while its left adjoint

$$\mathcal{M} \longmapsto \mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{M}$$

respects arbitrary colimits.

Exponentials (or "inner $\mathcal{H}om$") and tensor products

Definition:

(i) For any object S of any category $\mathcal{C}$,
the relative category $\mathcal{C}/S$ is the category
whose objects are morphisms $X \xrightarrow{p} S$ of $\mathcal{C}$
and whose morphisms $(X_1 \xrightarrow{p_1} S) \to (X_2 \xrightarrow{p_2} S)$
are commutative triangles of $\mathcal{C}$:

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & S & \end{array}$$

(ii) For any topology J on a (ess.) small category $\mathcal{C}$
and any object X of $\mathcal{C}$,
the induced topology J_X on $\mathcal{C}/X$ is defined by the property
that a sieve on an object of $\mathcal{C}/X$ belongs to J_X
if its image by the forgetful functor

$$\begin{array}{ccc} \mathcal{C}/X & \longrightarrow & \mathcal{C}\,, \\ (X' \to X) & \longmapsto & X' \end{array}$$

belongs to J.

(iii) In this situation, composition with $\mathcal{C}/X \to \mathcal{C}$
defines a functor $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}/X}$
which restricts to a functor called the restriction functor

$$\begin{array}{cccc} r_X & : & \widehat{\mathcal{C}}_J & \longrightarrow & \widehat{(\mathcal{C}/X)}_{J_X}\,, \\ & & F & \longmapsto & F_{|X} = F_X\,. \end{array}$$

Remarks:

(i) Restriction functors respect arbitrary limits and colimits. In particular, they transform any ring object $\mathcal{O}$ of $\widehat{\mathcal{C}}_J$ into ring objects $\mathcal{O}_X$ of each $\mathcal{C}/X$ and induce additive exact functors

$$\mathcal{M}od_{\mathcal{O}} \longrightarrow \mathcal{M}od_{\mathcal{O}_X} .$$

(ii) For any sheaves F_1 and F_2 on $(\mathcal{C}, J)$, the presheaf

$$X \longmapsto \mathrm{Hom}(F_{1|X}, F_{2|X})$$

is a sheaf denoted $F_2^{F_1}$ or $\mathcal{H}om(F_1, F_2)$.
It is characterized by the property that, for any sheaf G,

$$\mathrm{Hom}(G, \mathcal{H}om(F_1, F_2)) = \mathrm{Hom}(G \times F_1, F_2) .$$

(iii) In the same way, for any ring object $\mathcal{O}$ of $\widehat{\mathcal{C}}_J$ and any $\mathcal{O}$-Modules $\mathcal{M}_1, \mathcal{M}_2$, the presheaf

$$X \longmapsto \mathrm{Hom}_{\mathcal{O}_X}(F_{1|X}, F_{2|X})$$

is a sheaf denoted $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}_1, \mathcal{M}_2)$.

Proposition:

Let $(\mathcal{C}, J, \mathcal{O})$ = commutative ringed site

= site $(\mathcal{C}, J)$+ commutative ring object $\mathcal{O}$ of $\widehat{\mathcal{C}}_J$,

$\mathcal{N}$ = $\mathcal{O}$-Module in $\widehat{\mathcal{C}}_J$.

Then the functor

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}} & \longrightarrow & \mathcal{M}od_{\mathcal{O}}\,, \\ \mathcal{L} & \longmapsto & \mathcal{H}om_{\mathcal{O}}(\mathcal{N}, \mathcal{L}) \end{array}$$

has a left adjoint denoted

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}} & \longrightarrow & \mathcal{M}od_{\mathcal{O}}\,, \\ \mathcal{M} & \longmapsto & \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}\,. \end{array}$$

Furthermore, $\otimes$ extends as a double functor

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}} \times \mathcal{M}od_{\mathcal{O}} & \longrightarrow & \mathcal{M}od_{\mathcal{O}}\,, \\ (\mathcal{M}, \mathcal{N}) & \longmapsto & \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N} \end{array}$$

such that the two triple functors

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}} & \longrightarrow & \mathcal{O}(X)\text{-modules} \\ (\mathcal{M}, \mathcal{N}, \mathcal{L}) & \longmapsto & \mathrm{Hom}_{\mathcal{O}}(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}, \mathcal{L})\,, \\ (\mathcal{M}, \mathcal{N}, \mathcal{L}) & \longmapsto & \mathrm{Hom}_{\mathcal{O}}(\mathcal{M}, \mathcal{H}om_{\mathcal{O}}(\mathcal{N}, \mathcal{L})) \end{array}$$

are isomorphic.

Remarks:

(i) The tensor product $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ is constructed as the sheafification of the functor

$$X \longmapsto \mathcal{M}(X) \otimes_{\mathcal{O}(X)} \mathcal{N}(X)\,.$$

(ii) The two functors $\mathcal{M}od_{\mathcal{O}} \times \mathcal{M}od_{\mathcal{O}} \to \mathcal{M}od_{\mathcal{O}}$

$$\begin{array}{rccc} & (\mathcal{M},\mathcal{N}) & \longmapsto & \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N} \\ \text{and} & (\mathcal{M},\mathcal{N}) & \longmapsto & \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M} \end{array}$$

are canonically isomorphic.

(iii) The double functor

$$(\mathcal{M},\mathcal{N}) \longmapsto \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$$

respects arbitrary colimits in $\mathcal{M}$ or $\mathcal{N}$, while the double functor

$$(\mathcal{N},\mathcal{L}) \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{N},\mathcal{L})$$

respects arbitrary limits in $\mathcal{L}$ and transforms arbitrary colimits in $\mathcal{N}$ into limits.

Push-forward and pull-back functors

Definition:

(i) A category $\mathcal{E}$ is called a topos
if it is equivalent to the category $\widehat{\mathcal{C}}_J$ of sheaves on some site $(\mathcal{C}, J)$.

(ii) A (geometric) morphism of toposes $\mathcal{E}_1 \to \mathcal{E}_2$
is a pair of adjoint functors $\left(\mathcal{E}_2 \xrightarrow{f^{-1}} \mathcal{E}_1, \mathcal{E}_1 \xrightarrow{f_*} \mathcal{E}_2\right)$
whose left component f^{-1} respects finite limits (as well as arbitrary colimits).

(iii) A morphism between two morphisms of toposes $\mathcal{E}_1 \rightrightarrows \mathcal{E}_2$

$$(f^{-1}, f_*) \longrightarrow (g^{-1}, g_*)$$

is a natural transformation of functors

$$\alpha : f^{-1} \longrightarrow g^{-1} .$$

Remarks:

(i) If (f^{-1}, f_*) is a topos morphism, f^{-1} is called the pull-back component and f_* the push-forward component.

(ii) The composite of two morphisms of toposes

$$\mathcal{E}_1 \xrightarrow{(f^{-1}, f_*)} \mathcal{E}_2 \xrightarrow{(g^{-1}, g_*)} \mathcal{E}_3$$

is defined as the pair $(f^{-1} \circ g^{-1}, g_* \circ f_*)$.

(iii) Morphisms from a topos $\mathcal{E}_1$ to a topos $\mathcal{E}_2$ make up a category denoted

$$\mathcal{G}eom(\mathcal{E}_1, \mathcal{E}_2)\,.$$

(iv) Any morphism of toposes

$$\mathcal{E}_1' \longrightarrow \mathcal{E}_1 \qquad \text{[resp.} \quad \mathcal{E}_2 \longrightarrow \mathcal{E}_2' \text{]}$$

induces a functor defined by composition

$$\begin{array}{lccc} & \mathcal{G}eom(\mathcal{E}_1, \mathcal{E}_2) & \longrightarrow & \mathcal{G}eom(\mathcal{E}_1', \mathcal{E}_2) \\ \text{[resp.} & \mathcal{G}eom(\mathcal{E}_1, \mathcal{E}_2) & \longrightarrow & \mathcal{G}eom(\mathcal{E}_1, \mathcal{E}_2') \text{].} \end{array}$$

(v) If $\mathcal{E}$ is a topos and 1 denotes its terminal object, there is a unique morphism of toposes $\mathcal{E} \xrightarrow{(p^{-1}, p_*)} \mathrm{Set}$ defined by

$$\begin{array}{rrcll} & p^{-1}I & = & \coprod_{i \in I} 1 & \text{(“constant” objects of } \mathcal{E}) \\ \text{and} & p_* F & = & \mathrm{Hom}(1, F) & \text{(“global sections” functor).} \end{array}$$

(vi) A morphism of toposes $\mathrm{Set} \xrightarrow{(x^{-1}, x_*)} \mathcal{E}$ is called a “point” of $\mathcal{E}$ and its left component $x^{-1} : \mathcal{E} \to \mathrm{Set}$ the “fiber functor” at the point.

(vii) Points of a topos $\mathcal{E}$ make up a category $\mathcal{P}t(\mathcal{E}) = \mathcal{G}eom(\mathrm{Set}, \mathcal{E})$.

(viii) Any morphisms of toposes $\mathcal{E}_1 \to \mathcal{E}_2$ induces a functor

$$\mathcal{P}t(\mathcal{E}_1) \longrightarrow \mathcal{P}t(\mathcal{E}_2)\,.$$

Lemma: For any morphism of toposes

$$(f^{-1}, f_*) : \mathcal{E}_1 \to \mathcal{E}_2 \,,$$

both functors f^{-1} and f_* transform
group objects into group objects,
ring objects into ring objects
and module objects over a ring object into module objects over the transform of this ring object.

Sketch of proof:
This is because both functors f^{-1} and f_*
respect finite limits, in particular finite products.

Definition:

(i) A ringed topos is a topos $\mathcal{E}$ endowed with a ring object $\mathcal{O}$.

(ii) A morphism of ringed toposes

$$(\mathcal{E}_1, \mathcal{O}_1) \longrightarrow (\mathcal{E}_2, \mathcal{O}_2)$$

is a morphism of toposes

$$(f^{-1}, f_*) : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$$

completed with a morphism of ring objects

$$f^{-1}\mathcal{O}_2 \longrightarrow \mathcal{O}_1 \qquad \text{or, equivalently,} \qquad \mathcal{O}_2 \longrightarrow f_*\mathcal{O}_1 \,.$$

Corollary:

Let $(\mathcal{E}_1, \mathcal{O}_1) \to (\mathcal{E}_2, \mathcal{O}_2)$

$=$ morphism of ringed toposes

consisting in $\mathcal{E}_1 \xrightarrow{(f^{-1}, f_*)} \mathcal{E}_2$

and $f^{-1}\mathcal{O}_2 \longrightarrow \mathcal{O}_1$.

Then:

(i) The composition of the functor

$$f_* : \mathcal{M}od_{\mathcal{O}_1} \longrightarrow \mathcal{M}od_{f_*\mathcal{O}_1}$$

and of the forgetful functor defined by $\mathcal{O}_2 \to f_*\mathcal{O}_1$

$$\mathcal{M}od_{f_*\mathcal{O}_1} \longrightarrow \mathcal{M}od_{\mathcal{O}_2}$$

defines a functor

$$f_* : \mathcal{M}od_{\mathcal{O}_1} \longrightarrow \mathcal{M}od_{\mathcal{O}_2} .$$

(ii) This functor $f_* : \mathcal{M}od_{\mathcal{O}_1} \longrightarrow \mathcal{M}od_{\mathcal{O}_2}$ has a left adjoint functor

$$f^* : \mathcal{M}od_{\mathcal{O}_2} \longrightarrow \mathcal{M}od_{\mathcal{O}_1}$$

constructed as the composite of the functors

$$f^{-1} : \mathcal{M}od_{\mathcal{O}_2} \longrightarrow \mathcal{M}od_{f^{-1}\mathcal{O}_2}$$

and

$$\begin{array}{ccc} \mathcal{M}od_{f^{-1}\mathcal{O}_2} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_1} , \\ \mathcal{M} & \longmapsto & \mathcal{O}_1 \otimes_{f^{-1}\mathcal{O}_2} \mathcal{M} . \end{array}$$

Remark:

$f_* : \mathcal{M}od_{\mathcal{O}_1} \longrightarrow \mathcal{M}od_{\mathcal{O}_2}$ respects limits,

$f^* : \mathcal{M}od_{\mathcal{O}_2} \longrightarrow \mathcal{M}od_{\mathcal{O}_1}$ respects colimits.

A concrete process to generate some morphisms of toposes

Proposition:
Let $\mathcal{E}_1, \mathcal{E}_2$
$=$ two toposes defined by two sites $(\mathcal{C}_1, J_1), (\mathcal{C}_2, J_2)$
such that $\mathcal{C}_2$ has arbitrary finite limits,
and $\rho : \mathcal{C}_2 \to \mathcal{C}_1$
$=$ functor such that

$$\begin{cases} \bullet \quad \rho \text{ respects finite limits,} \\ \bullet \quad \rho \text{ transforms } J_2\text{-covering families} \\ \qquad\qquad \text{into } J_1\text{-covering families.} \end{cases}$$

Then ρ defines a toposes morphism

in the following way:
$$(f^*, f_*) : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$$

- For any sheaf F_1 on $(\mathcal{C}_1, J_1)$, $f_* F_1$ is the sheaf on $(\mathcal{C}_2, J_2)$
$$X_2 \longmapsto F_1(\rho(X_2)) .$$
- For any sheaf F_2 on $(\mathcal{C}_2, J_2)$, $f^* F_2$ is the sheafification of the presheaf
$$X_1 \longmapsto \varinjlim_{X_2 \in (X_1 \backslash_\rho \mathcal{C}_2)} F_2(X_2)$$
where $X_1 \backslash_\rho \mathcal{C}_2$ is the category of objects X_2 of $\mathcal{C}_2$ endowed with a morphism $X_1 \to \rho(X_2)$ in $\mathcal{C}_1$.

Remarks:

(i) This generalises the construction of the topos morphism

$$(f^*, f_*) : \mathrm{Sh}(X_1) \longrightarrow \mathrm{Sh}(X_2)$$

associated to a continuous maps $f : X_1 \to X_2$
between topological spaces X_1, X_2.
Indeed, f defines $\rho = f^{-1} : 0(X_2) \to 0(X_1)$.

(ii) Even if $\mathcal{E}_1, \mathcal{E}_2$ are two toposes defined by sites $(\mathcal{C}_1, J_1)$, $(\mathcal{C}_2, J_2)$
such that $\mathcal{C}_2$ has finite limits,
not all morphisms of toposes $\mathcal{E}_1 \to \mathcal{E}_2$ are constructed in this way.

(iii) Nevertheless, it will be enough for the étale toposes of schemes.

Sketch of proof of the proposition:

- If F_1 is a sheaf on $(\mathcal{C}_1, J_1)$,

$$X_2 \longmapsto F_1(\rho(X_2))$$

  is a sheaf on $(\mathcal{C}_2, J_2)$
  because ρ transforms J_2-covering families into J_1-covering families.

- It is clear that f^* is left adjoint to f_*.
  We only need to prove that it respects finite limits.
  For this it is enough to prove that for any X_1 the functor

$$F_2 \longmapsto \varinjlim_{X_2 \in (X_1 \backslash_\rho \mathcal{C}_2)} F_2(X_2)$$

  respects finite limits.

  This is because the category $X_1 \backslash_\rho \mathcal{C}_2$ is filtering,
  as $\mathcal{C}_2$ has finite limits and they are respected by ρ.

Corollary:

For $\mathcal{E}_1 = \widehat{(\mathcal{C}_1)}_{J_1}$, $\mathcal{E}_2 = \widehat{(\mathcal{C}_2)}_{J_2}$
and $(f^{-1}, f_*) : \mathcal{E}_1 \to \mathcal{E}_2$ defined by $\rho : \mathcal{C}_2 \to \mathcal{C}_1$
as in the previous proposition,
let $\mathcal{O}_1, \mathcal{O}_2 =$ ring objects of $\mathcal{E}_1, \mathcal{E}_2$ related by a morphism
$f^{-1}\mathcal{O}_2 \to \mathcal{O}_1$ or, equivalently, $\mathcal{O}_2 \to f_*\mathcal{O}_1$
consisting in a compatible family of ring morphisms

$$\mathcal{O}_2(X_2) \longrightarrow \mathcal{O}_1(\rho(X_2))\,, \qquad X_2 \in \mathrm{Ob}(\mathcal{C}_2)\,.$$

Then $(f^{-1}, f_*) : \mathcal{E}_1 \to \mathcal{E}_2$ defines adjoint additive functors

$$f_* : \mathcal{M}od_{\mathcal{O}_1} \longrightarrow \mathcal{M}od_{\mathcal{O}_2}$$

and

$$\begin{array}{rcccl} f^* & : & \mathcal{M}od_{\mathcal{O}_2} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_1}\,, \\ & & \mathcal{M} & \longmapsto & \mathcal{O}_1 \otimes_{f^{-1}\mathcal{O}_2} \mathcal{M}\,. \end{array}$$

Localisation of toposes

Proposition:
Let $\mathcal{E}$ = topos.

(i) For any object F of $\mathcal{E}$, the relative category

$$\mathcal{E}/F$$

is a topos, called the localisation of $\mathcal{E}$ at F.
More precisely, if $\mathcal{E} = \widehat{\mathcal{C}}_J$, then $\mathcal{E}/F = \widehat{(\mathcal{C}/F)}_{J_F}$
where:

- $\mathcal{C}/F$ is the category whose objects are pairs

$$(X, a) \quad \text{with} \quad X \in \mathrm{Ob}(\mathcal{C}), \ a \in F(X),$$

  and whose morphisms $(X_1, a_1) \to (X_2, a_2)$ are morphisms of $\mathcal{C}$

$$f : X_1 \longrightarrow X_2 \quad \text{such that} \quad F(f)(a_2) = a_1 ,$$

- J_F is the "induced" topology on $\mathcal{C}/F$ such that
  a family of morphisms $(X_i, a_i) \xrightarrow{f_i} (X, a)$ is J_F-covering
  if and only if the family $X_i \xrightarrow{f_i} X$ is J-covering.

(ii) For any morphism $f : F_1 \to F_2$ of $\mathcal{E}$, the functor

$$\begin{array}{rcccl} f^{-1} & : & \mathcal{E}/F_2 & \longrightarrow & \mathcal{E}/F_1 \,, \\ & & (F \to F_2) & \longmapsto & (F \times_{F_2} F_1 \to F_1) \end{array}$$

has a left adjoint

$$\begin{array}{rcccl} f_! & : & \mathcal{E}/F_1 & \longrightarrow & \mathcal{E}/F_2 \\ & & (F \xrightarrow{g} F_1) & \longmapsto & (F \xrightarrow{f \circ g} F_2) \end{array}$$

and a right adjoint

$$f_* : \mathcal{E}/F_1 \longrightarrow \mathcal{E}/F_2$$

so it defines a morphism of toposes

$$(f^{-1}, f_*) : \mathcal{E}/F_1 \longrightarrow \mathcal{E}/F_2 \,.$$

Remarks:

(i) If 1 is the terminal object of $\mathcal{E}$, $\mathcal{E}/1$ identifies with $\mathcal{E}$.

(ii) If $\mathcal{E} = \widehat{\mathcal{C}}_J$ and $\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J$ is the canonical functor, then for any object X of $\mathcal{C}$, the restriction functor

$$\mathcal{E}/\ell(X) = (\widehat{\mathcal{C}/\ell(X)})_{J(\ell(X))} \longrightarrow \widehat{(\mathcal{C}/X)}_{J_X}$$

is an equivalence, so that $\mathcal{E}/\ell(X)$ and $\widehat{(\mathcal{C}/X)}_{J_X}$ identify.

Sketch of proof of the proposition:

(i) It is enough to prove that for any presheaf P on $\mathcal{C}$,

$$\widehat{\mathcal{C}}/P \quad \text{and} \quad \widehat{\mathcal{C}/P} \quad \text{are equivalent.}$$

A natural equivalence is defined by the two functors

$$\begin{array}{ccc} \widehat{\mathcal{C}}/P & \longrightarrow & \widehat{\mathcal{C}/P} \\ (p : P' \to P) & \longmapsto & P_p = \left[\begin{array}{ccc} (X,a) & \longmapsto & \text{fiber of } P'(X) \xrightarrow{p_X} P(X) \\ \pitchfork & & \\ \mathrm{Ob}(\mathcal{C}/P) & & \text{over the element } a \in P(X) \end{array} \right] \end{array}$$

and

$$\begin{array}{ccc} \widehat{\mathcal{C}/P} & \longrightarrow & \widehat{\mathcal{C}}/P \\ Q & \longmapsto & (P_Q \to P) = \left[\begin{array}{ccc} X & \longmapsto & \coprod\limits_{a \in P(X)} Q((X,a)) \\ \pitchfork & & \\ \mathrm{Ob}(\mathcal{C}) & & \end{array} \right]. \end{array}$$

(ii) The functor $f_! : (F \to F_1) \mapsto (F \to F_2)$ is left adjoint to $f^{-1} : (F \to F_2) \mapsto (F \times_{F_2} F_1 \to F_1)$ by definition of fiber products.
In a topos, functors $F \mapsto F \times_{F_2} F_1$ respect arbitrary colimits.
Indeed, this is true in Set, therefore in $\widehat{\mathcal{C}}$ and lastly in $\widehat{\mathcal{C}}_J$ as $j^* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}_J$ respects arbitrary colimits and finite limits.
So f^{-1} has a right adjoint functor f_*
(and defines a topos morphism $(f^{-1}, f_*) : \mathcal{E}/F_1 \to \mathcal{E}/F_2$)
according to the following theorem:

Theorem:

Let $\rho : \mathcal{E} \to \mathcal{D}$

$=$ functor from a topos $\mathcal{E}$ to a category $\mathcal{D}$.

Then ρ has a right adjoint if and only if it respects colimits.

Furthermore, if $\mathcal{E} = \widehat{\mathcal{C}}_J$ endowed with $\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J$,
the right adjoint of ρ is

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \widehat{\mathcal{C}}_J \\ Y & \longmapsto & F_Y = \left[\begin{array}{ccc} X & \longmapsto & \mathrm{Hom}(\rho \circ \ell(X), Y) \\ \pitchfork & & \\ \mathrm{Ob}(\mathcal{C}) & & \end{array} \right]. \end{array}$$

Remark:

It can also be proved that if $\mathcal{E}$ is a topos, a functor $\rho : \mathcal{E} \to \mathcal{D}$ has a left adjoint if and only if it respects limits.

Proof of the theorem:

The condition is necessary for any functor between categories.
Conversely, suppose $\mathcal{E}$ is a topos $\widehat{\mathcal{C}}_J$ and ρ respects colimits.
For any covering sieve S of an object X of $\mathcal{C}$, we have

$$\ell(X) = \varinjlim_{(X' \to X) \in S} \ell(X') \quad \text{in} \quad \widehat{\mathcal{C}}_J$$

so $\rho \circ \ell(X) = \varinjlim_{(X' \to X) \in S} \rho \circ \ell(X')$ in $\mathcal{D}$
and $F_Y(X) = \varprojlim_{(X' \to X) \in S} F_Y(X')$, which means F_Y is a sheaf.
Furthermore, for any sheaf F on $\mathcal{C}$, we have $F = \varinjlim_{(X,a) \in \mathcal{C}/F} \ell(X)$ and so

$$\mathrm{Hom}(F, F_Y) = \varprojlim_{(X,a) \in \mathcal{C}/F} \mathrm{Hom}(\ell(X), F_Y) = \varprojlim_{(X,a) \in \mathcal{C}/F} \mathrm{Hom}(\rho \circ \ell(X), Y) = \mathrm{Hom}(\rho(F), Y)\,.$$

Corollary:

Let $\mathcal{E}$ = topos endowed with a ring object $\mathcal{O}$.

For any object F of $\mathcal{E}$, let

$$\mathcal{O}_F = \text{ring object } \mathcal{O} \times F \text{ of } \mathcal{E}/F\,.$$

Then any morphism $f : F_1 \to F_2$ of $\mathcal{E}$ induces an additive functor

$$f^* = f^{-1} : \mathcal{M}od_{\mathcal{O}_{F_2}} \longrightarrow \mathcal{M}od_{\mathcal{O}_{F_1}}$$

which has a right adjoint

$$f_* : \mathcal{M}od_{\mathcal{O}_{F_1}} \longrightarrow \mathcal{M}od_{\mathcal{O}_{F_2}}$$

and a left adjoint

$$f_! : \mathcal{M}od_{\mathcal{O}_{F_1}} \longrightarrow \mathcal{M}od_{\mathcal{O}_{F_2}}\,.$$

Remark: Suppose $\mathcal{E}$ is $\widehat{\mathcal{C}}_J$ endowed with $\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J$. Then:

- For any object X of $\mathcal{C}$, the ring object $\mathcal{O}_X = \mathcal{O}_{\ell(X)}$ of $\mathcal{E}/\ell(X) \cong \widehat{(\mathcal{C}/X)}_{J_X}$ is the sheaf
$$(X' \longrightarrow X) \longmapsto \mathcal{O}(X').$$
- For any morphism $f : X_1 \to X_2$ of $\mathcal{C}$, the functor $f^* = f^{-1} : \mathcal{M}od_{\mathcal{O}_{X_2}} \to \mathcal{M}od_{\mathcal{O}_{X_1}}$ associates to any $\mathcal{O}_{X_2}$-Module $\mathcal{M}$ on $\mathcal{C}/X_2$ the sheaf
$$(X \xrightarrow{g} X_1) \longmapsto \mathcal{M}(X \xrightarrow{f \circ g} X_2).$$
- Its right adjoint $f_* : \mathcal{M}od_{\mathcal{O}_{X_1}} \to \mathcal{M}od_{\mathcal{O}_{X_2}}$ associates to any $\mathcal{O}_{X_1}$-Module $\mathcal{M}$ on $\mathcal{C}/X_1$ the sheaf
$$(X \longrightarrow X_2) \longmapsto \varprojlim_{\begin{pmatrix} X' & \to & X \\ \downarrow & & \downarrow \\ X_1 & \to & X_2 \end{pmatrix} = \text{commutative square}} \mathcal{M}(X' \longrightarrow X_1).$$
- Its left adjoint $f_! : \mathcal{M}od_{\mathcal{O}_{X_1}} \to \mathcal{M}od_{\mathcal{O}_{X_2}}$ associates to any $\mathcal{O}_{X_1}$-Module $\mathcal{M}$ on $\mathcal{C}/X_1$ the sheafification of the presheaf
$$(X \longrightarrow X_2) \longmapsto \bigoplus_{\substack{(X \xrightarrow{g} X_1 \xrightarrow{f} X_2) \\ = \text{factorisation of } X \to X_2}} \mathcal{M}(X \xrightarrow{g} X_1).$$

So the functor $f_!$ is exact.

Subtoposes and open subtoposes

Definition:

(i) A morphism of toposes $(f^{-1}, f_*) : \mathcal{E}_1 \to \mathcal{E}_2$
is called an embedding, and $\mathcal{E}_1$ is called a subtopos of $\mathcal{E}_2$,
if its push-forward component

$$f_* : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$$

is fully faithful.

(ii) A subtopos $(f^{-1}, f_*) : \mathcal{E}_1 \to \mathcal{E}_2$ is called open
if it identifies with a localisation $(p^{-1}, p_*) : \mathcal{E}_2/F \to \mathcal{E}_2$
for some object F of $\mathcal{E}_2$ endowed with $p : F \to 1$.

Remarks:

(i) For any site $(\mathcal{C}, J)$, $(j^*, j_*) : \widehat{\mathcal{C}}_J \to \widehat{\mathcal{C}}$ is a subtopos.

(ii) Conversely, one can prove that any subtopos of $\widehat{\mathcal{C}}$
has the form $\widehat{\mathcal{C}}_J$ for a unique topology J on $\mathcal{C}$.

(iii) This implies that subtoposes of a topos $\widehat{\mathcal{C}}_J$
correspond to topologies J' on $\mathcal{C}$ which contain J.

Lemma: Let $\mathcal{E}$ = topos,

$(F_1 \xrightarrow{f} F_2)$ = morphism of $\mathcal{E}$.

Then the morphism of toposes

$$(f^{-1}, f_*) : \mathcal{E}/F_1 \longrightarrow \mathcal{E}/F_2$$

is an embedding if and only if the morphism

$$f : F_1 \longrightarrow F_2$$

is a monomorphism.

Remark: If $\mathcal{E} = \widehat{\mathcal{C}}_J$ endowed with $\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J$,
any monomorphism $i : X_1 \hookrightarrow X_2$ of $\mathcal{C}$
yields a monomorphism $\ell(X_1) \hookrightarrow \ell(X_2)$ of $\widehat{\mathcal{C}}_J$
and so an open embedding of toposes

$$(\widehat{\mathcal{C}/X_1})_{J_{X_1}} = \mathcal{E}/\ell(X_1) \longrightarrow \mathcal{E}/\ell(X_2) = (\widehat{\mathcal{C}/X_2})_{J_{X_2}} .$$

Proof of the lemma: The following conditions are equivalent:

(1) f_* is fully faithful.

(2) The morphism $f^* \circ f_* \to \mathrm{id}$ is an isomorphism.

(3) The morphism $\mathrm{id} \to f^* \circ f_!$ is an isomorphism.

(4) $f_! : (F \xrightarrow{g} F_1) \mapsto (F \xrightarrow{f \circ g} F_2)$ is fully faithful.

(5) f is a monomorphism of $\mathcal{E}$.

Remark:

Suppose $\mathcal{E} = \widehat{\mathcal{C}}_J$ endowed with $\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J$
and $\mathcal{O}$ = ring object of $\mathcal{E}$
inducing a ring object $\mathcal{O}_X = \mathcal{O}_{\ell(X)}$
of $\mathcal{E}/\ell(X) \cong \widehat{(\mathcal{C}/X)}_{J_X}$ for any object X of $\mathcal{C}$.

Then, for any monomorphism $i : X_1 \hookrightarrow X_2$ of $\mathcal{C}$, the functor

$$i_! : \mathcal{M}od_{\mathcal{O}_{X_1}} \longrightarrow \mathcal{M}od_{\mathcal{O}_{X_2}}$$

associates to any $\mathcal{O}_{X_1}$-Module $\mathcal{M}$ on $\mathcal{C}/X_1$
the sheafification of the presheaf on $\mathcal{C}/X_2$

$$(X \longrightarrow X_2) \longmapsto \begin{cases} \mathcal{M}(X \to X_1) & \text{if} \quad X \to X_2 \text{ factorises as } X \to X_1 \hookrightarrow X_2\,, \\ 0 \text{ otherwise.} \end{cases}$$

So the functor $i_!$ can be called “extension by 0” as in the case of topological spaces.

Derived categories of modules in toposes

Definition:

Let $(\mathcal{E}, \mathcal{O})$ = ringed topos
= topos $\mathcal{E}$ endowed with a ring object $\mathcal{O}$.

Then one denotes

$D(\mathcal{M}od_{\mathcal{O}})$,
$D^+(\mathcal{M}od_{\mathcal{O}})$,
$D^-(\mathcal{M}od_{\mathcal{O}})$,
$D^b(\mathcal{M}od_{\mathcal{O}})$

the derived categories of the abelian category $\mathcal{M}od_{\mathcal{O}}$ of modules over $\mathcal{O}$ in $\mathcal{E}$.

Remark:

If $\mathcal{E}$ is written $\widehat{\mathcal{C}}_J$,
the objects of these derived categories
can be seen as complexes of linear sheaves on $(\mathcal{C}, J)$.

The additive functors we have introduced
induce functors between derived categories when they are exact:

Corollary:

(i) For any morphism of toposes $(f^{-1}, f_*) : \mathcal{E}_1 \to \mathcal{E}_2$
and any ring object $\mathcal{O}_2$ of $\mathcal{E}_2$,
the exact functor $f^{-1} : \mathcal{M}od_{\mathcal{O}_2} \to \mathcal{M}od_{f^{-1}\mathcal{O}_2}$ defines an additive functor

$$f^{-1} : D(\mathcal{M}od_{\mathcal{O}_2}) \longrightarrow D(\mathcal{M}od_{f^{-1}\mathcal{O}_2})$$

which respects distinguished triangles and commutes with each $[m]$.

(ii) For any morphism $f : F_1 \to F_2$
in a topos $\mathcal{E}$ endowed with a ring object $\mathcal{O}$,
the exact functor $f_! : \mathcal{M}od_{\mathcal{O}_{F_1}} \to \mathcal{M}od_{\mathcal{O}_{F_2}}$
between the abelian categories of modules over $\mathcal{O}_{F_1}$ and $\mathcal{O}_{F_2}$
in the localised toposes $\mathcal{E}/F_1$ and $\mathcal{E}/F_2$
defines an additive functor

$$f_! : D(\mathcal{M}od_{\mathcal{O}_{F_1}}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}_{F_2}})$$

which respects distinguished triangles and commutes with each $[m]$.

Flat and injective modules in toposes

We recall:

Definition:
Let $\mathcal{E} =$ topos,
$\mathcal{O} =$ ring object of $\mathcal{E}$.

(i) An object $\mathcal{M}$ of $\mathcal{M}od_{\mathcal{O}}$ is called "flat" if the functor $\bullet \otimes_{\mathcal{O}} \mathcal{M}$ is exact.

(ii) An object $\mathcal{I}$ of $\mathcal{M}od_{\mathcal{O}}$ is called "injective" if the functor $\mathrm{Hom}_{\mathcal{O}}(\bullet, \mathcal{I})$ is exact.

Remark:
These definitions make sense even if $\mathcal{O}$ is not necessarily a commutative ring object of $\mathcal{E}$.
In that case, $\bullet \otimes_{\mathcal{O}} \mathcal{M}$ is an additive functor from the abelian category $\mathcal{M}od_{\mathcal{O}^{\mathrm{op}}}$ of right $\mathcal{O}$-Modules in $\mathcal{E}$ to the category $\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}}}$ of abelian objects of $\mathcal{E}$.

Theorem:

Let $\mathcal{E}$ = topos,
$\mathcal{O}$ = ring object of $\mathcal{E}$.

Then:

(i) For any $\mathcal{O}$-Module $\mathcal{M}$ in $\mathcal{E}$, there is an epimorphism

$$\mathcal{M}_0 \twoheadrightarrow \mathcal{M}$$

from a flat $\mathcal{O}$-Module $\mathcal{M}_0$.

(ii) For any $\mathcal{O}$-Module $\mathcal{M}$ in $\mathcal{E}$, there is a monomorphism

$$\mathcal{M} \hookrightarrow \mathcal{I}$$

to an injective $\mathcal{O}$-Module $\mathcal{I}$.

Proof of (i):
Let $\mathcal{E} = \widehat{\mathcal{C}}_J$ for some small site $(\mathcal{C}, J)$ endowed with $\ell : \mathcal{C} \to \widehat{\mathcal{C}}_J$.
For any object X of $\mathcal{C}$, consider the localisation morphism

$$(i_X^*, i_{X,*}) : \mathcal{E}/\ell(X) = \widehat{(\mathcal{C}/X)}_{J_X} \longrightarrow \mathcal{E} = \widehat{\mathcal{C}}_J \,,$$

the restricted ring object $\mathcal{O}_X = i_X^* \mathcal{O}$ in $\mathcal{E}/\ell(X)$ and the left adjoint

$$\begin{array}{rrcl} & i_{X,!} \cdot \mathcal{M}od_{\mathcal{O}_X} & \longrightarrow & \mathcal{M}od_{\mathcal{O}} \\ \text{of} & i_X^* : \mathcal{M}od_{\mathcal{O}} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_X} \,. \end{array}$$

Any section $m \in \mathcal{M}(X)$ of an $\mathcal{O}$-Module $\mathcal{M}$ can be seen as a morphism

$$i_{X,!}\mathcal{O}_X \longrightarrow \mathcal{M}$$

and so there is a canonical epimorphism

$$\mathcal{M}_0 = \bigoplus_X \bigoplus_{m \in \mathcal{M}(X)} i_{X,!}\mathcal{O}_X \twoheadrightarrow \mathcal{M} \,.$$

Lastly, the $\mathcal{O}$-Module $\mathcal{M}_0$ is flat because for any X the functor

$$\mathcal{N} \longmapsto \mathcal{N} \otimes_{\mathcal{O}} i_{X,!}\mathcal{O}_X$$

identifies with the composite exact functor

$$\mathcal{N} \longmapsto i_{X,!} i_X^* \mathcal{N} \,.$$

Proof of (ii):
Choose in $\mathcal{M}od_{\mathcal{O}}$ a "generator" $\mathcal{A}$ in the sense that,
for any monomorphism $\mathcal{M}' \hookrightarrow \mathcal{M}$ of $\mathcal{M}od_{\mathcal{O}}$ with $\mathcal{M}/\mathcal{M}' \neq 0$,
there is a morphism $\mathcal{A} \to \mathcal{M}$ which does not factorise through $\mathcal{M}'$.
For instance, if $\mathcal{E} = \widehat{\mathcal{C}}_J$ for some small site $(\mathcal{C}, J)$, one can take

$$\mathcal{A} = \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} i_{X,!} i_X^* \mathcal{O} \, .$$

We first prove:

Lemma:
An $\mathcal{O}$-Module $\mathcal{I}$ in $\mathcal{E}$ is injective if and only if,
for any subobject $\mathcal{B} \to \mathcal{A}$ of the generator $\mathcal{A}$,
any morphism $\mathcal{B} \to \mathcal{I}$ extends to a morphism $\mathcal{A} \to \mathcal{I}$.

Proof of the lemma:
The condition is obviously necessary.
In the reverse direction, consider a monomorphism of $\mathcal{M}od_{\mathcal{O}}$

$$\mathcal{M}' \hookrightarrow \mathcal{M}$$

and a morphism $f : \mathcal{M}' \to \mathcal{I}$.
We have to prove that f extends to a morphism $\mathcal{M} \to \mathcal{I}$.

Consider the set I of pairs $(\mathcal{M}_1, f_1)$ consisting in a subobject $\mathcal{M}_1 \hookrightarrow \mathcal{M}$ containing $\mathcal{M}'$ and a morphism $f_1 : \mathcal{M}_1 \to \mathcal{I}$ which extends f.
For two elements $(\mathcal{M}_1, f_1)$, $(\mathcal{M}_2, f_2)$ we say that

$$(\mathcal{M}_1, f_1) \leq (\mathcal{M}_2, f_2)$$

if $\mathcal{M}_2$ contains $\mathcal{M}_1$ and f_2 extends f_1. For any totally ordered subset I' of I,

$$\mathcal{M}_2 = \varinjlim_{(\mathcal{M}_1, f_1) \in I'} \mathcal{M}_1$$

is a subobject of $\mathcal{M}$ and it is endowed with a morphism

$$f_2 : \mathcal{M}_2 \to \mathcal{I}$$

such that $(\mathcal{M}_1, f_1) \leq (\mathcal{M}_2, f_2), \forall (\mathcal{M}_1, f_1) \in I'$.
According to Zorn's lemma, I has a maximal element $(\mathcal{M}_1, f_1)$.
For any morphism $\mathcal{A} \to \mathcal{M}$, consider $\mathcal{B} = \mathcal{M}_1 \times_{\mathcal{M}} \mathcal{A}$.
By hypothesis, the composed morphism $\mathcal{B} \to \mathcal{M}_1 \xrightarrow{f_1} \mathcal{I}$ extends to a morphism $\mathcal{A} \to \mathcal{I}$.
This defines a morphism

$$\mathcal{B} \backslash (\mathcal{M}_1 \oplus \mathcal{A}) = \mathcal{M}_2 \xrightarrow{f_2} \mathcal{I}$$

which extends $f_1 : \mathcal{M}_1 \to \mathcal{I}$ to $\mathcal{M}_2$. On the other hand, $\mathcal{M}_2$ is a subobject of $\mathcal{M}$.
As $(\mathcal{M}_1, f_1)$ is maximal, this implies that $\mathcal{M}_2 = \mathcal{M}_1$ or, equivalently, that $\mathcal{A} \to \mathcal{M}$ factorises through $\mathcal{M}_1$.
As $\mathcal{A}$ is a generator, this means that $\mathcal{M}_1 = \mathcal{M}$.

We also prove:

Lemma:
For any $\mathcal{O}$-Module $\mathcal{M}$, there is a monomorphism

$$\mathcal{M} \hookrightarrow \mathcal{M}_1$$

such that, for any subobject $\mathcal{B}$ of the generator $\mathcal{A}$, any morphism

$$\mathcal{B} \longrightarrow \mathcal{M}$$

extends to a morphism

$$\mathcal{A} \longrightarrow \mathcal{M}_1 .$$

Proof of the lemma:
The subobjects of any object of $\mathcal{M}od_{\mathcal{O}}$ make up a set.
In particular, the subobjects of $\mathcal{A}$ make up a set S.
One can take for $\mathcal{M}_1$ the quotient of

$$\mathcal{M} \oplus \Big(\bigoplus_{\mathcal{B} \in S} \; \bigoplus_{f \in \mathrm{Hom}(\mathcal{B}, \mathcal{M})} \mathcal{A} \Big)$$

by

$$\Big(\bigoplus_{\mathcal{B} \in S} \; \bigoplus_{f \in \mathrm{Hom}(\mathcal{B}, \mathcal{M})} \mathcal{B} \Big).$$

Conclusion of the proof of (ii): Starting from $\mathcal{M}_0 = \mathcal{M}$, let's define an inductive system of $\mathcal{O}$-Modules

$$\mathcal{M}_i \quad \text{indexed by the ordinals} \quad i$$

and related by monomorphisms $\mathcal{M}_i \hookrightarrow \mathcal{M}_j$ for $i \leq j$.
The construction is by transfinite induction:

$$\begin{cases} \bullet \quad \text{if } j = i+1, \mathcal{M}_j \text{ is deduced from } \mathcal{M}_i \text{ by the construction of the previous lemma.} \\ \bullet \quad \text{if } j \text{ is the limit of the } i < j \text{, we take } \mathcal{M}_j = \varinjlim_{i<j} \mathcal{M}_i \,. \end{cases}$$

Let k be an ordinal whose cardinality is strictly bigger than the cardinality of the set of subobjects of $\mathcal{A}$ and which is the limit of the $i < k$.
For any morphism $f : \mathcal{B} \to \mathcal{M}_k$ defined on a subobject $\mathcal{B}$ of $\mathcal{A}$, the formula

$$\mathcal{M}_k = \varinjlim_{i<k} \mathcal{M}_i \qquad \text{implies} \qquad \mathcal{B} = \varinjlim_{i<k} f^{-1}(\mathcal{M}_i)\,.$$

As the cardinality of k is strictly bigger than the cardinality of the set of subobjects of $\mathcal{B}$, this implies that $f : \mathcal{B} \to \mathcal{M}_k$ factorises as

$$\mathcal{B} \longrightarrow \mathcal{M}_i$$

for some $i < k$ and so it extends to some morphism

$$\mathcal{A} \longrightarrow \mathcal{M}_k\,.$$

The $\mathcal{O}$-Module $\mathcal{M}_k$ is injective according to the first lemma.

Remark:

Suppose a topos $\mathcal{E}$ has a set P of points $x = (x^*, x_*) : \mathrm{Set} \to \mathcal{E}$ which is conservative in the sense that a morphism of $\mathcal{E}$

$$F_1 \longrightarrow F_2$$

is an isomorphism if and only if $x^* F_1 \to x^* F_2$ is one-to-one for any $x \in F$.

Then, for any ring object $\mathcal{O}$ of $\mathcal{E}$, any $\mathcal{O}$-Module $\mathcal{M}$ has the canonical embedding

$$\mathcal{M} \hookrightarrow \prod_{x \in P} x_* \circ x^* \mathcal{M} \, .$$

Each $x^* \mathcal{M}$ is a module over the ring $\mathcal{O}_x = x^* \mathcal{O}$ and can be embedded into an injective $\mathcal{O}_x$-module, for instance

$$I_x = \mathrm{Hom}(M_x, \mathbb{Q}/\mathbb{Z})$$

for any free $\mathcal{O}_x$-module M_x endowed with an epimorphism

$$M_x \twoheadrightarrow \mathrm{Hom}(x^* \mathcal{M}, \mathbb{Q}/\mathbb{Z})) \, .$$

Then there is an induced embedding

$$\mathcal{M} \hookrightarrow \prod_{x \in P} x_* I_x \, .$$

The $\mathcal{O}$-Module $\prod\limits_{x \in P} x_* I_x$ is injective as, for any $x \in P$, I_x is an injective $\mathcal{O}_x$-module and the functor x^* is exact.

In order to derive the functors f^* and $\otimes$, we need to complete the previous theorem with:

Proposition:

Let $\mathcal{E}$ = topos,
$\mathcal{O}$ = ring object of $\mathcal{E}$.

Then, for any short exact sequence of $\mathcal{M}od_{\mathcal{O}}$

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0\,,$$

we have:

(i) For any Module $\mathcal{N}$, the induced sequence

$$0 \longrightarrow \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M}_1 \longrightarrow \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M}_2 \longrightarrow \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M}_3 \longrightarrow 0$$

is exact if $\mathcal{M}_3$ is flat.

(ii) If $\mathcal{M}_2$ and $\mathcal{M}_3$ are flat, $\mathcal{M}_1$ is flat as well.

Proof:

(i) For any Module $\mathcal{N}$, we can choose an epimorphism

$$\mathcal{N}' \twoheadrightarrow \mathcal{N}$$

from a flat Module $\mathcal{N}'$ and denote $\mathcal{N}'' = \mathrm{Ker}(\mathcal{N}' \to \mathcal{N})$.
As $\mathcal{N}'$ is flat, $\mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_1 \to \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_2$ is a monomorphism and we deduce from the commutative square

$$\begin{array}{ccc} \mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_1 & \longrightarrow & \mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_2 \\ \downarrow & & \downarrow \\ \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_1 & \longrightarrow & \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_2 \end{array}$$

that $\mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_1 \to \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_1$ factorises through

$$\mathcal{L} = \mathrm{Im}(\mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_1 \longrightarrow \mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_2)\,.$$

So we have a short exact sequence of complexes

$$0 \longrightarrow \begin{pmatrix} \mathcal{L} \\ \downarrow \\ \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_2 \\ \downarrow \\ \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_3 \\ \downarrow \\ \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_3 \end{pmatrix} \longrightarrow 0\,.$$

As $\mathcal{M}_3$ is flat, $\mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_3 \to \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_3$ is a monomorphism and the associated long exact sequence of cohomology yields a short exact sequence

$$0 \longrightarrow \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M}_1 \longrightarrow \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M}_2 \longrightarrow \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M}_3 \longrightarrow 0\,.$$

(ii) If $\mathcal{M}_2$ and $\mathcal{M}_3$ are flat, we have for any short exact sequence of Modules

$$0 \longrightarrow \mathcal{N}'' \longrightarrow \mathcal{N}' \longrightarrow \mathcal{N} \longrightarrow 0$$

an associated short exact sequence of complexes

$$0 \longrightarrow \begin{pmatrix} \mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_2 \\ \downarrow \\ \mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_2 \\ \downarrow \\ \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M}_2 \\ \downarrow \\ \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M}_3 \end{pmatrix} \longrightarrow 0\,.$$

As $\mathcal{M}_3$ is flat, the associated long exact sequence of cohomology reduces to the short exact sequence

$$0 \longrightarrow \mathcal{N}'' \otimes_{\mathcal{O}} \mathcal{M}_1 \longrightarrow \mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}_1 \longrightarrow \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M}_1 \longrightarrow 0\,.$$

This means that $\mathcal{M}_1$ also is flat.

Corollary:
Let $f : (\mathcal{E}_1, \mathcal{O}_1) \to (\mathcal{E}_2, \mathcal{O}_2)$
$=$ morphism of ringed toposes.
Then:

(i) The right-exact functor

$$f^* : \mathcal{M}od_{\mathcal{O}_2} \to \mathcal{M}od_{\mathcal{O}_1}$$

has a left derived functor

$$\mathrm{L}f^* : D^-(\mathcal{M}od_{\mathcal{O}_2}) \longrightarrow D^-(\mathcal{M}od_{\mathcal{O}_1})$$

whose restriction to complexes of flat Modules (or more generally f^*-acyclic Modules) is induced by f^*.

(ii) The left-exact functor

$$f_* : \mathcal{M}od_{\mathcal{O}_1} \longrightarrow \mathcal{M}od_{\mathcal{O}_2}$$

has a right exact functor

$$\mathrm{R}f_* : D^+(\mathcal{M}od_{\mathcal{O}_1}) \longrightarrow D^+(\mathcal{M}od_{\mathcal{O}_2})$$

whose restriction to complexes of injective Modules (or more generally f_*-acyclic Modules) is induces by f_*.

Remarks:

(i) If f^* has finite cohomological dimension, it even has a derived functor

$$\mathrm{L}f^* : D(\mathcal{M}od_{\mathcal{O}_2}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}_1})$$

whose restriction to complexes of f^*-acyclic objects is induced by f^*. It restricts to a functor

$$\mathrm{L}f^* : D^+(\mathcal{M}od_{\mathcal{O}_2}) \longrightarrow D^+(\mathcal{M}od_{\mathcal{O}_1})$$

which is left adjoint to $\mathrm{R}f_*$.

(ii) If f_* has finite cohomological dimension, it even has a derived functor

$$\mathrm{R}f_* : D(\mathcal{M}od_{\mathcal{O}_1}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}_2})$$

whose restriction to complexes of f_*-acyclic objects is induced by f_*. It restricts to a functor

$$\mathrm{R}f_* : D^-(\mathcal{M}od_{\mathcal{O}_1}) \longrightarrow D^-(\mathcal{M}od_{\mathcal{O}_2})$$

which is right adjoint to $\mathrm{L}f^*$.

(iii) For any morphisms of ringed toposes

$$(\mathcal{E}_1, \mathcal{O}_1) \xrightarrow{f} (\mathcal{E}_2, \mathcal{O}_2) \xrightarrow{g} (\mathcal{E}_3, \mathcal{O}_3)\,,$$

the functor

$$g^* = \mathcal{O}_2 \otimes_{g^{-1}\mathcal{O}_3} \bullet$$

transforms flat $\mathcal{O}_3$-Modules into flat $\mathcal{O}_2$-Modules.
Therefore the canonical morphism

$$\mathrm{L}f^* \circ \mathrm{L}g^* \longrightarrow \mathrm{L}(g \circ f)^*$$

is an isomorphism.

(iv) In the same situation, the canonical functor

$$\mathrm{R}(g \circ f)_* \longrightarrow \mathrm{R}g_* \circ \mathrm{R}f_*$$

is also an isomorphism.
Indeed, we first remark that this statement is true if $\mathcal{O}_1 = f^{-1}\mathcal{O}_2$ as, in that case, $f^* = f^{-1} : \mathcal{M}od_{\mathcal{O}_2} \to \mathcal{M}od_{\mathcal{O}_1}$ is exact and its right adjoint $f_* : \mathcal{M}od_{\mathcal{O}_1} \to \mathcal{M}od_{\mathcal{O}_2}$ transforms injective $\mathcal{O}_1$-Modules in injective $\mathcal{O}_2$-Modules.

The general case follows from this remark and the following lemma:

Lemma:
For any morphism of ringed toposes

$$(\mathcal{E}_1, \mathcal{O}_1) \xrightarrow{f} (\mathcal{E}_2, \mathcal{O}_2)\,,$$

the diagram

$$\begin{array}{ccc} D^+(\mathcal{M}od_{\mathcal{O}_1}) & \longrightarrow & D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_1}}) \\ \downarrow{\scriptstyle \mathrm{R}f_*} & \searrow^{\mathrm{R}f_*} & \downarrow{\scriptstyle \mathrm{R}f_*} \\ D^+(\mathcal{M}od_{\mathcal{O}_2}) & \longrightarrow & D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_2}}) \end{array}$$

is commutative up to isomorphism.

Remark:
For any topos $\mathcal{E}$ and its canonical morphism $\mathcal{E} \xrightarrow{(p^{-1},\, p_*)} \mathrm{Set}$ and any ring R, we denote $\mathrm{R}_{\mathcal{E}} = p^{-1}\mathrm{R}$.
In particular, $\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}}}$ is the category of abelian objects of $\mathcal{E}$.
We know that any submodule of a flat $\mathbb{Z}$-module is flat.
So the subcategory of $\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}}}$ on flat $\mathbb{Z}_{\mathcal{E}}$-Modules has codimension ≤ 1.

Proof of the lemma: The triangle

$$\begin{array}{ccc} D^+(\mathcal{M}od_{\mathcal{O}_1}) & & \\ {\scriptstyle \mathrm{R}f_*}\downarrow & \searrow^{\mathrm{R}f_*} & \\ D^+(\mathcal{M}od_{\mathcal{O}_2}) & \longrightarrow & D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_2}}) \end{array}$$

is commutative because the forgetful functor

$$\mathcal{M}od_{\mathcal{O}_2} \longrightarrow \mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_2}}$$

is exact. The functor $f^{-1} : \mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_2}} \to \mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_1}}$ is exact and the functors

$$\begin{array}{rcccl} f^* & : & \mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_2}} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_1} \\ \mathcal{O}_1 \otimes_{\mathbb{Z}_{\mathcal{E}_1}} \bullet & : & \mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_1}} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_1} \end{array}$$

have finite cohomological dimension. So they have derived functors

$$\begin{array}{rcccl} f^{-1} & : & D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_2}}) & \longrightarrow & D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_1}}), \\ \mathrm{L}f^* & : & D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_2}}) & \longrightarrow & D^+(\mathcal{M}od_{\mathcal{O}_1}), \\ \mathcal{O}_1 \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_{\mathcal{E}_1}} \bullet & : & D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_1}}) & \longrightarrow & D^+(\mathcal{M}od_{\mathcal{O}_1}) \end{array}$$

and we already know that the canonical morphism

$$\mathcal{O}_1 \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_{\mathcal{E}_1}} f^{-1}(\bullet) \longrightarrow \mathrm{L}f^*$$

is an isomorphism. Taking right adjoints, $\mathrm{R}f_* : D^+(\mathcal{M}od_{\mathcal{O}_1}) \to D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_2}})$ is isomorphic to $\mathrm{R}f_* : D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_1}}) \to D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_2}})$ composed with $D^+(\mathcal{M}od_{\mathcal{O}_1}) \to D^+(\mathcal{M}od_{\mathbb{Z}_{\mathcal{E}_1}})$.

The previous theorem and proposition also imply:

Corollary:
Let $(\mathcal{E}, \mathcal{O})$ = commutative ringed topos
= topos $\mathcal{E}$ endowed with a commutative ring object $\mathcal{O}$.
Then:

(i) The right-exact additive bifunctor

$$\otimes_{\mathcal{O}} : \mathcal{M}od_{\mathcal{O}} \times \mathcal{M}od_{\mathcal{O}} \longrightarrow \mathcal{M}od_{\mathcal{O}}$$

has a left derived functor

$$\overset{\mathrm{L}}{\otimes}_{\mathcal{O}} : D(\mathcal{M}od_{\mathcal{O}}) \times D^{-}(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}})$$

constructed by factorising

$$K(\mathcal{M}od_{\mathcal{O}}) \times K^{-}(\mathcal{F}lat_{\mathcal{O}}) \xrightarrow{\bullet \otimes_{\mathcal{O}} \bullet} K(\mathcal{M}od_{\mathcal{O}}) \xrightarrow{Q} D(\mathcal{M}od_{\mathcal{O}})$$

if $\mathcal{F}lat_{\mathcal{O}}$ denotes the full additive subcategory of $\mathcal{M}od_{\mathcal{O}}$ on flat $\mathcal{O}$-Modules. Furthermore, if $\bullet \otimes_{\mathcal{O}} \bullet$ has finite cohomological dimension, it even has a derived functor

$$\overset{\mathrm{L}}{\otimes}_{\mathcal{O}} : D(\mathcal{M}od_{\mathcal{O}}) \times D(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}})$$

constructed by factorising

$$K(\mathcal{M}od_{\mathcal{O}}) \times K(\mathcal{F}lat_{\mathcal{O}}) \xrightarrow{\bullet \otimes_{\mathcal{O}} \bullet} K(\mathcal{M}od_{\mathcal{O}}) \xrightarrow{Q} D(\mathcal{M}od_{\mathcal{O}})\,.$$

(ii) The left-exact additive bifunctors

$$\begin{array}{rcll} \mathcal{H}om & : & \mathcal{M}od_{\mathcal{O}}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}} & \longrightarrow & \mathcal{M}od_{\mathcal{O}}\,, \\ \mathrm{Hom} & : & \mathcal{M}od_{\mathcal{O}}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}} & \longrightarrow & \mathrm{Ab} \end{array}$$

have right derived functors

$$\begin{array}{rcll} \mathrm{R}\mathcal{H}om & : & D(\mathcal{M}od_{\mathcal{O}})^{\mathrm{op}} \times D^{+}(\mathcal{M}od_{\mathcal{O}}) & \longrightarrow & D(\mathcal{M}od_{\mathcal{O}}) \\ \mathrm{RHom} & : & D(\mathcal{M}od_{\mathcal{O}})^{\mathrm{op}} \times D^{+}(\mathcal{M}od_{\mathcal{O}}) & \longrightarrow & D(\mathrm{Ab}) \end{array}$$

constructed by factorising

$$\begin{array}{lclcl} K(\mathcal{M}od_{\mathcal{O}})^{\mathrm{op}} \times K^{+}(\mathcal{I}nj_{\mathcal{O}}) & \xrightarrow{\mathcal{H}om} & K(\mathcal{M}od_{\mathcal{O}}) & \xrightarrow{Q} & D(\mathcal{M}od_{\mathcal{O}})\,, \\ K(\mathcal{M}od_{\mathcal{O}})^{\mathrm{op}} \times K^{+}(\mathcal{I}nj_{\mathcal{O}}) & \xrightarrow{\mathrm{Hom}} & K(\mathrm{Ab}) & \xrightarrow{Q} & D(\mathrm{Ab})\,. \end{array}$$

Remarks:

(i) Commutativity: The functors

$$(\mathcal{M}_1, \mathcal{M}_2) \longmapsto \mathcal{M}_1 \overset{\mathrm{L}}{\otimes} \mathcal{M}_2 \quad \text{and} \quad (\mathcal{M}_1, \mathcal{M}_2) \longmapsto \mathcal{M}_2 \overset{\mathrm{L}}{\otimes} \mathcal{M}_1$$

from $D^{-}(\mathcal{M}od_{\mathcal{O}}) \times D^{-}(\mathcal{M}od_{\mathcal{O}})$ to $D^{-}(\mathcal{M}od_{\mathcal{O}})$ are canonically isomorphic.

(ii) Associativity: The functors

$$(\bullet \overset{\mathrm{L}}{\otimes} \bullet) \overset{\mathrm{L}}{\otimes} \bullet \qquad \text{and} \qquad \bullet \overset{\mathrm{L}}{\otimes} (\bullet \overset{\mathrm{L}}{\otimes} \bullet)$$

from $D(\mathcal{M}od_{\mathcal{O}}) \times D^{-}(\mathcal{M}od_{\mathcal{O}}) \times D^{-}(\mathcal{M}od_{\mathcal{O}})$ to $D^{-}(\mathcal{M}od_{\mathcal{O}})$ are canonically isomorphic.

(iii) Compatibility with pull-back:
For any morphism of commutative ringed toposes

$$(\mathcal{E}_1, \mathcal{O}_1) \xrightarrow{f} (\mathcal{E}_2, \mathcal{O}_2)\,,$$

the functors

$$\mathrm{L}f^*(\bullet \overset{\mathrm{L}}{\otimes} \bullet) \quad \text{and} \quad \mathrm{L}f^*(\bullet) \overset{\mathrm{L}}{\otimes} \mathrm{L}f^*(\bullet)$$

from $D^-(\mathcal{M}od_{\mathcal{O}_2}) \times D^-(\mathcal{M}od_{\mathcal{O}_2})$ to $D^-(\mathcal{M}od_{\mathcal{O}_1})$ are canonically isomorphic.

(iv) If $\mathcal{M}$ is a flat $\mathcal{O}$-Module and $\mathcal{I}$ an injective $\mathcal{O}$-Module, then $\mathcal{H}om(\mathcal{M}, \mathcal{I})$ is an injective $\mathcal{O}$-Module.
This follows from the identification between the functors

$$\mathrm{Hom}(\bullet, \mathcal{H}om(\mathcal{M}, \mathcal{I})) \quad \text{and} \quad \mathrm{Hom}(\bullet \otimes_{\mathcal{O}} \mathcal{M}, \mathcal{I})$$

from $\mathcal{M}od_{\mathcal{O}}$ to Ab.

(v) The previous remark implies that the pairs of functors

$$\mathrm{R}\mathcal{H}om(\bullet, \mathrm{R}\mathcal{H}om(\bullet, \bullet)) \quad \text{and} \quad \mathrm{R}\mathcal{H}om(\bullet \overset{\mathrm{L}}{\otimes} \bullet, \bullet)$$

$$\text{or} \quad \mathrm{RHom}(\bullet, \mathrm{R}\mathcal{H}om(\bullet, \bullet)) \quad \text{and} \quad \mathrm{RHom}(\bullet \overset{\mathrm{L}}{\otimes} \bullet, \bullet)$$

$$\text{or} \quad \mathrm{Hom}(\bullet, \mathrm{R}\mathcal{H}om(\bullet, \bullet)) \quad \text{and} \quad \mathrm{Hom}(\bullet \overset{\mathrm{L}}{\otimes} \bullet, \bullet)$$

from $D(\mathcal{M}od_{\mathcal{O}}) \times D^-(\mathcal{M}od_{\mathcal{O}}) \times D^+(\mathcal{M}od_{\mathcal{O}})$ to $D(\mathcal{M}od_{\mathcal{O}})$, $D(\mathrm{Ab})$ or Ab are canonically isomorphic.

(vi) Remark (iv) also implies that if

$$(\mathcal{E}, \mathcal{O}) \xrightarrow{p} (\mathrm{Set}, \mathbb{Z})$$

is the canonical morphism of commutative ringed toposes, the functors

$$\mathrm{RHom} \qquad \text{and} \qquad \mathrm{R}p_* \circ \mathrm{R}\mathcal{H}om$$

from $D^-(\mathcal{M}od_{\mathcal{O}}) \times D^+(\mathcal{M}od_{\mathcal{O}})$ to $D^+(\mathrm{Ab})$ are canonically isomorphic.

(vii) If $f : (\mathcal{E}_1, \mathcal{O}_1) \to (\mathcal{E}_2, \mathcal{O}_2)$ is a morphism of commutative ringed toposes such that $\mathcal{O}_1$ is flat over $f^{-1}\mathcal{O}_2$,
then the functors

$$\mathrm{RHom}(f^*(\bullet), \bullet) \qquad \text{and} \qquad \mathrm{RHom}(\bullet, \mathrm{R}f_*(\bullet))$$

from $D(\mathcal{M}od_{\mathcal{O}_2})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathcal{O}_1})$ to $D(\mathrm{Ab})$
are canonically isomorphic,
as well as the functors

$$\mathrm{R}f_* \circ \mathrm{R}\mathcal{H}om(f^*(\bullet), \bullet) \qquad \text{and} \qquad \mathrm{R}\mathcal{H}om(\bullet, \mathrm{R}f_*(\bullet))$$

from $D(\mathcal{M}od_{\mathcal{O}_2})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathcal{O}_1})$ to $D(\mathcal{M}od_{\mathcal{O}_1})$.

Application to geometric categories

Suppose $\mathcal{G}$ is a geometric category endowed with maps

$$\begin{array}{ccc} X & \longmapsto & (\mathcal{E}_X, \mathcal{O}_X)\,, \\ \| & & \| \\ \text{object of } \mathcal{G} & & \text{commutative ringed topos} \\ (X \xrightarrow{f} Y) & \longmapsto & \left[(\mathcal{E}_X, \mathcal{O}_X) \xrightarrow{(f^{-1}, f_*, f^{-1}\mathcal{O}_Y \to \mathcal{O}_X)} (\mathcal{E}_Y, \mathcal{O}_Y)\right], \\ \| & & \| \\ \text{morphism of } \mathcal{G} & & \text{morphism of commutative ringed toposes} \\ (X \xrightarrow{f} Y \xrightarrow{g} Z) & \longmapsto & \left[(g \circ f)^{-1} \xrightarrow{\sim} f^{-1} \circ g^{-1}\right], \\ & & \| \\ & & \text{isomorphism which exchanges} \\ & & f^{-1} \circ g^{-1}\mathcal{O}_Z \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X \text{ and } (g \circ f)^{-1}\mathcal{O}_Z \to \mathcal{O}_X \end{array}$$

such that, for any $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, the isomorphisms

$$(h \circ g \circ f)^{-1} \xrightarrow{\sim} (g \circ f)^{-1} \circ h^{-1} \xrightarrow{\sim} (f^{-1} \circ g^{-1}) \circ h^{-1}\,,$$

$$(h \circ g \circ f)^{-1} \xrightarrow{\sim} f^{-1} \circ (h \circ g)^{-1} \xrightarrow{\sim} f^{-1} \circ (g^{-1} \circ h^{-1})$$

are equal.

We also suppose that, for any open embedding of $\mathcal{G}$,

$$i : U \longrightarrow X ,$$

the morphism of toposes

$$(i^{-1}, i_*) : \mathcal{E}_U \longrightarrow \mathcal{E}_X$$

identifies $\mathcal{E}_U$ with an open subtopos of $\mathcal{E}_X$
and the morphism

$$i^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_U$$

is an isomorphism.

Then one can associate to any object X of $\mathcal{G}$
the abelian category $\mathcal{M}od_{\mathcal{O}_X}$
endowed with the functors Hom, $\mathcal{H}om$, $\otimes$
and its derived categories

$$D(\mathcal{M}od_{\mathcal{O}_X}) , \; D^+(\mathcal{M}od_{\mathcal{O}_X}) , \; D^-(\mathcal{M}od_{\mathcal{O}_X}) , \; D^b(\mathcal{M}od_{\mathcal{O}_X})$$

together with the derived functors

$$\mathrm{RHom} , \; \mathrm{R}\mathcal{H}om , \; \overset{\mathrm{L}}{\otimes} .$$

One can also associate to any morphism of $\mathcal{G}$

$$f : X \longrightarrow Y$$

a pair of adjoint functors

$$f^* : \mathcal{M}od_{\mathcal{O}_Y} \longrightarrow \mathcal{M}od_{\mathcal{O}_X} \quad \text{and} \quad f_* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_Y}$$

together with derived functors

$$\mathrm{L}f^* \quad \text{and} \quad \mathrm{R}f_* \, .$$

If $i : U \hookrightarrow X$ is an open immersion,

$$i^* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_U}$$

also has a left adjoint

$$i_! : \mathcal{M}od_{\mathcal{O}_U} \longrightarrow \mathcal{M}od_{\mathcal{O}_X}$$

which is exact and induces a functor

$$D(\mathcal{M}od_{\mathcal{O}_U}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}_X}) \, .$$

All these functors

$$\mathrm{RHom} \, , \; \mathrm{R}\mathcal{H}om \, , \; \overset{\mathrm{L}}{\otimes} \, , \; \mathrm{L}f^* \, , \; \mathrm{R}f_* \, , \; i_!$$

verify the properties stated before
in the context of commutative ringed toposes.

For any commutative square of $\mathcal{G}$

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{s} & S \end{array}$$

such that s^*, x^* [resp. p_*, p'_*] have finite cohomological dimension, there is a canonical morphism of functors

$$\mathrm{L}s^* \circ \mathrm{R}p_* \longrightarrow \mathrm{R}p'_* \circ \mathrm{L}x^*$$

from $D^+(\mathcal{M}od_{\mathcal{O}_X})$ to $D^+(\mathcal{M}od_{\mathcal{O}_{S'}})$
[resp. from $D^-(\mathcal{M}od_{\mathcal{O}_X})$ to $D^-(\mathcal{M}od_{\mathcal{O}_{S'}})$].

Definition:
A morphism of $\mathcal{G}$

$$X \xrightarrow{p} S \qquad [\text{resp.} \quad S' \xrightarrow{s} S\,]$$

is called cohomologically proper [resp. coh. smooth] if:

- it is squarable in $\mathcal{G}$,
- for any cartesian square of $\mathcal{G}$

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{s} & S \end{array}$$

  completing $p : X \to S$ [resp. $s : S' \to S$],

  x^* always has finite cohomological dimension
  or p'_* always has finite cohomological dimension,
- for any such cartesian square, the canonical morphism

$$\mathrm{L}s^* \circ \mathrm{R}p_* \longrightarrow \mathrm{R}p'_* \circ \mathrm{L}x^*$$

  is an isomorphism.

The geometric category of schemes

Lemma:
Let $A =$ commutative ring.

(i) For any $f \in A$, the functor

$$B \longmapsto \{u \in \mathrm{Hom}(A, B) \mid u(f) \text{ is invertible in } B\}$$

is representable by

$$A_f = A[X]/(f \cdot X - 1)\,.$$

(ii) For any A-module M and any element $f \in A$,
elements of $A_f \otimes_A M = M_f$ can be written $f^{-n} \cdot m$ with $n \in \mathbb{N}$, $m \in M$.
Two elements $f^{-n} \cdot m$ and $f^{-n'} \cdot m'$ are equal in M_f if and only if there exists $N \in \mathbb{N}$ such that $f^N \cdot (f^{n'} \cdot m - f^n \cdot m') = 0$ in M.

(iii) For any elements f_i, $i \in I$, of A such that $\sum\limits_i f_i \cdot A = A$,
and any A-module M, the canonical morphism

$$M \longrightarrow \mathrm{Eq}\left(\prod_i M_{f_i} \rightrightarrows \prod_{i,j} M_{f_i f_j}\right)$$

is an isomorphism.

Proof:

(i) is obvious.

(ii) The A_f-module $M_f = A_f \otimes_A M$ is the quotient of the $A[X]$-module $A[X] \otimes_A M = \bigoplus_{n \in \mathbb{N}} X^n \otimes M$ by the submodule $(f \cdot X - 1) \cdot A[X] \otimes_A M$.

Any element of M_f can be represented by an expression

$$P = 1 \otimes m_0 + X \otimes m_1 + \cdots + X^n \otimes m_n$$

with $m_0, m_1, \ldots, m_n \in M$. Then $f^n \cdot P$ is also represented by

$$f^n \cdot m_0 + f^{n-1} \cdot m_1 + \cdots + f \cdot m_{n-1} + m_n \in M$$

as $f^k \cdot X^k = 1$ in A_f for any $k \in \mathbb{N}$.

If an element $m \in M$ is 0 in M_f, there exists an expression $P = 1 \otimes m_0 + X \otimes m_1 + \cdots + X^n \otimes m_n \in A[X] \otimes_A M$ such that

$$m = (f \cdot X - 1) \cdot P \qquad \text{in} \qquad A[X] \otimes_A M\,.$$

This implies $m = m_0$, $f \cdot m_0 = m_1, \ldots, f \cdot m_{n-1} = m_n$, $f \cdot m_n = 0$ and so $f^{n+1} \cdot m = 0$.

(iii) The equality $\sum_{i \in I} f_i \cdot A = A$ is equivalent to $1 \in \sum_{i \in I} f_i \cdot A$

so we can suppose that I is finite and equal to $\{1, \ldots, k\}$.

It is also equivalent to the property that, for any prime ideal p of A, there exists i such that $f_i \notin p$.

So each f_i can be replaced by an arbitrary power of f_i.

Consider an element $m \in M$ whose image in each M_{f_i} is 0. Then there exist integers $n_i \geq 1$ such that
$$f_i^{n_i} \cdot m = 0 \quad \text{in } M \text{ for any } i\,.$$
As there are elements $a_i \in A$ such that
$$a_1\, f_1^{n_1} + \cdots + a_k\, f_k^{n_k} = 1\,,$$
we conclude
$$m = a_1\, f_1^{n_1} \cdot m + \cdots + a_k f_k^{n_k} \cdot m = 0 \quad \text{in} \quad M\,.$$
This means we have an embedding
$$M \hookrightarrow \prod M_{f_i}\,.$$
Then consider a family of elements $f_i^{-n_i} \cdot m_i \in M_{f_i}$, $1 \leq i \leq k$, such that, for any i, j, $f_i^{-n_i} \cdot m_i = f_j^{-n_j} \cdot m_j$ in $M_{f_i f_j}$.
We can suppose all the integers n_i to be equal to some $n \in \mathbb{N}$.
Then there is an integer $N \geq 0$ such that, for any i, j,
$$(f_i\, f_j)^N f_j^n \cdot m_i = (f_i\, f_j)^N f_i^n \cdot m_j \quad \text{in} \quad M\,.$$
Replacing each m_i by $f_i^N \cdot m_i$ and each f_i by f_i^{N+n}, our elements are now written $f_i^{-1} \cdot m_i$ and verify the equalities
$$f_j \cdot m_i = f_i \cdot m_j \quad \text{in } M \text{ for any } i, j\,.$$
Choosing elements $a_i \in A$ such that $a_1 f_1 + \cdots + a_k f_k = 1$, we define the element of M
$$m = a_1 \cdot m_1 + \cdots + a_k \cdot m_k\,.$$
For any i, we have in M
$$f_i \cdot m = \sum_j a_j\, f_i \cdot m_j = \sum_j a_j\, f_j \cdot m_i = m_i$$
which means that $f_i^{-1} \cdot m_i = m$ in each M_{f_i}.

Corollary:

(i) Any commutative ring A defines a ringed space $\mathrm{Spec}(A)$ (called the spectrum of A) such that

- the underlying set of $\mathrm{Spec}(A)$ is the set of ideals $p \subset A$ which are prime (meaning: $a_1 a_2 \in p \Rightarrow a_1 \in p$ or $a_2 \in p$)
- open subsets of $\mathrm{Spec}(A)$ are unions of subsets of the form
$$\mathrm{Spec}(A_f) = \{p \text{ prime } \mid f \notin p\} \quad \text{with } f \in A\,,$$
- the structure sheaf of $\mathrm{Spec}(A)$ is the unique sheaf of rings $\mathcal{O}_A$ such that, for any $f \in A$,
$$\mathcal{O}_A(\mathrm{Spec}(A_f)) = A_f\,.$$

(ii) Any morphism $u : A \to B$ of commutative rings defines a morphism of ringed spaces
$$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$$
such that

- the underlying map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is
$$(q \subset B) \longmapsto (p = u^{-1}(q) \subset A)\,,$$
- for any $f \in A$, the pull-back of the open subset $\mathrm{Spec}(A_f)$ is the open subset $\{q \mid f \notin u^{-1}(q)\} = \mathrm{Spec}(B_{u(f)})$,
- for any $f \in A$, the morphism
$$\mathcal{O}_A(\mathrm{Spec}(A_f)) \longrightarrow \mathcal{O}_B(\mathrm{Spec}(B_{u(f)}))$$
is the morphism $A_f \to B_{u(f)}$ induced by $u : A \to B$.

Remarks:

(i) If Aff denotes the opposite category of the category of commutative rings, this defines a faithful functor
$$\mathrm{Aff} \longrightarrow \mathrm{Sp}$$
to the category Sp of ringed spaces.

(ii) The category Aff, which is called the category of affine schemes, has a terminal object $\mathrm{Spec}(\mathbb{Z})$ and arbitrary fiber products
$$\mathrm{Spec}(B_1) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B_2) = \mathrm{Spec}(B_1 \otimes_A B_2)\,.$$

(iii) For any point p of some affine scheme $\mathrm{Spec}(A)$, the fiber
$$\mathcal{O}_{A,p} = \varinjlim_{f \notin p} A_f = A_p$$
has a unique maximal ideal $p \cdot A_p$ and the quotient $A_p / p \cdot A_p = \kappa_p$ (called the residue field at p) is the fraction field of the domain A/p.

So $\mathrm{Spec}(A)$ is a locally ringed space.

(iv) For any morphism $u : A \to B$ inducing $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and any point $q \in \mathrm{Spec}(B)$ sent to $u^{-1}(q) = p \in \mathrm{Spec}(A)$, the induced morphism between the fibers
$$A_p = \mathcal{O}_{A,p} \longrightarrow \mathcal{O}_{B,q} = B_q$$
sends $p \cdot A_p$ to $q \cdot B_q$.

So $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of locally ringed spaces.

(v) Conversely, one can prove that any morphism of locally ringed spaces
$$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$$
is induced by a ring morphism $A \to B$.

Examples of affine schemes:

(i) For any family of polynomials

$$P_i \in A[X_1, \ldots, X_n]$$

with coefficients in a commutative ring A, the functor

$$\begin{array}{rcl} [\mathrm{Aff}/\mathrm{Spec}(A)]^{\mathrm{op}} & \longrightarrow & \mathrm{Set}\,, \\ (A \to B) & \longmapsto & \{(b_1, \ldots, b_n) \in B^n \mid P_i(b_1, \ldots, b_n) = 0\,,\ \forall i\} \end{array}$$

is represented by the affine scheme

$$\mathrm{Spec}(A[X_1, \ldots, X_n]/I)$$

associated to the A-algebra $A[X_1, \ldots, X_n]/I$
defined by the ideal $I = \sum\limits_i P_i \cdot A[X_1, \ldots, X_n]$.

(ii) In particular, the functor

$$\begin{array}{rcl} \mathrm{Aff}^{\mathrm{op}} & \longrightarrow & \mathrm{Set}\,, \\ A & \longmapsto & A^n \end{array}$$

is represented by the affine scheme

$$\mathbb{A}^n = \mathrm{Spec}(\mathbb{Z}[X_1, \ldots, X_n])\,.$$

(iii) The functor

$$\begin{array}{rcl} \mathrm{Aff}^{\mathrm{op}} & \longrightarrow & \mathrm{Set}\,, \\ A & \longmapsto & A^{\times} = \mathrm{GL}_1(A) \end{array}$$

is represented by the affine scheme

$$\mathbb{G}_m = \mathrm{GL}_1 = \mathrm{Spec}(\mathbb{Z}[X, X^{-1}])\,.$$

(iv) More generally, for any $r \geq 1$, the functor

$$\begin{array}{rcl} \mathrm{Aff}^{\mathrm{op}} & \longrightarrow & \mathrm{Set}\,, \\ A & \longmapsto & \mathrm{GL}_r(A) \end{array}$$

is represented by the affine scheme

$$\mathrm{GL}_r = \mathrm{Spec}(\mathbb{Z}[(X_{i,j})_{1 \leq i,j \leq r}, Y]/(Y \cdot \det(X_{i,j}) - 1)\,.$$

Corollary:
Let $A =$ commutative ring,
$M = A$-module.
Then there is a unique $\mathcal{O}_A$-Module $\widetilde{M}$ on $\mathrm{Spec}(A)$ such that, for any $f \in A$,

$$\widetilde{M}(\mathrm{Spec}(A_f)) = M_f = A_f \otimes_A M\,.$$

Remark:

(i) The functor

$$\begin{array}{rcl} \mathrm{Mod}_A & \longrightarrow & \mathcal{M}od_{\mathcal{O}_A}\,, \\ M & \longmapsto & \widetilde{M} \end{array}$$

is fully faithful, it is left-adjoint to the functor

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}_A} & \longrightarrow & \mathrm{Mod}_A\,, \\ \mathcal{M} & \longmapsto & \mathcal{M}(\mathrm{Spec}(A))\,. \end{array}$$

(ii) An $\mathcal{O}_A$-Module $\mathcal{M}$ on $\mathrm{Spec}(A)$ is called "quasi-coherent" [resp. "coherent"] if it is isomorphic to $\widetilde{M}$ for some A-module M [resp. some finitely presentable A-module M].

The quasi-coherent sheaf of relative differentials

Proposition:

Let $X = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = Y$
be a morphism of affine schemes.

Then there is a quasi-coherent $\mathcal{O}_X$-Module on X,
called the sheaf $\Omega_{X/Y}$ of relative differentials,
such that for any $f \in B$,

$$\Omega_{X/Y}(\mathrm{Spec}(B_f)) = \Omega_{B_f/A}\,.$$

Remarks:

(i) Recall that for any $A \xrightarrow{u} B$, $\Omega_{B/A}$ represents the functor

$$\begin{array}{ccl} \mathrm{Mod}_B & \longrightarrow & \mathrm{Set}\,, \\ M & \longmapsto & \left\{ \mathrm{d} : B \to M \;\middle|\; \begin{array}{ll} \mathrm{d}(b_1 + b_2) = \mathrm{d}b_1 + \mathrm{d}b_2\,, & \forall\, b_1, b_2\,, \\ \mathrm{d}(b_1 \cdot b_2) = b_1 \cdot \mathrm{d}b_2 + b_2 \cdot \mathrm{d}b_1\,, & \forall\, b_1, b_2\,, \\ \mathrm{d}u(a) = 0\,, & \forall\, a \in A \end{array} \right\}. \end{array}$$

(ii) If B is finitely presentable over A, i.e. isomorphic to

$$A[X_1, \ldots, X_n] / \Big(\sum_{1 \le i \le k} P_i \cdot A[X_1, \ldots, X_n] \Big),$$

then $\Omega_{B/A}$ is the quotient of the free module

$$\bigoplus_j B \cdot \mathrm{d}X_j$$

by the submodule generated by the elements

$$\sum_j \frac{\partial P_i}{\partial X_j} \cdot \mathrm{d}X_j\,, \quad 1 \le i \le k\,.$$

So $\Omega_{B/A}$ is a finitely presentable B-module
and $\Omega_{\mathrm{Spec}(B)/\mathrm{Spec}(A)}$ is a coherent $\mathcal{O}_B$-Module.

Proof of the proposition:
We just have to check that for any element $f \in B$,
the B_f-module $\Omega_{B_f/A}$ identifies with $B_f \otimes_B \Omega_{B/A}$.
By definition, $\Omega_{B_f/A}$ represents the functor

$$\begin{array}{ccc} \mathrm{Mod}_{B_f} & \longrightarrow & \mathrm{Set}\,, \\ M & \longmapsto & \left\{ \begin{array}{l} \text{differentials } \mathrm{d} : B_f \to M \\ \text{such that } \mathrm{d}u(a) = 0, \forall\, a \in A \end{array} \right\}. \end{array}$$

For any differential $\mathrm{d} : B_f \to M$, the composite

$$B \longrightarrow B_f \longrightarrow M$$

is also a differential and uniquely factorises as a morphism

$$\Omega_{B/A} \longrightarrow M$$

of B-modules.
As the forgetful functor $\mathrm{Mod}_{B_f} \to \mathrm{Mod}_B$ is right adjoint to the functor $B_f \otimes_B \bullet$,
this morphism of B-modules corresponds to a morphism of B_f-modules

$$B_f \otimes_B \Omega_{B/A} \longrightarrow M\,.$$

Conversely, any such morphism $B_f \otimes_B \Omega_{B/A} \to M$ defines a differential

$$\mathrm{d} : B \longrightarrow M$$

which uniquely extends to

$$\mathrm{d} : B_f = B[X]/(f \cdot X - 1) \longrightarrow M$$

by the formula $f \cdot \mathrm{d}X + X \cdot \mathrm{d}f = 0$ or, equivalently, $\mathrm{d}X = -f^{-2} \cdot \mathrm{d}f$.

Definition:

(i) A scheme is a ringed space $(X, \mathcal{O}_X)$
which has a covering by open subspaces $(U_i, \mathcal{O}_{U_i})$
which are isomorphic to some affine schemes $\mathrm{Spec}(A_i)$.

(ii) A morphism of schemes

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a morphism of ringed spaces
such that, for any point $x \in X$ there are affine open neighborhoods

$$x \in U \cong \mathrm{Spec}(B) \qquad \text{and} \qquad f(x) \in V \cong \mathrm{Spec}(A)$$

with $U \subset f^{-1}(V)$ and a morphism of affine schemes

$$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$$

which corresponds to the restriction $(U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$ of f.

Remarks:

(i) The category Sch of schemes
is a geometric subcategory of the category Sp of ringed spaces.

(ii) It is a full subcategory of the category of locally ringed spaces.

(iii) For any scheme X, its topology on the underlying set
is called the Zariski topology.

Lemma:

(i) Any scheme X defines a contravariant functor

$$\begin{array}{rcl} \mathrm{Aff}^{\mathrm{op}} & \longrightarrow & \mathrm{Set}\,, \\ A & \longmapsto & \mathrm{Hom}(\mathrm{Spec}(A), X) = X(A)\,. \end{array}$$

(ii) This defines a fully faithful functor

$$\mathrm{Sch} \longrightarrow [\mathrm{Aff}^{\mathrm{op}}, \mathrm{Set}]\,.$$

(iii) A contravariant functor $F : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Set}$ is a scheme if and only if there exist morphisms

$$x_i : \mathrm{Hom}(\bullet, \mathrm{Spec}(A_i)) \longrightarrow F$$

from representable functors such that:

- each x_i is open in the sense that for any morphism

  $$\mathrm{Hom}(\bullet, \mathrm{Spec}(A)) \longrightarrow F$$

  from a representable functor, the fiber product

  $$\mathrm{Hom}(\bullet, \mathrm{Spec}(A_i)) \times_F \mathrm{Hom}(\bullet, \mathrm{Spec}(A))$$

  is representable by an open subspace $\mathrm{Spec}(A_i) \times_F \mathrm{Spec}(A)$ of the ringed space $\mathrm{Spec}(A)$,
- the family (x_i) is a covering in the sense that for any

  $$\mathrm{Hom}(\bullet, \mathrm{Spec}(A)) \longrightarrow F\,,$$

  the open subspaces $\mathrm{Spec}(A_i) \times_F \mathrm{Spec}(A)$ make up an open covering of $\mathrm{Spec}(A)$.

Remark: The set $X(A) = \mathrm{Hom}(\mathrm{Spec}(A), X)$ is called the set of points of the scheme X with coefficients in the commutative ring A.

Corollary:

(i) The category Sch has arbitrary finite limits and disjoint sums.

(ii) The embedding functor
$$\mathrm{Aff} \longrightarrow \mathrm{Sch}$$
preserves finite limits.

Proof:

(ii) The statement follows from the fact that for any scheme $(X, \mathcal{O}_X)$ and any affine scheme $\mathrm{Spec}(A)$, the map
$$\mathrm{Hom}(X, \mathrm{Spec}(A)) \longrightarrow \mathrm{Hom}(A, \mathcal{O}_X(X))$$
is a bijection.

(i) It follows from (ii) that the terminal object $\mathrm{Spec}(\mathbb{Z})$ of Aff is also a terminal object in Sch. So it is enough to show that for morphisms of schemes
$$f : X \longrightarrow S \qquad \text{and} \qquad g : Y \longrightarrow S\,,$$
the fiber product $X \times_S Y$ in $[\mathrm{Aff}^{\mathrm{op}}, \mathrm{Set}]$ is a scheme.

Let's consider an open covering of S by affine schemes S_i and, for any i, open coverings of $f^{-1}(S_i)$ and $g^{-1}(S_i)$ by affine schemes $X_{i,j}$ and $Y_{i,k}$.

Then the fiber products $X_{i,j} \times_{S_i} Y_{i,k}$ in Aff make up an open covering of the presheaf $X \times_S Y$.

Examples of schemes

(i) For any n, the union of the open affine subschemes

$$\mathrm{Spec}(\mathbb{Z}[X_1, \ldots, X_n, X_i^{-1}]) \, , \quad 1 \leq i \leq n \, ,$$

of $\mathbb{A}^n = \mathrm{Spec}(\mathbb{Z}[X_1, \ldots, X_n])$ is an open subscheme

$$\mathbb{A}^n - \{0\} \hookrightarrow \mathbb{A}^n \, .$$

It is endowed with a free action of $\mathbb{G}_m$

$$\begin{array}{rcl} \mathbb{G}_m \times (\mathbb{A}^n - \{0\}) & \longrightarrow & \mathbb{A}^n - \{0\} \, , \\ (a, (a_1, \ldots, a_n)) & \longmapsto & (a \cdot a_1, \ldots, a \cdot a_n) \, . \end{array}$$

(ii) The contravariant functor

$$\begin{array}{rcl} \mathrm{Aff}^{\mathrm{op}} & \longrightarrow & \mathrm{Set}\,, \\ A & \longmapsto & \mathbb{G}_m(A)\backslash(\mathbb{A}^{n+1} - \{0\})(A) \end{array}$$

is separated for the Zariski topology (in the sense that sections coincide if they coincide locally).
Its sheafification is representable by a scheme $\mathbb{P}^n$
called the projective space of dimension n.

The commutative square

$$\begin{array}{ccc} \mathbb{G}_m \times (\mathbb{A}^{n+1} - \{0\}) & \longrightarrow & \mathbb{A}^{n+1} - \{0\} \\ \downarrow & & \downarrow \\ \mathbb{A}^{n+1} - \{0\} & \longrightarrow & \mathbb{P}^n \end{array}$$

is both cartesian and cocartesian.

If $\mathbb{A}^{n+1} = \mathrm{Spec}(\mathbb{Z}[X_0, \ldots, X_n])$, the affine schemes
$\mathrm{Spec}\left(\mathbb{Z}\left[\frac{X_0}{X_i}, \ldots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \ldots, \frac{X_n}{X_i}\right]\right)$ make up an open covering of $\mathbb{P}^n$.

Definition:

Let $(X, \mathcal{O}_X) =$ scheme.
An $\mathcal{O}_X$-Module $\mathcal{M}$ is called quasi-coherent [resp. coherent] if, for any affine open subscheme $U = \mathrm{Spec}(A)$ of X, the restriction of $\mathcal{M}$ to U is quasi-coherent [resp. coherent].

Remarks:

(i) An $\mathcal{O}_X$-Module $\mathcal{M}$ is quasi-coherent [resp. coherent] if and only if there exists an open covering of X by schemes U_i such that the restriction of $\mathcal{M}$ to each U_i is quasi-coherent [resp. coherent].

(ii) For any morphism of schemes $f : X \to Y$, $f^* : \mathcal{M}od_{\mathcal{O}_Y} \to \mathcal{M}od_{\mathcal{O}_X}$ transforms quasi-coherent [resp. coherent] $\mathcal{O}_Y$-Modules into quasi-coherent [resp. coherent] $\mathcal{O}_X$-Modules and $f_* : \mathcal{M}od_{\mathcal{O}_X} \to \mathcal{M}od_{\mathcal{O}_Y}$ transforms quasi-coherent $\mathcal{O}_X$-Modules into quasi-coherent $\mathcal{O}_Y$-Modules.

(iii) For any morphism of schemes $f : X \to Y$, the derived functors

$$\mathrm{L}^k f^* : \mathcal{M}od_{\mathcal{O}_Y} \longrightarrow \mathcal{M}od_{\mathcal{O}_X}$$

transform quasi-coherent $\mathcal{O}_Y$-Modules into quasi-coherent $\mathcal{O}_X$-Modules. Indeed, any quasi-coherent Module on an affine scheme has a resolution by flat quasi-coherent Modules.

(iv) One can prove that for any morphism of affine schemes $X = \mathrm{Spec}(B) \xrightarrow{f} \mathrm{Spec}(A) = Y$ we have

$$\mathrm{R}^k f_* \mathcal{M} = 0$$

for any quasi-coherent $\mathcal{O}_X$-Module $\mathcal{M}$ and any $k \geq 1$.
One can deduce from this property that for any morphism of schemes $f : X \to Y$, the derived functors

$$\mathrm{R}^k f_* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_Y}$$

transform quasi-coherent $\mathcal{O}_X$-Modules into quasi-coherent $\mathcal{O}_Y$-Modules.

(v) One can prove that for any base scheme S and any $n \geq 0$ defining the projective projection

$$p : \mathbb{P}^n \times S \longrightarrow S\,,$$

the derived functors

$$\mathrm{R}^k p_* : \mathcal{M}od_{\mathcal{O}_{\mathbb{P}^n \times S}} \longrightarrow \mathcal{M}od_{\mathcal{O}_S}\,, \quad k \geq 0\,,$$

transform coherent $\mathcal{O}_{\mathbb{P}^n \times S}$-Modules into coherent $\mathcal{O}_S$-Modules.
Moreover we have

$$\mathrm{R}^k p_* \mathcal{M} = 0$$

for any quasi-coherent $\mathcal{O}_{\mathbb{P}^n \times S}$-Module $\mathcal{M}$ and any $k > n$.

Example of quasi-coherent Module: the sheaf of differentials

Definition:
Let $f : X \to Y$ be a morphism of schemes.
We denote

$$\Omega_{X/Y}$$

the unique quasi-coherent $\mathcal{O}_X$-Module such that, for any open subschemes $U = \mathrm{Spec}(B)$ of X and $V = \mathrm{Spec}(A)$ of Y with $U \subset f^{-1}(V)$, we have

$$\Omega_{X/Y}(U) = \Omega_{B/A}\,.$$

Remark:
The sheaves of higher differentials

$$\Omega^k_{X/Y} = \Lambda^k \Omega_{X/Y}$$

are also quasi-coherent $\mathcal{O}_X$-Modules.
The De Rham complex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega^1_{X/Y} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^k_{X/Y} \xrightarrow{\mathrm{d}} \cdots$$

is a complex of $f^{-1}\mathcal{O}_Y$-Modules.

Lemma:

(i) Any morphisms of schemes

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

yield an exact sequence of quasi-coherent $\mathcal{O}_X$-Modules

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0\,.$$

(ii) For any cartesian square of schemes

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y \end{array}$$

$\Omega_{X'/Y'}$ identifies with $x^*\Omega_{X/Y}$.

Proof:

(i) Any morphisms of commutative rings

$$A \longrightarrow B \longrightarrow C$$

yield an exact sequence of C-modules

$$C \otimes_B \Omega_{B/A} \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0\,.$$

Indeed, for any C-module M, a B-linear differential $\mathrm{d} : C \to M$ is an A-linear differential $C \to M$ whose composite with $B \to C$ is 0.

(ii) For any ring morphisms $A \to B$ and $A \to A'$, $\Omega_{A' \otimes_A B/A'}$ identifies with $(A' \otimes_A B) \otimes_B \Omega_{B/A} = A' \otimes_A \Omega_{B/A}$.
Indeed, for any module M over $A' \otimes_A B$,
an $A' \otimes_A B$-linear morphism $\Omega_{A' \otimes_A B/A'} \to M$
corresponds to an A'-linear differential $\mathrm{d} : A' \otimes_A B \to M$
or, equivalently, to an A-linear differential $\mathrm{d} : B \to M$.
This corresponds to a B-linear morphism $\Omega_{B/A} \to M$
or, equivalently, to an $A' \otimes_A B$-linear morphism $A' \otimes_A \Omega_{B/A} \to M$.

Properties of morphisms of schemes

Definition:

A morphism of schemes $X \xrightarrow{f} Y$ is called

(1) quasi-compact
if, for any open subset $V \subset Y$ which is quasi-compact
(in the sense that any open covering has a finite subcovering),
$f^{-1}(V) \subset X$ is quasi-compact,

(2) locally of finite type [resp. locally of finite presentation]
if Y has a covering by affine open subschemes $V_i = \mathrm{Spec}(A_i)$
and each $f^{-1}(V_i)$ has a covering by affine open subschemes $U = \mathrm{Spec}(B)$
such that B is an A_i-algebra of finite type [resp. of finite presentation],

(3) of finite type [resp. of finite presentation] if it is quasi-compact
and locally of finite type [resp. locally of finite presentation].

Remarks:

(i) These properties are universal (i.e. stable by base change), stable by composition and local on the base.

(ii) The properties (2) are even local on the source.

(iii) An affine scheme $\mathrm{Spec}(A)$ is always quasi-compact.

(iv) An affine scheme $\mathrm{Spec}(A)$ is called nœtherian
if any finitely generated A-module is finitely presentable
(or, equivalently, if any ideal of A is finitely generated).
A scheme is called locally nœtherian if it has a covering by nœtherian affine open subschemes.
If $X \xrightarrow{f} Y$ is locally of finite type and Y is locally nœtherian,
X is also locally nœtherian and f is locally of finite presentation.

Definition: A morphism of schemes $X \xrightarrow{f} Y$ is called

(4) affine if for any morphism $\mathrm{Spec}(A) \to Y$ from an affine scheme, the fiber product $\mathrm{Spec}(A) \times_Y X$ is affine,

(5) finite [resp. a closed immersion] if it is affine and for any morphism $\mathrm{Spec}(A) \to Y$ with $\mathrm{Spec}(A) \times_Y X = \mathrm{Spec}(B)$, B is finitely generated as an A-module [resp. $A \to B$ is surjective],

(6) a locally closed immersion if it is the composite of a closed immersion and an open embedding.

Remarks:

(i) These properties are universal, stable by composition and local on the base.

(ii) If $j : Z \hookrightarrow X$ is a closed immersion, the induced morphism of $\mathcal{O}_X$-Modules $\mathcal{O}_X \to j_*\mathcal{O}_Z$ is an epimorphism and its kernel is a sheaf of ideals of $\mathcal{O}_X$, called the defining Ideal of Z.

Conversely, any sheaf of ideals $\mathcal{I} \hookrightarrow \mathcal{O}_X$ defines a closed subscheme $Z \hookrightarrow X$.

(iii) If $Z \hookrightarrow X$ is a locally closed immersion factorised as the composition

$$Z \overset{j}{\hookrightarrow} U \overset{i}{\hookrightarrow} X$$

of a closed immersion j and an open embedding i, and $\mathcal{I}$ is the defining Ideal of Z in U, the $\mathcal{O}_Z$-Module

$$j^*\mathcal{I} = \mathcal{N}_{Z/X}$$

is called the normal sheaf of Z in X.

Lemma:

Let $X \xrightarrow{f} Y$
$= $ morphism of schemes.

Then:

(i) The diagonal morphism $X \to X \times_Y X$ is a locally closed immersion.

(ii) Its normal sheaf identifies with the $\mathcal{O}_X$-Module $\Omega_{X/Y}$.

Proof:

(i) It is enough to consider the case when $Y = \mathrm{Spec}(A)$.
Consider a covering of X by affine open subschemes $\mathrm{Spec}(B_i)$.
The morphism $X \to X \times_Y X$ factorises through the union of the open subschemes $\mathrm{Spec}(B_i) \times_Y \mathrm{Spec}(B_i) = \mathrm{Spec}(B_i \otimes_A B_i)$ and, by the base changes $\mathrm{Spec}(B_i \otimes_A B_i) \to X \times_Y X$ it becomes

$$\mathrm{Spec}(B_i) \longrightarrow \mathrm{Spec}(B_i \otimes_A B_i)$$

which are closed immersions as the canonical morphisms

$$B_i \otimes_A B_i \longrightarrow B_i$$

are surjective.

(ii) If $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and I is the kernel of the canonical epimorphism $B \otimes_A B \to B$, $\Omega_{B/A}$ identifies with I/I^2 endowed with the differential

$$\begin{array}{rccl} \mathrm{d} \; : & B & \longrightarrow & I/I^2\,, \\ & b & \longmapsto & b \otimes 1 - 1 \otimes b\,. \end{array}$$

Definition:

A morphism of schemes $X \xrightarrow{f} Y$ is called

(7) separated
if the diagonal embedding $X \hookrightarrow X \times_Y X$ is a closed immersion,

(8) proper if

- it is separated,
- it is of finite type,
- it is universally closed (i.e. for any $Y' \to Y$, the morphism $X \times_Y Y' \to Y'$ transforms closed subsets of $X \times_Y Y'$ in closed subsets of Y').

Remarks:

(i) These properties are universal, stable by composition
and local on the base.

(ii) One can prove that if $f : X \to Y$ is proper, the derived functors

$$\mathrm{R}^k f_* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_Y}$$

transform coherent $\mathcal{O}_X$-Modules in coherent $\mathcal{O}_Y$-Modules.

Examples of separated and proper morphisms:

- **(i)** Any locally closed immersion is separated.
- **(ii)** Any affine morphism is separated.
- **(iii)** Any finite morphism (in particular, any closed immersion) is proper.
- **(iv)** For any n, the projection
$$\mathbb{P}^n \longrightarrow \mathrm{Spec}(\mathbb{Z})$$
is a proper morphism.
- **(v)** A scheme X over some base scheme S is called projective [resp. quasi-projective] over S if the morphism $X \to S$ factorises as the composite of a closed [resp. locally closed] immersion
$$X \hookrightarrow \mathbb{P}^n \times S$$
followed by the projection
$$\mathbb{P}^n \times S \longrightarrow S\,.$$
This implies that
$$X \longrightarrow S$$
is proper [resp. separated].

Definition: A morphism of schemes $X \xrightarrow{f} Y$ is called

(9) flat [resp. faithfully flat]
if $\mathcal{O}_X$ is flat as a Module over $f^{-1}\mathcal{O}_Y$
[resp. and the underlying map $X \to Y$ is surjective],

(10) smooth of dimension d [resp. étale] if

- it is locally of finite presentation,
- it is flat,
- the sheaf of relative differentials $\Omega_{X/Y}$ is locally free of rank d as on $\mathcal{O}_X$-Module [resp. is 0].

Remarks:

(i) These properties are universal and local on the base and on the source (except for faithful flatness which is only local on the base).

(ii) The properties (9) are stable by composition.

(iii) If $X \xrightarrow{f} Y$ is smooth of dimension d [resp. étale]
and $Y \xrightarrow{g} Z$ is smooth of dimension d' [resp. étale],
then $g \circ f$ is smooth of dimension $d + d'$ [resp. étale].

(iv) One can prove that if $f : X \to Y$ is smooth of dimension d, x is a point of X, $f_1, \ldots, f_n$ are sections of $\mathcal{O}_X$ in an open neighborhood U of x such that $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ is a basis of $\Omega_{X/Y}$ on U, then the morphism they define $U \to \mathbb{A}^n \times Y$ is étale.

Examples of flat, étale and smooth morphisms:

(i) Any open immersion is étale.

(ii) The schemes $\mathbb{A}^n$ and $\mathbb{P}^n$ are smooth of dimension n over $\mathrm{Spec}(\mathbb{Z})$. The group scheme GL_n is smooth of dimension n^2.

(iii) For any commutative ring A and any polynomial P of the form

$$P = X^d + a_{d-1} \cdot X^{d-1} + \cdots + a_1 \cdot X + a_0 \quad \text{in} \quad A[X]\,,$$

with $B = A[X]/(P)$, the morphism

$$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$$

is finite and flat.

It is étale if and only if P and P' generate the full ideal $A[X]$.

(iv) More generally, if

$$B = A[X_1, \ldots, X_n]/I$$

for some ideal I of $A[X_1, \ldots, X_n]$ generated by polynomials

$$P_j(X_1, \ldots, X_n)\,, \quad 1 \leq j \leq k\,,$$

then $\mathrm{Spec}(B)$ is smooth of dimension $n - k$ over $\mathrm{Spec}(A)$ if and only if the ideal of B generated by the k-minors of the matrix

$$\left(\frac{\partial P_j}{\partial X_i}(X_1, \ldots, X_n) \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}}$$

is the whole B.

Proposition:

(i) If M is an A-module, the quasi-coherent $\mathcal{O}_A$-Module $\widetilde{M}$ on $\mathrm{Spec}(A)$ is flat if and only if M is flat.

(ii) A finitely generated A-module M is flat if it is locally free on $\mathrm{Spec}(A)$. The converse is true if A is nœtherian.

(iii) In particular, a finite morphism $X \xrightarrow{f} Y$ is flat if $f_*\mathcal{O}_X$ is locally free as an $\mathcal{O}_Y$-Module, and the converse is true if Y is locally nœtherian.

(iv) If a scheme morphism $f : X \to Y$ is locally of finite type, $\Omega_{X/Y} = 0$ if and only if $X \to X \times_Y X$ is an open immersion.

(v) A finite morphism $X \xrightarrow{f} Y$ such that $f^*\mathcal{O}_X$ is locally free of rank d over $\mathcal{O}_Y$ is étale if and only if there exists a finite étale surjective [resp. quasi compact faithfully flat] morphism $Y' \to Y$ such that $X \times_Y Y' \longrightarrow Y'$ is isomorphic to the trivial cover

$$\coprod_{1 \le i \le d} Y' \longrightarrow Y' .$$

Proof of (i):

If M is a flat A-module, $M_f = A_f \otimes_A M$ is flat over A_f for any $f \in A$, so $\widetilde{M}$ is a flat $\mathcal{O}_A$-Module.

The functors

$$\begin{array}{rcl} M & \longmapsto & \widetilde{M} \\ \text{and} \qquad \mathcal{M} & \longmapsto & \mathcal{M}(\mathrm{Spec}(A)) \end{array}$$

define an equivalence between the abelian category of A-modules and the abelian category of quasi-coherent $\mathcal{O}_A$-Modules.

In particular, they are exact.

Furthermore, they commute with tensor products.

So, M is a flat A-module if $\widetilde{M}$ is a flat $\mathcal{O}_A$-Module.

Proof of (iv):

We can suppose that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$.

Let's denote I the kernel of $B \otimes_A B \to B$ so that $\Omega_{B/A}$ identifies with I/I^2.

As B is of finite type over A, I is finitely generated.

If $X \to X \times_Y X$ is an open immersion, I is 0 in an open neighborhood of $X = \mathrm{Spec}(B)$ and a fortiori $\Omega_{B/A} = 0$.

Conversely, $I = I^2$ implies that $I = 0$ in an open neighborhood of $\mathrm{Spec}(B)$ as follows from the lemma:

Lemma:

Let $I =$ ideal of a commutative ring A,
$M =$ finitely generated A-module such that $I \cdot M = M$.
Then there exists an element $a \in I$ such that

$$(1 + a) \cdot m = 0 \,, \quad \forall \, m \in M \,.$$

In particular, M is 0 in the open neighborhood $\mathrm{Spec}(A_{(1+a)})$ of $\mathrm{Spec}(A/I)$ in $\mathrm{Spec}(A)$.

Proof of the lemma:

Consider a finite family of generators $m_1, \ldots, m_k$ of M.
Any m_i, $1 \leq i \leq k$, can be written

$$m_i = \sum_{1 \leq j \leq k} a_{i,j} \cdot m_j$$

for some coefficients $a_{i,j} \in I$.
The determinant of the matrix

$$\mathrm{Id} - (a_{i,j})_{1 \leq i,j \leq k}$$

has the form

$$1 + a \qquad \text{for some } a \in I \,,$$

and we have

$$(1 + a) \cdot m_i = 0 \,, \quad 1 \leq i \leq k \,.$$

Proof of (ii) and (iii):
(iii) is a particular case of (ii).
(ii) According to (i), the property of flatness is local
so M is flat if it is locally free.
Conversely, suppose M is finitely generated and flat and A is nœtherian.
Consider a point $x \in \mathrm{Spec}(A)$ corresponding to a prime ideal p
and the residue field $\kappa_p = A_p / p \cdot A_p$.
Choose a finite basis over κ_p of the vector space $\kappa_p \otimes_A M$ and lift is to a family of sections $m_1, \ldots, m_d$ of $\widetilde{M}$ in an open neighborhood of x.
They induce a morphism $\mathcal{O}_X^d \to \widetilde{M}$ whose cokernel has the form $\widetilde{N}$ for some finitely generated N such that $\kappa_p \otimes N = 0$. According to the previous lemma $\widetilde{N} = 0$ in an open neighborhood of x and $\mathcal{O}_X^d \to \widetilde{M}$ is an epimorphism there.
Its kernel has the form $\widetilde{K}$ for some module K which is finitely generated as A is nœtherian.
As M is flat, the exact sequence $0 \to \widetilde{K} \to \mathcal{O}_X^d \to \widetilde{M} \to 0$ yields an exact sequence

$$0 \longrightarrow \kappa_p \otimes \widetilde{K} \longrightarrow \kappa_p^d \longrightarrow \kappa_p \otimes \widetilde{M} \longrightarrow 0\,.$$

It means that $\kappa_p \otimes \widetilde{K} = 0$ as $\kappa_p^d \to \kappa_p \otimes \widetilde{M}$ is an isomorphism.
According to the previous lemma, $\widetilde{K} = 0$ is an open neighborhood of x
and $\mathcal{O}_X^d \to \widetilde{M}$ is an isomorphism there.

Proof of (v):

Suppose $X \xrightarrow{f} Y$ is étale.
The diagonal morphism $X \to X \times_Y X$ is an open embedding and also a closed immersion as the finite morphism $X \to Y$ is separated.

The scheme $X \times_Y X$ over X can be written as the disjoint union of $X \xrightarrow{\text{id}} X$ and a finite étale morphism

$$f_1 : X_1 \longrightarrow Y_1 = X$$

such that $(f_1)_* \mathcal{O}_{X_1}$ is locally free of rank $d - 1$.
We get by induction on the rank d that there exist a finite étale morphism

$$Y_d \longrightarrow Y$$

such that the morphism

$$X \times_Y Y_d \longrightarrow Y_d$$

is isomorphic to

$$\coprod_{1 \leq i \leq d} Y_d \longrightarrow Y_d \, .$$

Conversely, suppose that there exists a quasi-compact and faithfully flat morphism $Y' \to Y$ such that $X \times_Y Y' \to Y'$ is isomorphic to $\coprod_{1 \leq i \leq d} Y' \to Y'$.

We can suppose that $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(B)$.
The conclusion follows from the lemma:

Lemma:

Let $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$
$= $ faithfully flat morphism of affine schemes.

Then:

(i) For any A-module, the canonical morphism

$$M \longrightarrow B \otimes_A M$$

is a monomorphism. In particular, M is 0 if and only if $B \otimes_A M$ is 0.

(ii) A complex of A-modules

$$M_1 \longrightarrow M_2 \longrightarrow M_3$$

is exact if and only if the complex of B-modules

$$B \otimes_A M_1 \longrightarrow B \otimes_A M_2 \longrightarrow B \otimes_A M_3$$

is exact.

Proof:

(i) A non zero element m of M can be seen as a non zero morphism

$$A \longrightarrow M\,.$$

Its kernel I is an ideal contained in a prime ideal p and, by hypothesis, there exists a prime ideal q of B such that $p = u^{-1}(q)$ for $u : A \to B$. Then B/q is a quotient of $B \otimes_A A/I$. As u is flat, $B \otimes_A A/I \to B \otimes_A M$ is a monomorphism. So the image of m in $B \otimes_A M$ is non zero.

(ii) Let $H = \mathrm{Ker}(M_2 \to M_3)/\mathrm{Im}(M_1 \to M_2)$. As $A \to B$ is flat, we also have

$$B \otimes_A H = \mathrm{Ker}(B \otimes_A M_2 \longrightarrow B \otimes_A M_3)/\mathrm{Im}(B \otimes_A M_1 \longrightarrow B \otimes_A M_2)\,.$$

According to (i), H is 0 if and only if $B \otimes_A H$ is 0.

Definition:

(i) A sieve on an object X of Sch is called a covering sieve for the “étale” topology [resp. for the faithfully flat quasi-compact (fpqc) topology] if it contains a family of morphisms

$$X_i \longrightarrow X\,, \qquad i \in I\,,$$

such that the morphism

$$\coprod_{i\in I} X_i \longrightarrow X$$

is quasi-compact, étale [resp. flat] and surjective.

(ii) The “big” étale [resp. fppf] site of a scheme X consists in the essentially small category

$$\mathrm{Sch}_{\mathrm{fp}/X}$$

of morphisms $X' \to X$ of finite presentation, endowed with the étale [resp. fpqc] topology. The “big” étale [resp. fpqc] topos of X is the associated topos.
It can be denoted $\mathrm{\acute{E}t}_X$ [resp. Fl_X].

(iii) The “small” étale [resp. fppf] site of a scheme X consists in the subcategory of $\mathrm{Sch}_{\mathrm{fp}/X}$ on étale [resp. flat] morphisms $X' \to X$ endowed with the étale [resp. fpqc] topology.
The “small” étale [resp. flat] topos of X is the associated topos.
It can be denoted $\mathrm{\acute{e}t}_X$ [resp. fl_X].

Remarks:

(i) For any scheme X, there is a commutative square of morphisms of toposes

$$\begin{array}{ccc} \mathrm{Fl}_X & \longrightarrow & \mathrm{fl}_X \\ \big\downarrow & & \big\downarrow \\ \mathrm{\acute{E}t}_X & \longrightarrow & \mathrm{\acute{e}t}_X \end{array}$$

whose push-forward components are restriction functors.
Furthermore, $\mathrm{Fl}_X \hookrightarrow \mathrm{\acute{E}t}_X$ is a subtopos.

(ii) For any morphism of schemes $X \xrightarrow{f} Y$, the functor

$$(Y' \longrightarrow Y) \longmapsto (Y' \times_Y X \longrightarrow X)$$

respects finite limits and disjoint sums.
It preserves the property of morphisms to be étale, flat, quasi-compact, surjective or of finite presentation.
So it induces morphisms of toposes

$$\begin{array}{cccccc} (f^*, f_*) & : & \mathrm{Fl}_X & \longrightarrow & \mathrm{Fl}_Y \,, \\ & & \mathrm{fl}_X & \longrightarrow & \mathrm{fl}_Y \,, \\ & & \mathrm{\acute{E}t}_X & \longrightarrow & \mathrm{\acute{E}t}_Y \,, \\ & & \mathrm{\acute{e}t}_X & \longrightarrow & \mathrm{\acute{e}t}_Y \,. \end{array}$$

Proposition:

(i) For any scheme X, the associated presheaf

$$\begin{array}{ccc} \mathrm{Sch}^{\mathrm{op}} & \longrightarrow & \mathrm{Set} \\ Y & \longmapsto & \mathrm{Hom}(Y, X) \end{array}$$

is a sheaf for the fpqc topology.

(ii) All properties (1)–(10) of morphisms of schemes

$$X \longrightarrow Y$$

are local on the base for the fpqc topology.

(iii) For any quasi-compact faithfully flat morphism

$$X' \longrightarrow X,$$

the category of quasi-coherent $\mathcal{O}_X$-Modules on X is equivalent to the category of quasi-coherent $\mathcal{O}_{X'}$-Modules $\mathcal{M}'$ on X' endowed with an isomorphism

$$\sigma : p_1^* \mathcal{M}' \xrightarrow{\sim} p_2^* \mathcal{M}' \quad \text{for the two projections} \quad X' \times_X X' \overset{p_1}{\underset{p_2}{\rightrightarrows}} X',$$

such that the triangle associated with the three projections

$$q_1, q_2, q_3 : X' \times_X X' \times_X X' \rightrightarrows X'$$

$$\begin{array}{ccc} q_1^* \mathcal{M}' & \xrightarrow{\sim} & q_3^* \mathcal{M}' \\ & \searrow^{\sim} \quad \nearrow^{\sim} & \\ & q_2^* \mathcal{M}' & \end{array}$$

is commutative.

The proof follows from the previous lemma completed with:

Lemma:
Let $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$
$=$ faithfully flat morphism of affine schemes.
Then:

(i) Any A-module M identifies with

$$\mathrm{Eq}\,(B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M)\,.$$

(ii) Any B-module M' endowed with an isomorphism

$$\sigma : (B \otimes_A B) \otimes_{p_1,B} M' \xrightarrow{\sim} (B \otimes_A B) \otimes_{p_2,B} M'$$

such that the triangle

$$\begin{array}{ccc} (B \otimes_A B \otimes_A B) \otimes_{q_1,B} M' & \longrightarrow & (B \otimes_A B \otimes_A B) \otimes_{q_3,B} M' \\ & \searrow \quad (B \otimes_A B \otimes_A B) \otimes_{q_2,B} M' \quad \nearrow & \end{array}$$

is commutative, identifies with

$$B \otimes_A M$$

for $M = \mathrm{Eq}\,(M' \rightrightarrows B \otimes_A M')$.

Proof:

(i) The morphism $M \to \mathrm{Eq}\,(B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M)$ is an isomorphism because the functor $B \otimes_A \bullet$ transforms it into an isomorphism. Indeed, the sequence

$$0 \longrightarrow B \longrightarrow B \otimes_A B \longrightarrow B \otimes_A B \otimes_A B$$

consisting in the morphisms
$b \mapsto b \otimes 1$ and $b \otimes b' \mapsto b \otimes b' \otimes 1 - b \otimes 1 \otimes b'$
is split exact, with the splitting

$$\begin{array}{ccc} B \otimes_A B & \longrightarrow & B \\ b \otimes b' & \longmapsto & bb' \,. \end{array}$$

(ii) The morphism $B \otimes_A \mathrm{Eq}\,(M' \rightrightarrows B \otimes_A M') \to M'$ is an isomorphism because, according to (i), $B \otimes_A \bullet$ transforms it into an isomorphism.

Points of small étale sites

Proposition: If $Y = \mathrm{Spec}(\overline{k})$ for an algebraically closed field $\overline{k}$, any quasi-compact étale morphism

$$X \longrightarrow Y = \mathrm{Spec}(\overline{k})$$

is isomorphic to some

$$\coprod_{1 \leq i \leq d} \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k}) .$$

Proof: Let's consider a $\overline{k}$-algebra of finite presentation

$$A = \overline{k}[X_1, \ldots, X_n]/(P_1, \ldots, P_k)$$

which is étale over $\overline{k}$.
For any maximal ideal m of A, the morphism

$$\overline{k} \longrightarrow A/m$$

is an isomorphism as $\overline{k}$ is algebraically closed and A is finitely generated.
Furthermore, the closed embedding

$$\mathrm{Spec}(\overline{k}) = \mathrm{Spec}(A/m) \hookrightarrow \mathrm{Spec}(A)$$

is also an open embedding as A is étale.
For distinct maximal ideals $m_1, \ldots, n_d$, A decomposes as a product

$$A \cong (A/m_1) \times \cdots \times (A/m_d) \times A' .$$

As A is generated by n elements, it yields $d \leq n$. If d is maximal, we get

$$A \cong (A/m_1) \times \cdots \times (A/m_d) \cong \overline{k}^d .$$

Corollary:

(i) If $X = \mathrm{Spec}(\overline{k})$ for an algebraically closed field, the topos $\acute{\mathrm{e}}\mathrm{t}_X$ identifies with Set.

(ii) For any scheme X, any "geometric point" of X

$$\overline{x} : \mathrm{Spec}(\overline{k}) \longrightarrow X$$

(where $\overline{k}$ is an algebraically closed field) defines a point

$$(\overline{x}^*, \overline{x}_*) : \mathrm{Set} \longrightarrow \acute{\mathrm{e}}\mathrm{t}_X$$

of the small étale topos of X.

Proof:

(i) follows from the previous proposition.

(ii) follows from (i).

The étale fundamental group

Theorem:
Let $X =$ connected scheme endowed with a geometric point

$$\overline{x} : \mathrm{Spec}(\overline{k}) \longrightarrow X\,,$$

$\mathrm{Cov}_X =$ category of finite étale morphisms $X' \to X$
such that $p_* \mathcal{O}_{X'}$ is locally free over $\mathcal{O}_X$,
$\pi_1(X, \overline{x}) =$ group of automorphisms of the functor

$$\begin{array}{ccl} \mathrm{Cov}_X & \longrightarrow & \mathrm{Set}_f = \text{category of finite sets,} \\ (X' \to X) & \longmapsto & \mathrm{Hom}_{\overline{x}}(\mathrm{Spec}(\overline{k}), X') = F_{\overline{x}}(X') \end{array}$$

endowed with the smallest topology
for which its action on each finite set $F_{\overline{x}}(X')$
is continuous.

Then the functor

$$(X' \to X) \longmapsto F_{\overline{x}}(X')$$

is an equivalence from the category Cov_X
to the category of finite sets endowed with
a continuous action of the profinite group $\pi_1(X, \overline{x})$.

Remark: If $X = \mathrm{Spec}(k)$ for some field k, this equivalence is Galois theory.

Sketch of proof:
For any object $p : X' \to X$ of Cov_X, the locally free $\mathcal{O}_X$-Module $p_*\mathcal{O}_{X'}$ has a constant rank d as X is connected, and there is a finite étale surjective morphism

$$Y \longrightarrow X$$

such that $X' \times_X Y \to Y$ is isomorphic to

$$\coprod_{1 \leq i \leq d} Y \longrightarrow Y\,.$$

In the other direction, for any finite étale surjective morphism

$$Y \longrightarrow X\,,$$

let $\mathrm{Cov}_X^Y =$ full subcategory of Cov_X
on objects $X' \to X$ such that
$X' \times_X Y \to Y$ is isomorphic to some $\coprod_{1 \leq i \leq d} Y \to Y$.

So, Cov_X is the filtering union of its full subcategories Cov_X^Y and we have

$$\pi_1(\overline{X}, x) = \varprojlim_{Y} \pi_1^Y(X, \overline{x})$$

where, for any Y, $\pi_1^Y(X, \overline{x})$ is the automorphism group of the restricted functor

$$F_{\overline{x}} : \mathrm{Cov}_X^Y \longrightarrow \mathrm{Set}_f\,.$$

For any such Y, there exists a finite étale surjective morphism $Y' \to Y$ such that

$$Y' \times_X Y \cong \coprod_{1 \le i \le d} Y' .$$

Furthermore, Y' can be constructed as $Y' = Y_d$ where $Y_d \to Y_{d-1} \to \cdots \to Y_1 \to Y_0$ is the sequence of finite étale morphisms defined by $Y_0 = Y$ and, for any $i < d$,

$$Y_{i+1} = Y_i \times_X Y - Y_i$$

as $Y_i \hookrightarrow X_i \times_X Y$ is a closed and open subscheme.
So $Y' = Y_d$ is a closed and open subscheme of $Y \times_X \cdots \times_X Y$ (d times) and $Y' \times_X Y'$ is a closed and open subscheme of $Y' \times_X Y \times_X \cdots \times_X Y$ which is a disjoint sum of copies of Y'.
So we are reduced to the study of functors

$$F_{\overline{x}} : \mathrm{Cov}_X^Y \longrightarrow \mathrm{Set}_f$$

when

$$Y \times_X Y \cong \coprod_{1 \le i \le d} Y .$$

We can even suppose that Y is connected.
Then $Y \times_X Y$ is the sum of the graphs of the automorphisms

$$\sigma \in G = \mathrm{Aut}_X(Y) .$$

The category

$$\mathrm{Cov}_X^Y$$

is equivalent to the category of finite sets I
endowed with an isomorphism over $Y \times_X Y$

$$\tau : Y \times_X \left(\coprod_{i \in I} Y\right) \xrightarrow{\sim} \left(\coprod_{i \in I} Y\right) \times_X Y$$

such that the triangle

$$\begin{array}{ccc} Y \times_X Y \times_X \left(\coprod_{i \in I} Y\right) & \xrightarrow{\quad\sim\quad} & \left(\coprod_{i \in I} Y\right) \times_X Y \times_X Y \\ & {\scriptstyle\sim}\searrow \quad \nearrow{\scriptstyle\sim} & \\ & Y \times_X \left(\coprod_{i \in I} Y\right) \times_X Y & \end{array}$$

is commutative.
As $Y \times_X Y$ is the sum of the graphs of the automorphisms

$$\sigma \in G\,,$$

Cov_X^Y is equivalent to the category

$$[G, \mathrm{Set}_f]$$

of finite sets I endowed with an action of G.
We conclude by observing that the group of automorphisms of the forgetful functor

$$[G, \mathrm{Set}_f] \longrightarrow \mathrm{Set}_f$$

identifies with G.

Locally constant and constructible étale sheaves

Definition: Let $X =$ scheme.

(i) An étale sheaf F over X is called locally constant and finite if it is representable by a finite étale morphism

$$p : X' \longrightarrow X$$

such that $p_* \mathcal{O}_{X'}$ is locally free as an $\mathcal{O}_X$-Module.

(ii) An étale sheaf F on X is called constructible if, on any quasi-compact open subscheme U of X, there exists a finite sequence of closed subschemes of U

$$\emptyset = X_{d+1} \hookrightarrow X_d \hookrightarrow \cdots \hookrightarrow X_1 \hookrightarrow X_0 = U$$

such that the restriction of F on each locally closed subscheme

$$X_i - X_{i+1} \hookrightarrow X$$

is locally constant and finite.

Remarks:

(i) If X is connected and $\overline{x}$ is a geometric point of X, a locally constant finite étale sheaf F on X corresponds to a finite set endowed with an action of $\pi_1(X, \overline{x})$.

(ii) If X is a nœtherian scheme, any decreasing sequence of closed subschemes of X is finite.

(iii) Any finite limit or colimit of locally constant and finite [resp. constructible] sheaves is locally constant and finite [resp. constructible].

Etale cohomology

Definition: Let $R =$ commutative ring.
For any scheme X, let

$$\mathcal{M}od_{\mathrm{R}_X} = \mathcal{M}od^{\text{ét}}_{\mathrm{R}_X} \qquad [\text{resp.} \quad \mathcal{M}od^{\mathrm{fl}}_{\mathrm{R}_X}, \text{ resp. } \mathcal{M}od^{\mathrm{Zar}}_{\mathrm{R}_X}\ ]$$

be the abelian category of Modules on the constant ring object R_X defined by R in the topos

ét_X [resp. ft_X, resp. the topos Zar_X of sheaves on the topological space X].

Remarks:

(i) The categories $\mathcal{M}od_{\mathrm{R}_X}$ [resp. $\mathcal{M}od^{\mathrm{fl}}_{\mathrm{R}_X}$, resp. $\mathcal{M}od^{\mathrm{Zar}}_{\mathrm{R}_X}$] have arbitrary limits and colimits.
They are endowed with functors $\otimes$, $\mathcal{H}om$ and Hom.
They have enough injective objects and enough R_X-flat objects so that $\otimes$, $\mathcal{H}om$ and Hom have derived functors $\overset{\mathrm{L}}{\otimes}$, $\mathrm{R}\mathcal{H}om$ and RHom.

(ii) These categories are related by restriction functors

$$\mathcal{M}od^{\mathrm{fl}}_{\mathrm{R}_X} \longrightarrow \mathcal{M}od_{\mathrm{R}_X} \longrightarrow \mathcal{M}od^{\mathrm{Zar}}_{\mathrm{R}_X}$$

which have exact left adjoint functors.

(iii) Any morphism of schemes $X \xrightarrow{f} Y$
induces direct image functors f_*
which are compatible in the sense that the diagram

$$\begin{array}{ccccc}
\mathcal{M}od^{\mathrm{fl}}_{\mathrm{R}_X} & \longrightarrow & \mathcal{M}od_{\mathrm{R}_X} & \longrightarrow & \mathcal{M}od^{\mathrm{Zar}}_{\mathrm{R}_X} \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
\mathcal{M}od^{\mathrm{fl}}_{\mathrm{R}_Y} & \longrightarrow & \mathcal{M}od_{\mathrm{R}_Y} & \longrightarrow & \mathcal{M}od^{\mathrm{Zar}}_{\mathrm{R}_Y}
\end{array}$$

is commutative.
They have exact left adjoint functors f^*
and they have derived functors $\mathrm{R}f_*$
which are right adjoint to f^*.

(iv) If $f : X \to Y$ is an étale morphism
[resp. is flat and finitely presentable, resp. is an open embedding],
the functor f^* also has an exact left adjoint

$$\begin{array}{rcccc}
 & f_! & : & \mathcal{M}od_{\mathrm{R}_X} & \longrightarrow & \mathcal{M}od_{\mathrm{R}_Y} \\
[\text{resp.} & f_! & : & \mathcal{M}od^{\mathrm{fl}}_{\mathrm{R}_X} & \longrightarrow & \mathcal{M}od^{\mathrm{fl}}_{\mathrm{R}_Y}, \\
\text{resp.} & f_! & : & \mathcal{M}od^{\mathrm{Zar}}_{\mathrm{R}_X} & \longrightarrow & \mathcal{M}od^{\mathrm{Zar}}_{\mathrm{R}_Y}].
\end{array}$$

Quick presentation of Čech cohomology

Proposition:

Let $(\mathcal{C}, J)$ = site endowed with a sheaf of rings $\mathcal{O}$,
X = object of $\mathcal{C}$,
$(U_i \to X)_{i \in I}$ = J-covering family of X such that each $U_i \to X$ is squarable in $\mathcal{C}$,
$\mathcal{M}$ = sheaf of modules over $\mathcal{O}$.

Then:

(i) In the derived category $D^+(\mathrm{Mod}_{\mathcal{O}(X)})$, there is a canonical morphism from the complex

$$\prod_{i_0 \in I} \mathcal{M}(U_{i_0}) \to \prod_{i_0, i_1 \in I} \mathcal{M}(U_{i_0} \times_X U_{i_1}) \to \cdots \to \prod_{i_0, \ldots, i_n \in I} \mathcal{M}(U_{i_0} \times_X \cdots \times_X U_{i_n}) \to \cdots$$

to the object $\mathrm{R}\Gamma(X, \mathcal{M})$.

(ii) This morphism is an isomorphism if

$$\mathrm{R}^k\Gamma(U_{i_0} \times_X \cdots \times_X U_{i_n}, \mathcal{M}) = 0\,, \quad \forall\, k \geq 1\,, \forall\, i_0, \ldots, i_n \in I\,.$$

Remarks:

(i) Part (ii) applies in particular if
$\mathcal{C}$ is the category $O(X)$ of open subsets of a topological space X,
J is the usual notion of open covering,
any connected component of any intersection $U_{i_0} \cap \cdots \cap U_{i_n}$, $n \geq 0$,
is contractible.

(ii) Part (ii) also applies if
$\mathcal{C}$ is the category of open subschemes of a scheme X
which is separated over $\mathrm{Spec}(\mathbb{Z})$,
J is the usual notion of open covering,
$\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-Module,
the U_i's are affine open subschemes of X
(so that, as X is separated over $\mathrm{Spec}(\mathbb{Z})$,
all intersections $U_{i_0} \times_X \cdots \times_X U_{i_n}$ are also affine).

Proof of the proposition:

(i) For any morphism $i : U \to X$ of $\mathcal{C}$, denote $\mathcal{O}_U = i_! i^* \mathcal{O}$. The complex of $\mathcal{O}$-Modules

$$\cdots \longrightarrow \bigoplus_{i_0,\ldots,i_n \in I} \mathcal{O}_{U_{i_0} \times_X \cdots \times_X U_{i_n}} \longrightarrow \cdots \longrightarrow \bigoplus_{i_0, i_1 \in I} \mathcal{O}_{U_{i_0} \times_X U_{i_1}} \longrightarrow \bigoplus_{i_0 \in I} \mathcal{O}_{U_{i_0}} \longrightarrow \mathcal{O}_X$$

is exact: indeed, its restriction to any U_i is homotopic to 0.

So, for any injective $\mathcal{O}$-Module $\mathcal{I}$, the morphism from the complex $\mathcal{I}(X)$ (concentrated in degree 0) to the complex

$$\prod_{i_0} \mathcal{I}(U_{i_0}) \longrightarrow \prod_{i_0, i_1} \mathcal{I}(U_{i_0} \times_X U_{i_1}) \longrightarrow \cdots \longrightarrow \prod_{i_0,\ldots,i_n} \mathcal{I}(U_{i_0} \times_X \cdots \times_X U_{i_n}) \longrightarrow \cdots$$

is a quasi-isomorphism.

Therefore, if $\mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to \cdots$ is an injective resolution of $\mathcal{M}$, $\mathrm{R}\Gamma(X, \mathcal{M})$ is represented by the simple complex associated to the double complex

$$\begin{array}{ccccccc}
\prod_{i_0} \mathcal{I}_0(U_{i_0}) & \longrightarrow & \prod_{i_0, i_1} \mathcal{I}_0(U_{i_0} \times_X U_{i_1}) & \longrightarrow \cdots \longrightarrow & \prod_{i_0,\ldots,i_n} \mathcal{I}_0(U_{i_0} \times_X \cdots \times_X U_{i_n}) & \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\prod_{i_0} \mathcal{I}_1(U_{i_0}) & \longrightarrow & \prod_{i_0, i_1} \mathcal{I}_1(U_{i_0} \times_X U_{i_1}) & \longrightarrow \cdots \longrightarrow & \prod_{i_0,\ldots,i_n} \mathcal{I}_1(U_{i_0} \times_X \cdots \times_X U_{i_n}) & \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\vdots & & \vdots & & \vdots &
\end{array}$$

It is endowed with a canonical morphism from the simple complex:

$$\prod_{i_0} \mathcal{M}(U_{i_0}) \longrightarrow \prod_{i_0, i_1} \mathcal{M}(U_{i_0} \times_X U_{i_1}) \longrightarrow \cdots \longrightarrow \prod_{i_0, \ldots, i_n} \mathcal{M}(U_{i_0} \times_X \cdots \times_X U_{i_n}) \longrightarrow \cdots$$

(ii) This morphism of complexes is a quasi-isomorphism if, for any $i_0, \ldots, i_n$, the morphism of complexes from

$$\mathcal{M}(U_{i_0} \times_X \cdots \times_X U_{i_n}) \qquad \text{concentrated in degree 0}$$

to

$$\mathcal{I}_0(U_{i_0} \times_X \cdots \times_X U_{i_n}) \longrightarrow \mathcal{I}_1(U_{i_0} \times_X \cdots \times_X U_{i_n}) \longrightarrow \mathcal{I}_2(U_{i_0} \times_X \cdots \times_X U_{i_n}) \longrightarrow \cdots$$

is a quasi-isomorphism.

It is equivalent to ask that

$$\mathrm{R}^k \Gamma(U_{i_0} \times_X \cdots \times_X U_{i_n}, \mathcal{M}) = 0, \qquad \forall\, k \geq 1.$$

Proposition:

Let $\mathcal{C}$ = small category with arbitrary fiber products,
J = topology on $\mathcal{C}$,
$\mathcal{O}$ = sheaf of rings on $(\mathcal{C}, J)$,
$\mathcal{M}$ = $\mathcal{O}$-Module in $\widehat{\mathcal{C}}_J$,
X = object of $\mathcal{C}$.

For any J-covering family $U_\bullet = (U_i \to X)_{i \in I}$, note

$$H^n(U_\bullet, \mathcal{M})$$

the cohomology modules of the complex:

$$\prod_{i_0 \in I} \mathcal{M}(U_{i_0}) \longrightarrow \prod_{i_0, i_1 \in I} \mathcal{M}(U_{i_0} \times_X U_{i_1}) \longrightarrow \cdots \longrightarrow \prod_{i_0, \ldots, i_n \in I} \mathcal{M}(U_{i_0} \times_X \cdots \times_X U_{i_n}) \longrightarrow \cdots$$

Then:

(i) Each $H^n(U_\bullet, \mathcal{M})$ only depends on the sieve $S \in J(X)$ generated by $U_\bullet$ and can be denoted $H^n(S, \mathcal{M})$.

(ii) The canonical morphism

$$\varinjlim_{S \in J(X)} H^n(S, \mathcal{M}) \longrightarrow \mathrm{R}^n\Gamma(X, \mathcal{M})$$

is an isomorphism for $n = 1$.

Remark: As $\mathcal{M}$ is a sheaf,

$$H^0(U_\bullet, \mathcal{M}) \longrightarrow \Gamma(X, \mathcal{M})$$

is an isomorphism for any $U_\bullet$.

Proof of the proposition:

(i) If S is the sieve generated by $U_\bullet$, the canonical morphism from the complex

$$\prod_{i_0 \in I} \mathcal{M}(U_{i_0}) \longrightarrow \prod_{i_0, i_1 \in I} \mathcal{M}(U_{i_0} \times_X U_{i_1}) \longrightarrow \cdots$$

to the complex

$$\prod_{U_0 \in S} \mathcal{M}(U_0) \longrightarrow \prod_{U_0, U_1 \in S} \mathcal{M}(U_0 \times_X U_1) \longrightarrow \cdots$$

is an homotopy equivalence.

(ii) Let $\mathcal{M} \hookrightarrow \mathcal{I}$ be an embedding into an injective $\mathcal{O}$-Module and $\mathcal{M}' = \mathcal{I}/\mathcal{M}$.
Then the short exact sequence $0 \to \mathcal{M} \to \mathcal{I} \to \mathcal{M}' \to 0$ yields an isomorphism $\mathrm{Coker}(\mathcal{M}(X) \to \mathcal{I}(X)) \xrightarrow{\sim} \mathrm{R}^1\Gamma(X, \mathcal{M})$.

On the other hand, we have a commutative diagram

$$
\begin{array}{ccccccc}
0 & \rightarrow & \varinjlim\limits_{S} \prod\limits_{U_0 \in S} \mathcal{M}(U_0) & \rightarrow & \varinjlim\limits_{S} \prod\limits_{U_0 \in S} \mathcal{I}(U_0) & \rightarrow & \varinjlim\limits_{S} \prod\limits_{U_0 \in S} \mathcal{M}'(U_0) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \varinjlim\limits_{S} \prod\limits_{U_0, U_1 \in S} \mathcal{M}(U_0 \times_S U_1) & \rightarrow & \varinjlim\limits_{S} \prod\limits_{U_0, U_1 \in S} \mathcal{I}(U_0 \times_X U_1) & \rightarrow & \varinjlim\limits_{S} \prod\limits_{U_0, U_1 \in S} \mathcal{M}'(U_0 \times_X U_1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \varinjlim\limits_{S} \prod\limits_{U_0, U_1, U_2 \in S} \mathcal{M}(U_0 \times_X U_1 \times_X U_2) & \rightarrow & \varinjlim \prod\limits_{U_0, U_1, U_2 \in S} \mathcal{I}(U_0 \times_X U_1 \times_X U_2) & \rightarrow & \varinjlim\limits_{S} \prod\limits_{U_0, U_1, U_2 \in S} \mathcal{M}'(U_0 \times_X U_1 \times_X U_2)
\end{array}
$$

whose lines are exact as the colimit $\varinjlim\limits_{S}$ is filtering.

The middle column is also exact as $\mathcal{I}$ is injective.
As $\mathcal{I}(X) = \mathrm{Ker}\Big(\prod\limits_{U_0 \in S} \mathcal{I}(U_0) \rightarrow \prod\limits_{U_0, U_1 \in S} \mathcal{I}(U_0 \times_X U_1) \Big)$,
$\mathcal{M}'(X) = \mathrm{Ker}\Big(\prod\limits_{U_0 \in S} \mathcal{M}'(U_0) \rightarrow \prod\limits_{U_0, U_1 \in S} \mathcal{M}'(U_0 \times_X U_1) \Big)$ for any S
and $\mathcal{M}'(X)$ is contained in the image of $\varinjlim\limits_{S} \prod\limits_{U_0 \in S} \mathcal{I}(U_0)$,
we get an isomorphism

$$
\mathrm{Coker}(\mathcal{I}(X) \longrightarrow \mathcal{M}'(X)) \xrightarrow{\sim} \varinjlim\limits_{S} H^1(S, \mathcal{M}) .
$$

Corollary:

Let $(\mathcal{C}, J) =$ site endowed with a sheaf of commutative rings $\mathcal{O}$,
$X =$ object of $\mathcal{C}$.
Then the cohomology group

$$H^1(X, \mathcal{O}^\times) = \mathrm{R}^1\Gamma(X, \mathcal{O}^\times)$$

of the sheaf of abelian groups

$$\mathcal{O}^\times : X' \longmapsto \mathcal{O}(X')^\times$$

identifies with the abelian group of isomorphism classes
of $\mathcal{O}$-Modules $\mathcal{L}$ on $(\mathcal{C}/X, J_X)$
which are locally isomorphic to $\mathcal{O}$,
endowed with the group law defined by $\otimes$.

Proof:

For any J-covering family $U_\bullet = (U_i \to X)_{i \in I}$, the group

$$H^1(U_\bullet, \mathcal{O}^\times)$$

identifies with the group of $\mathcal{O}$-Modules $\mathcal{L}$ on $(\mathcal{C}/X, J_X)$
whose restriction to any $\mathcal{C}/U_i$ is isomorphic to $\mathcal{O}$.
Indeed, for any object X' of $\mathcal{C}$, $\mathcal{O}^\times(X')$ is the automorphism group
of the restriction of the sheaf $\mathcal{O}$ to the relative category $\mathcal{C}/X'$.

Corollary:

Let X = scheme endowed with the sheaf

$$\mathcal{O}_X^\times = \mathrm{Hom}(\bullet, \mathbb{G}_m)$$

for the fppf, étale or Zariski topology.

Then the canonical morphisms

$$H^1_{\mathrm{Zar}}(X, \mathcal{O}_X^\times) \longrightarrow H^1_{\text{ét}}(\mathcal{O}_X^\times) \longrightarrow H^1_{\mathrm{fppf}}(\mathcal{O}_X^\times)$$

are isomorphisms.

Proof: We have to prove that any $\mathcal{O}_X$-Module $\mathcal{L}$ for the fppf [resp. étale] topology which is locally isomorphic to $\mathcal{O}_X$ is also isomorphic to $\mathcal{O}_X$ for the Zariski topology.

We can suppose X is an affine scheme $\mathrm{Spec}(A)$.

First, $\mathcal{L}$ is locally quasi-coherent for the fppf [resp. étale] topology so it is quasi-coherent: there exists an A-module L such that, for any $X' \xrightarrow{p} X$, $\mathcal{L}(X')$ identifies with $p^*\widetilde{L}(X')$.

Secondly, L is a flat A-module and it is finitely generated as it is so locally.

Lastly, as $B \otimes_A L$ is isomorphic to B for some faithfully flat [resp. étale] A-algebra B of finite presentation, we can suppose that L has the form $A \otimes_{A'} L'$ for some ring A' finitely generated over $\mathbb{Z}$, some morphism $A' \to A$ and some A'-module L' locally isomorphic to A' for the fppf [resp. étale] topology.

As A' is nœtherian, L' is locally isomorphic to A' for the Zariski topology.

The expression of arbitrary $H^k(X, \mathcal{M}) = \mathrm{R}^k\Gamma(X, \mathcal{M})$ in terms of Čech cohomology requires the notion of hypercovering. It is based on:

Lemma:

Let $\mathcal{E}$ = category with finite limits,

Δ = simplicial category whose objects are denoted $[n]$, $n \in \mathbb{N}$, and whose morphisms $[m] \to [n]$ are increasing maps

$$\{0, 1, \ldots, m\} \longrightarrow \{0, 1, \ldots, n\},$$

Δ_n = full subcategory of Δ on objects $[0], [1], \ldots, [n]$.

Then the restriction functor

$$\mathrm{sk}_n : [\Delta^{\mathrm{op}}, \mathcal{E}] \longrightarrow [\Delta_n^{\mathrm{op}}, \mathcal{E}]$$

has a right adjoint

$$\begin{array}{rcccl} \mathrm{cosk}_n & : & [\Delta_n^{\mathrm{op}}, \mathcal{E}] & \longrightarrow & [\Delta^{\mathrm{op}}, \mathcal{E}]\,, \\ & & F_\bullet & \longmapsto & \mathrm{cosk}_n F_\bullet = F'_\bullet \end{array}$$

defined by the formula

$$F'_m = \varprojlim_{(\alpha:[m']\to[m])} F_{m'}\,, \quad \forall\, m \in \mathbb{N}\,,$$

where the limit is computed on the category $\Delta_n/[m]$ of objects $[m']$ of Δ_n endowed with a morphism $[m'] \to [m]$.

Definition:

Let $(\mathcal{C}, J) =$ small site endowed with $\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J$,
$X =$ object of $\mathcal{C}$.

An hypercovering of X is a simplicial object

$$P_\bullet : \Delta^{\mathrm{op}} \longrightarrow \widehat{\mathcal{C}}/y(X)$$

such that

- for any n, the presheaf P_n has the form

  $$P_n \cong \coprod_{i \in I} y(X_i)$$

  where each X_i is an object of $\mathcal{C}$ endowed with a morphism $X_i \to X$,

- for any n, the transform by j^* of the canonical morphism

  $$P_{n+1} \longrightarrow \left[\operatorname{cosk}_n \circ \operatorname{sk}_n(P_\bullet)\right]_{n+1}$$

  is an epimorphism of $\widehat{\mathcal{C}}_J/\ell(X)$.

Remarks:

(i) If K is an infinite cardinal containing the cardinal of $\mathcal{C}$, one can restrict to sums $P_n \cong \coprod_{i \in I} y(X_i)$ indexed by subsets I of K.

(ii) If $\mathcal{C}$ has finite limits and any J-covering family of an object of $\mathcal{C}$ has a finite subcovering (which is the case for the fppf, étale or Zariski topology over a quasi-compact scheme), one can restrict to finite sums

$$P_n \cong \coprod_{1 \le i \le k} y(X_i) \, .$$

Theorem:
Let $(\mathcal{C}, J)$ = small site,
$\mathcal{O}$ = sheaf of rings on $(\mathcal{C}, J)$,
$\mathcal{M}$ = sheaf of $\mathcal{O}$-modules on $(\mathcal{C}, J)$,
X = object of $\mathcal{C}$.
Then there are canonical isomorphisms

$$\varinjlim_{P_\bullet} H^k(P_\bullet, \mathcal{M}) \xrightarrow{\sim} H^k(X, \mathcal{M}) = \mathrm{R}^k\Gamma(X, \mathcal{M}), \quad \forall\, k \geq 0,$$

where:

- the colimits are taken on the filtered category of hypercoverings $P_\bullet$ of X,
- for any hypercovering $P_\bullet$ of X, the $H^k(P_\bullet, \mathcal{M})$ are the cohomology modules of the complex:

$$\mathrm{Hom}(P_0, \mathcal{M}) \longrightarrow \mathrm{Hom}(P_1, \mathcal{M}) \longrightarrow \cdots \longrightarrow \mathrm{Hom}(P_k, \mathcal{M}) \longrightarrow \cdots$$

Remark: For any $P_n \simeq \coprod_{i \in I} y(X_i)$, $\mathrm{Hom}(P_n, \mathcal{M})$ identifies with

$$\prod_{i \in I} \mathcal{M}(X_i).$$

Proof: See Chapter VI.

Corollary:

Let $\mathcal{C}$ = essentially small category with finite limits,
J = topology on $\mathcal{C}$ such that any J-covering family contains a finite subcovering,
$\mathcal{O}$ = sheaf of rings on $(\mathcal{C}, J)$,
X = object of $\mathcal{C}$.

Then the functors

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}} & \longrightarrow & \mathrm{Mod}_{\mathcal{O}(X)}\,, \\ \mathcal{M} & \longmapsto & H^k(X, \mathcal{M}) \end{array}$$

respect arbitrary filtered colimits.

Remark: This corollary applies in particular to the fppf, étale or Zariski topology of quasi-compact schemes.

Proof: We know $H^k(X, \mathcal{M}) \cong \varinjlim_{P_\bullet} H^k(P_\bullet, \mathcal{M})$ where the filtered colimit is taken over hypercoverings $P_\bullet$ such that each P_n is a finite sum $\coprod_{1 \leq i \leq k} y(X_i)$,

and, therefore, the functor $\mathcal{M} \mapsto \mathrm{Hom}(P_n, \mathcal{M}) = \prod_{1 \leq i \leq k} \mathcal{M}(X_i)$ respects colimits.

As colimits respect colimits and filtered colimits are exact functors, the conclusion follows.

The notion of geometric dimension

Definition:

(i) The dimension (or Krull dimension) of a scheme X is

$$\dim(X) = \sup\{\ell \in \mathbb{N} \mid \exists\, x_0, x_1, \ldots, x_\ell \in X \text{ such that } \overline{x}_0 \subsetneq \overline{x}_1 \subsetneq \ldots \subsetneq \overline{x}_\ell\}.$$

(ii) The (relative) dimension of a scheme morphism $X \to Y$ is

$$\dim(X/Y) = \sup\{\dim X_y \mid y = \mathrm{Spec}(k) = \text{ point of } y\,,\ X_y = X \times_Y y\}.$$

Remarks:

(i) A topological space is called irreducible if intersections of pairs of non empty open subsets are non empty.

For any point x of a topological space X, its closure $\overline{x}$ is irreducible.

A topological space is called sober if any irreducible closed subset of the closure of a unique point. Any scheme is sober.

(ii) If a scheme X is a union of open subschemes U_i, $i \in I$,

$$\dim(X) = \sup_{i \in I} \dim(U_i)\,.$$

(iii) If $X = \mathrm{Spec}(A)$ is a scheme, $\dim(X) = \dim(A)$ is

$$\sup\{\ell \in \mathbb{N} \mid \exists\, p_0, \ldots, p_\ell = \text{ prime ideals of } A \text{ such that } p_0 \supsetneq p_1 \supsetneq \cdots \supsetneq p_\ell\}\,.$$

Basic facts about dimensions

(i) $\mathrm{Spec}(\mathbb{Z})$ has dimension 1 and, for any field k, $\mathrm{Spec}(k)$ has dimension 0.

(ii) If $\mathrm{Spec}(A)$ is an affine scheme of dimension d, $\mathrm{Spec}(A[X_1, \ldots, X_n])$ has dimension $n + d$.

(iii) If $Z \hookrightarrow X$ is a locally closed subscheme,

$$\dim(Z) \leq \dim(X)\,.$$

Therefore, any scheme of finite type over a scheme of finite dimension has finite dimension.

(iv) If $U \subset X$ is a dense open subscheme,

$$\dim(X) = \dim(U)\,.$$

(v) For any morphism $X \to Y$,

$$\dim(X) \leq \dim(Y) + \dim(X/Y)\,.$$

(vi) For any scheme X over a field k and any field k' containing k,

$$\dim(X) = \dim(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')).$$

Therefore, for any morphisms $X \to Y$ and $Y' \to Y$,

$$\dim(X \times_Y Y'/Y') \leq \dim(X/Y)$$

and one even has an equality if $Y' \to Y$ is surjective.

(vii) For any finitely presented and flat morphism

$$X \longrightarrow Y\,,$$

the map

$$\begin{array}{ccc} y & \longmapsto & \dim(X_y) \\ \| & & \| \\ \text{point of } Y & & \text{fiber } X \times_Y y \text{ of } X \text{ over } y \end{array}$$

is locally constant on Y.

(viii) For any finitely presented morphism $X \xrightarrow{f} Y$ of relative dimension d, the Zariski topology derived functors

$$\mathcal{M} \longmapsto \mathrm{R}^k f_* \mathcal{M}$$

are 0 on all quasi-coherent $\mathcal{O}_X$-Modules $\mathcal{M}$ for all $k > d$.

Relative curves

Definition:
A relative curve over a base scheme Y
is a finitely presented and flat morphism

$$X \longrightarrow Y$$

such that, for any point $y = \mathrm{Spec}(k)$ of Y, the fiber

$$X_y = X \times_Y y = X \times_Y \mathrm{Spec}(k)$$

has dimension 1.

Remark:
One can prove that a relative curve $X \to Y$
is proper if and only if, for any affine open subscheme

$$\mathrm{Spec}(A) = V \subset Y\,,$$

the curve $X \times_Y V$ over $V = \mathrm{Spec}(A)$ is projective,
in the sense that $X \times_Y V \to V$ factorises as the
composition of some closed immersion

$$X \times_Y V \hookrightarrow \mathbb{P}^n \times V$$

and the projection $\mathbb{P}^n \times V \longrightarrow V$.

Relative jacobians

Proposition: Let $X \xrightarrow{p} Y$ be a relative curve such that

$$\begin{cases} \bullet \quad p \text{ is proper and smooth (of dimension 1),} \\ \bullet \quad \text{the fibers of } p \text{ are "geometrically connected"} \\ \quad\;\; \text{in the sense that, for any morphism } \overline{y} = \mathrm{Spec}(\overline{k}) \to Y \\ \quad\;\; \text{from an algebraically closed field } \overline{k}, \text{ the fiber } X_{\overline{y}} = X \times_Y \overline{y} \text{ is connected.} \end{cases}$$

Then the images $\mathrm{R}^k p_* \mathbb{G}_m$ of the étale sheaf $\mathbb{G}_m$ on X by the étale direct image cohomology functors are:

(i) $\mathrm{R}^k p_* \mathbb{G}_m$ is 0 if $k \geq 2$,

(ii) $p_* \mathbb{G}_m$ is the étale sheaf $\mathbb{G}_m$ on Y,

(iii) $\mathrm{R}^1 p_* \mathbb{G}_m$ associates to any étale morphism $Y' \to Y$ the cokernel of the morphism:

$$\begin{array}{ccc} H^1_{\mathrm{Zar}}(Y', \mathcal{O}^\times_{Y'}) & \longrightarrow & H^1_{\mathrm{Zar}}(X \times_Y Y', \mathcal{O}^\times_{X \times_Y Y'}) \\ \| & & \| \\ \text{group of } \mathcal{O}_{Y'}\text{-Modules} & & \text{group of } \mathcal{O}_{X \times_Y Y'}\text{-Modules} \\ \text{locally isomorphic to } \mathcal{O}_{Y'} & & \text{locally isomorphic to } \mathcal{O}_{X \times_Y Y'} \end{array}$$

Theorem: In the same situation, the functor

$$\begin{array}{ccc} \mathrm{Sch}/Y & \longrightarrow & \mathrm{Set} \\ (Y' \to Y) & \longmapsto & H^1_{\mathrm{Zar}}(X \times_Y Y', \mathcal{O}^\times_{X \times_Y Y'})/H^1_{\mathrm{Zar}}(Y', \mathcal{O}^\times_{Y'}) \end{array}$$

is representable by a locally finitely presented scheme over Y

$$\mathcal{P}ic_{X/Y} \longrightarrow Y \quad \text{(called the Picard scheme of } X \text{ over } Y\text{)}$$

endowed with an abelian group scheme structure and a short sequence of abelian group schemes over Y

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}ac_{X/Y} & \longrightarrow & \mathcal{P}ic_{X/Y} & \xrightarrow{\deg} & \mathbb{Z}_Y & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \coprod_{d \in \mathbb{Z}} Y & & \end{array}$$

such that

- this sequence is exact for the étale topology,
- $\mathcal{J}ac_{X/Y}$ is proper (even projective over any affine open subscheme of Y) and smooth over Y, and its fibers are geometrically connected.

Remark:

- $\mathcal{J}ac_{X/Y}$ is called the relative jacobian of the relative curve X/Y,
- the relative dimension g of $\mathcal{J}ac_{X/Y}$ over Y is locally constant, it is called the "genus" of the relative curve X over Y,
- the morphism $\mathcal{P}ic_{X/Y} \xrightarrow{\deg} \mathbb{Z}_Y$ is called the degree map.

Étale cohomology of relative curves

Definition:
On any scheme X, one denotes μ_n the étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules defined as the kernel of

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_m\,, \\ \lambda & \longmapsto & \lambda^n\,. \end{array}$$

Remark:
If n is invertible on X or, equivalently,
if X is a scheme over $\mathbb{Z}_{(n)} = \mathbb{Z}[X]/(n \cdot X - 1) = \mathbb{Z}\left[\frac{1}{n}\right]$,
the $(\mathbb{Z}/n\mathbb{Z})$-Module μ_n is isomorphic to the constant Module $(\mathbb{Z}/n\mathbb{Z})$ on the finite étale cover

$$X \times_{\mathrm{Spec}(\mathbb{Z}_{(n)})} \mathrm{Spec}(\mathbb{Z}_{(n)}[X]/(X^n - 1))$$

of X.

Proposition:
Suppose n is invertible on a scheme Y.
Let $X \to Y$ be a smooth and proper curve with geometrically connected fibers such that the smooth proper morphism

$$\mathcal{J}ac_{X/Y} \longrightarrow Y$$

has constant relative dimension g.

Then the scheme over Y defined as the kernel of the morphism

$$\begin{array}{rcccl} n & : & \mathcal{J}ac_{X/Y} & \longrightarrow & \mathcal{J}ac_{X/Y}\,, \\ & & \mathcal{L} & \longmapsto & \mathcal{L}^{\otimes n} \end{array}$$

is a finite étale scheme over Y

$$\mathcal{J}ac_{X/Y}[n]$$

which is locally isomorphic to the constant $(\mathbb{Z}/n\mathbb{Z})$-Module $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

Corollary:
In the same situation of a smooth proper curve

$$p : X \longrightarrow Y$$

with geometrically connected fibers and constant genus g,
the étale direct images

$$\mathrm{R}^k p_* \mu_n$$

of the locally constant étale $\mathbb{Z}/n\mathbb{Z}$-Module μ_n are:

(i) $\mathrm{R}^k p_* \mu_n$ is 0 for any $k \geq 3$,

(ii) $\mathrm{R}^2 p_* \mu_n$ identifies with the constant sheaf $\mathbb{Z}/n\mathbb{Z}$,

(iii) $\mathrm{R}^1 p_* \mu_n$ identifies with the locally constant finite étale $\mathbb{Z}/n\mathbb{Z}$-Module

$$\mathcal{J}ac_{X/Y}[n]$$

which is locally isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$,

(iv) $p_* \mu_n$ identifies with the locally constant étale sheaf μ_n on Y.

Grothendieck's six operations for étale cohomology

Definition:
A morphism of schemes $X \to S$ is called "compactifiable" if it factorises as the composite

$$X \overset{i}{\hookrightarrow} \overline{X} \overset{p}{\to} S$$

of an open embedding i and a proper morphism p.

Remarks:

(i) Any compactifiable morphism is locally of finite type.

(ii) If S is a base scheme, let's denote

$$\mathrm{Sch}_c/S$$

the full subcategory of Sch/S on compactifiable morphisms $X \to S$.

(iii) If S is quasi-compact, all objects $X \to S$ of Sch_c/S have finite relative dimension and, more generally, all morphisms $X \to Y$ of Sch_c/S have finite relative dimension.

Choice of torsion coefficients

Let $n =$ integer which is invertible in $\mathcal{O}_S(S)$. We consider:

- for any object $X \to S$ of Sch_c/S the category

$$\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}$$

  of étale $(\mathbb{Z}/n\mathbb{Z})_X$-Modules on X, together with the functors $\otimes$, $\mathcal{H}om$, Hom and their derived functors $\overset{\mathrm{L}}{\otimes}$, $\mathrm{R}\mathcal{H}om$, RHom,
- for any morphism $f : X \to Y$ of Sch_c/S the pair of adjoint functors $f^* = f^{-1}, f_*$ between $\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}$ and $\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}$ and their derived functors $f^* = f^{-1}, \mathrm{R}f_*$,
- for any étale morphism $i : X \to Y$ of Sch_c/S, the exact left adjoint $i_!$ of i^*.

Remark:
If $n = \ell_1^{m_1} \cdots \ell_k^{m_k}$ is the prime decomposition of n,
we have for any X a canonical decomposition

$$\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X} = \mathcal{M}od_{(\mathbb{Z}/\ell_1^{m_1}\mathbb{Z})_X} \times \cdots \times \mathcal{M}od_{(\mathbb{Z}/\ell_k^{m_k}\mathbb{Z})_X}.$$

So there is no restriction in supposing that

$$n = \ell^m$$

is a power of a prime ℓ.

The main theorems

Theorem:
Let $S =$ quasi-compact base scheme,
$n =$ integer which is invertible in $\mathcal{O}_S(S)$.
Consider a proper morphism of Sch_c/S

$$f : X \to Y\,.$$

Then:

(i) (Proper base change theorem)
For any cartesian square of Sch completing f

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y \end{array}$$

the canonical morphisms

$$\begin{array}{rcl} y^* \circ f_* & \longrightarrow & f'_* \circ x^*\,, \\ y^* \circ \mathrm{R}f_* & \longrightarrow & \mathrm{R}f'_* \circ x^* \end{array}$$

of functors from $\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}$ to $\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_{Y'}}$ or from $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X})$ to $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_{Y'}})$ are isomorphisms.

(ii) If d is the relative dimension of $X \xrightarrow{f} Y$, f_* has cohomological dimension $\leq 2d$. In other words,

$$\mathrm{R}^k f_* = 0$$

for any $k > 2d$.

(iii) The functors

$$f_*, \mathrm{R}^k f_* : \mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X} \longrightarrow \mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}$$

transform constructible $(\mathbb{Z}/n\mathbb{Z})_X$-Modules into constructible $(\mathbb{Z}/n\mathbb{Z})_Y$-Modules.

Remark:

(ii) implies that $\mathrm{R}f_*$ is well-defined as a functor from $D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X})$ to $D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y})$.

In the situation of (i), there is a morphism of functors from $D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X})$ to $D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y})$

$$y^* \circ \mathrm{R}f_* \longrightarrow \mathrm{R}f'_* \circ x^*$$

and it is an isomorphism.

Corollary (of the proper base change theorem):
We can associate to any morphism of Sch_c/S

$$f : X \to Y$$

a functor

$$\mathrm{R}f_! : D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}) \longrightarrow D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y})$$

$$\text{(or even:} \qquad D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}) \longrightarrow D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}))$$

such that:

- for any factorisation of f

  $$X \overset{i}{\hookrightarrow} \overline{X} \overset{p}{\longrightarrow} Y$$

  as the composite of an open embedding i and a proper morphism p, there is a canonical isomorphism

  $$\mathrm{R}f_! \cong \mathrm{R}p_* \circ i_!\,,$$

- for any pair of morphisms of Sch_c/S

  $$X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z\,,$$

  there is a canonical isomorphism

  $$\mathrm{R}(g \circ f)_! \cong \mathrm{R}g_! \circ \mathrm{R}f_!\,.$$

Remarks:

(i) Any morphism $f : X \to Y$ of Sch_c/S factorises as the composite of an open embedding followed by a proper morphism.
Indeed, $X \to S$ has such a factorisation

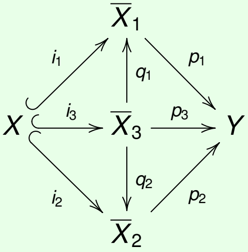
$$X \overset{i}{\hookrightarrow} \overline{X} \overset{p}{\longrightarrow} S\,.$$

Then, $\overline{X} \times_S Y \to Y$ is proper as well as $\overline{X}_1 \to Y$ if $\overline{X}_1$ is the smallest closed subscheme of $\overline{X} \times_S Y$ containing the image of

$$X \xhookrightarrow{(i,f)} \overline{X} \times_S Y\,.$$

So $X \xhookrightarrow{i_1} \overline{X}_1 \xrightarrow{p_1} Y$ is a factorisation of f as the composite of an open embedding i_1 and a proper morphism p_1.

(ii) If $X \overset{i_1}{\hookrightarrow} \overline{X}_1 \xrightarrow{p_1} Y$ and $X \overset{i_2}{\hookrightarrow} \overline{X}_2 \xrightarrow{p_2} Y$ are two such factorisations, there is a commutative diagram



such that

i_3 is an open embedding, just as i_1, i_2,
p_3, q_1, q_2 are proper, just as p_1, p_2,
$q_1^{-1}(i_1(X)) = i_3(X) = q_2^{-1}(i_2(X))$.

(iii) The corollary formally follows from the proper base change theorem combined with remarks (i) and (ii),
just as in the case of ringed topological spaces.

(iv) The functors $\mathrm{R}f_!$ commute with base change.

(v) We can associate to any morphism $f : X \to Y$ of Sch_c/S a functor

$$f_! : \mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X} \longrightarrow \mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}$$

such that:

$$\begin{cases} \bullet \quad \text{for any factorisation of } f \text{ as } X \overset{i}{\hookrightarrow} \overline{X} \overset{p}{\longrightarrow} Y,\ f_! \text{ identifies with } p_* \circ i_!, \\ \bullet \quad \text{for any pair of morphisms of } \mathrm{Sch}_c/S \\ \qquad\qquad X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z\,, \\ \quad (g \circ f)_! \text{ is canonically isomorphic to } g_! \circ f_!. \end{cases}$$

Nevertheless, in general, $\mathrm{R}f_!$ is not the derived functor of $f_!$.

The Künneth formula

Proposition:

(i) For any morphism $f : X \to Y$ of Sch_c/S
and objects $\mathcal{M}$ of $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X})$, $\mathcal{N}$ of $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y})$,

$$\mathrm{R}f_!(\mathcal{M} \overset{\mathrm{L}}{\otimes} f^{-1}\mathcal{N}) \quad \text{and} \quad \mathrm{R}f_!\mathcal{M} \overset{\mathrm{L}}{\otimes} \mathcal{N}$$

are canonically isomorphic.

(ii) For any cartesian square of Sch_c/S

$$\begin{array}{ccc} X_1 \times_Y X_2 & \xrightarrow{p_2} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ X_1 & \xrightarrow{q_1} & Y \end{array}$$

with $r = q_1 \circ p_1 = q_2 \circ p_2$,
and objects $\mathcal{M}_1$ of $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_{X_1}})$, $\mathcal{M}_2$ of $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_{X_2}})$,

$$\mathrm{R}r_!(p_1^{-1}\mathcal{M}_1 \overset{\mathrm{L}}{\otimes} p_2^{-1}\mathcal{M}_2) \quad \text{and} \quad \mathrm{R}(p_1)_!\mathcal{M}_1 \overset{\mathrm{L}}{\otimes} \mathrm{R}(p_2)_!\mathcal{M}_2$$

are canonically isomorphic.

Sketch of proof of the proposition:
It is similar to the case of topological spaces.
(ii) is a formal consequence of (i).
(i) is obvious when f is an open immersion.
So we can suppose that f is proper and $\mathrm{R}f_! = \mathrm{R}f_*$.
For any $\mathcal{M}$ and $\mathcal{N}$, the canonical morphism

$$f^{-1} \circ \mathrm{R}f_*\mathcal{M} \longrightarrow \mathcal{M}$$

yields a morphism

$$\begin{array}{ccc} (f^{-1} \circ \mathrm{R}f_*\mathcal{M}) \overset{\mathrm{L}}{\otimes} f^{-1}\mathcal{N} & \longrightarrow & \mathcal{M} \overset{\mathrm{L}}{\otimes} f^{-1}\mathcal{N} \\ \wr\| & & \\ f^{-1}(\mathrm{R}f_* \overset{\mathrm{L}}{\otimes} \mathcal{N}) & & \end{array}$$

and by adjunction, a morphism

$$\mathrm{R}f_* \overset{\mathrm{L}}{\otimes} \mathcal{N} \longrightarrow \mathrm{R}f_*(\mathcal{M} \overset{\mathrm{L}}{\otimes} f^{-1}\mathcal{N}) \,.$$

We have to check that this morphism is an isomorphism.
As $\mathrm{R}f_*$ commutes with base change, we can suppose that Y is a geometric point.
We can also suppose that $\mathcal{N}$ is a flat $(\mathbb{Z}/n\mathbb{Z})$-module.
Then $\bullet \otimes \mathcal{N}$ is an exact functor.
So, for any $(\mathbb{Z}/n\mathbb{Z})_X$-Module $\mathcal{M}$, $U \mapsto \mathcal{M}(U) \otimes \mathcal{N}$ is a $(\mathbb{Z}/n\mathbb{Z})_X$-Module
(in particular a sheaf) and it is f_*-acyclic if $\mathcal{M}$ is f_*-acyclic.
The conclusion follows.

The exceptionnal inverse image functor

Theorem:
Let $f : X \to Y$
$= $ morphism of Sch_c/S.
Then:

(i) The functor

$$\mathrm{R}f_! : D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}) \longrightarrow D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y})$$

has a right adjoint

$$f^! : D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}) \longrightarrow D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}).$$

(ii) The two functors

$$\begin{array}{rcl} D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}) \times D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}) & \longrightarrow & D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}), \\ (\mathcal{M}, \mathcal{N}) & \longmapsto & \mathrm{R}f_* \mathrm{R}\mathcal{H}om(\mathcal{M}, f^! \mathcal{N}), \\ (\mathcal{M}, \mathcal{N}) & \longmapsto & \mathrm{R}\mathcal{H}om(\mathrm{R}f_! \mathcal{M}, \mathcal{N}) \end{array}$$

are canonically isomorphic.

Remarks:

(i) Composing the isomorphism of (ii) with $\mathrm{R}\Gamma(Y, \bullet)$,
we get that the two functors

$$\begin{array}{rcl} D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}) \times D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}) & \longrightarrow & D(\mathcal{M}od_{\mathbb{Z}/n\mathbb{Z}}) \,, \\ (\mathcal{M}, \mathcal{N}) & \longmapsto & \mathrm{RHom}(\mathcal{M}, f^! \mathcal{N}) \,, \\ (\mathcal{M}, \mathcal{N}) & \longmapsto & \mathrm{RHom}(\mathrm{R}f_! \mathcal{M}, \mathcal{N}) \end{array}$$

are canonically isomorphic.

(ii) The isomorphism of (ii) also means that,
for any object $\mathcal{N}$ of $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y})$, the square

$$\begin{array}{ccc} D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}) & \xrightarrow{\mathrm{R}\mathcal{H}om(\bullet, f^! \mathcal{N})} & D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}) \\ \downarrow {\scriptstyle \mathrm{R}f_!} & & \downarrow {\scriptstyle \mathrm{R}f_*} \\ D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}) & \xrightarrow{\mathrm{R}\mathcal{H}om(\bullet, \mathcal{N})} & D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}) \end{array}$$

is commutative up to canonical isomorphism.

(iii) For any morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of Sch_c/S, $(g \circ f)^!$ is canonically
isomorphic to $f^! \circ g^!$.

(iv) If $f : X \to Y$ is an open immersion,
$\mathrm{R}f_!$ is the extension by 0 functor $f_!$ and so
$f^!$ is the restriction functor $f^* = f^{-1}$.
More generally, if $f : X \to Y$ is étale, $\mathrm{R}f_!$ is $f_!$ and so $f^!$ is $f^* = f^{-1}$.

(v) For any object $\mathcal{N}$ of $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y})$,
the identity morphism $f^!\mathcal{N} \to f^!\mathcal{N}$
corresponds by adjunction to a morphism

$$\mathrm{Tr} : \mathrm{R}f_! \circ f^!\mathcal{N} \longrightarrow \mathcal{N}$$

called the "trace morphism".

(vi) For any such object $\mathcal{N}$, the morphism

$$\mathrm{R}f_!(f^!(\mathbb{Z}/n\mathbb{Z})_Y \overset{\mathrm{L}}{\otimes} f^{-1}\mathcal{N} \simeq \mathrm{R}f_! \circ f^!(\mathbb{Z}/n\mathbb{Z})_Y \overset{\mathrm{L}}{\otimes} \mathcal{N} \longrightarrow \mathcal{N}$$

corresponds by adjunction to a morphism

$$f^!(\mathbb{Z}/n\mathbb{Z})_Y \overset{\mathrm{L}}{\otimes} f^{-1}\mathcal{N} \longrightarrow f^!\mathcal{N}\,.$$

Principles of the construction

They are very similar to the case of topological spaces.

- We can suppose that $f : X \to Y$ is proper of relative dimension d so that $\mathrm{R}f_! = \mathrm{R}f_*$ has dimension $\leq 2d$.
- There exists a finite resolution

$$0 \longrightarrow (\mathbb{Z}/n\mathbb{Z})_X \longrightarrow S^0 \longrightarrow S^1 \longrightarrow \cdots \longrightarrow S^{2d} \longrightarrow 0$$

  of $(\mathbb{Z}/n\mathbb{Z})_X$ by objects S^j of the full additive subcategory $\mathcal{S}_X$ of $\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}$ on $(\mathbb{Z}/n\mathbb{Z})_X$-Modules S which are flat and such that, for any étale morphism $U \xrightarrow{i} X$, $S_U = i_! i^* S$ is f_*-acyclic.
  For this we denote $|X|_f$ the set of points x of X which are closed in their fiber over Y and lift any $x \in |X|_f$ to a geometric point $\overline{x}$ of X.
  We define

$$C_0 = (\mathbb{Z}/n\mathbb{Z})_X,$$
$$C_j = S_{j-1}/C_{j-1} \text{ for } 1 \leq j \leq 2d,$$
$$S_j = \prod_{x \in |X|_f} \overline{x}_* \circ \overline{x}^* C_j \text{ for } 0 \leq j \leq 2d-1,$$
$$S_{2d} = C_{2d}$$

  so that there is an exact sequence

$$0 \longrightarrow (\mathbb{Z}/n\mathbb{Z})_X \longrightarrow S^0 \longrightarrow S^1 \longrightarrow \cdots \longrightarrow S^{2d} \longrightarrow 0\,.$$

We prove by induction on j that each C_j and S_j is flat over $(\mathbb{Z}/n\mathbb{Z})_X$.

For any étale morphism $U \xrightarrow{i} X$, the $(\mathbb{Z}/n\mathbb{Z})_X$-Modules

$$(S_j)_U = i_! \circ i^* S_j\,, \quad 0 \leq j \leq 2d-1\,,$$

are f_*-acyclic because they are products

$$\prod_{x \in |X|_f} \prod_{\substack{(\overline{x} \to U) \\ = \text{lift of } \overline{x} \to X}} \overline{x}_* \circ \overline{x}^* C_j\,.$$

Lastly, each $(S_{2d})_U$ is f_*-acyclic because the $(S_j)_U$, $0 \leq j < 2d$, are f_*-acyclic and f_* has cohomological dimension $\leq 2d$.

- For any object S of $\mathcal{S}_X$ and for any injective $(\mathbb{Z}/n\mathbb{Z})_X$-Module $\mathcal{I}$, the presheaf

$$(U \to X) \longmapsto \mathrm{Hom}_{(\mathbb{Z}/n\mathbb{Z})_Y}(f_*((\mathbb{Z}/n\mathbb{Z})_U \otimes_{(\mathbb{Z}/n\mathbb{Z})_X} S), \mathcal{I})$$

  is an injective $(\mathbb{Z}/n\mathbb{Z})_Y$-Module (in particular an étale sheaf) denoted $f_S^!(\mathcal{I})$.
- If $\mathcal{N}$ is an object of $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y})$, and $\mathcal{N} \to \mathcal{I}$ is an injective resolution of $\mathcal{N}$ by $\mathcal{I} = (\mathcal{I}^k)$, we define $f^!\mathcal{N}$ as the complex

$$\left(\bigoplus_{k-j=n} f_{S^j}^!(\mathcal{I}^k)\right)_{n \in \mathbb{Z}}.$$

Theorem:

Let $S =$ quasi-compact base scheme,

$n =$ integer which is invertible in $\mathcal{O}_S(S)$.

Consider a morphism of Sch_c/S

$$y : Y' \longrightarrow Y$$

which is smooth of dimension d.

Then:

(i) (Smooth base change theorem)
For any cartesian square of Sch_c/S completing y

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y \end{array}$$

the canonical morphisms

$$\begin{array}{rcl} y^* \circ f_* & \longrightarrow & f'_* \circ x^* \\ y^* \circ \mathrm{R}f_* & \longrightarrow & \mathrm{R}f'_* \circ x^* \end{array}$$

of functors from $\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}$ to $\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_{Y'}}$
or from $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X})$ to $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_{Y'}})$
are isomorphisms

(ii) The object of $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_{Y'}})$

$$f^!(\mathbb{Z}/n\mathbb{Z})_Y$$

is concentrated in degree $2d$
and quasi-isomorphic to

$$(\mu_n^{\otimes d})[-2d] = \big(\overbrace{\mu_n \otimes_{(\mathbb{Z}/n\mathbb{Z})_{Y'}} \cdots \otimes_{(\mathbb{Z}/n\mathbb{Z})_{Y'}} \mu_n}^{d \text{ times}}\big)[-2d]\,.$$

Furthermore, the functor

$$f^! : D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}) \longrightarrow D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_{Y'}})$$

is canonically isomorphic to the functor

$$\mathcal{N} \longmapsto f^!(\mathbb{Z}/n\mathbb{Z})_Y \otimes f^{-1}\mathcal{N}\,.$$

Remark:
In particular, if $y : Y' \to Y$ is étale, (ii) means that $f^! = f^{-1} = f^*$
or, equivalently, that $\mathrm{R}y_! = y_!$ is the exact functor of extension by 0.

Corollary:

Let $X \xrightarrow{f} Y$

$=$ smooth morphism of dimension d in Sch_c/S.

Then the square

$$\begin{array}{ccc} D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}) & \xrightarrow{\mathrm{R}\mathcal{H}om(\bullet,\mu_n^{\otimes d}[-2d])} & D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X}) \\ \downarrow {\scriptstyle \mathrm{R}f_!} & & \downarrow {\scriptstyle \mathrm{R}f_*} \\ D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}) & \xrightarrow{\mathrm{R}\mathcal{H}om(\bullet(\mathbb{Z}/n\mathbb{Z})_Y)} & D(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}) \end{array}$$

is commutative up to canonical isomorphism.

Remark: If $Y = \mathrm{Spec}(k)$ is a base field k, $\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}$ is the category of $(\mathbb{Z}/n\mathbb{Z})$-linear representations of the Galois group $\mathrm{Gal}_k = \mathrm{Aut}_k(\overline{k})$ for some algebraic closure $\overline{k}$ of k.

For any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y})$,

$$\mathrm{R}^{2d-i}f_*(\mathrm{R}\mathcal{H}om(\mathcal{M},\mu_n^{\otimes d}))$$

is the image of

$$\mathrm{R}f_!\mathcal{M}$$

by the duality functor $\mathrm{R}^i\mathcal{H}om(\bullet,(\mathbb{Z}/n\mathbb{Z})_Y)$

in the category of $(\mathbb{Z}/n\mathbb{Z})$-linear representations of Gal_k.

The case of proper and smooth morphisms

Theorem:
Let $X \xrightarrow{f} Y$
$=$ morphism of Sch_c/S which is both proper and smooth.
Then the functors

$$\mathrm{R}^k f_! = \mathrm{R}^k f_* = \mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_X} \longrightarrow \mathcal{M}od_{(\mathbb{Z}/n\mathbb{Z})_Y}$$

transform locally constant constructible $(\mathbb{Z}/n\mathbb{Z})_X$-Modules into locally constant constructible $(\mathbb{Z}/n\mathbb{Z})_Y$-Modules.

Remark:
In other words, if X and Y are connected,
$\overline{x}$ is a geometric point of X and $\overline{y}$ its composite with $f : X \rightarrow Y$, the functors

$$\mathrm{R}^k f_* = \mathrm{R}^k f_!$$

transform $(\mathbb{Z}/n\mathbb{Z})$-linear representations of the profinite group

$$\pi_1(X, \overline{x})$$

into $(\mathbb{Z}/n\mathbb{Z})$-linear representations of the profinite group

$$\pi_1(Y, \overline{y})\,.$$