Cohomology of toposes

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Chapter V:

Operations on linear sheaves on topological spaces, derived categories, derived functors and Grothendieck's six operations

Reminder on sheaves

Definition:
Let X = topological space,
$O(X)$ = ordered set of open subsets of X considered as a category.

(i) The category of presheaves on X

$$\mathrm{Psh}(X) = [O(X)^{\mathrm{op}}, \mathrm{Set}]$$

is the category of contravariant functors

$$\begin{array}{rcl} P : O(X)^{\mathrm{op}} & \longrightarrow & \mathrm{Set}\,, \\ U & \longmapsto & P(U) = \text{ set of “sections” of } P \text{ on } U, \\ (V \subseteq U) & \longmapsto & (P(U) \to P(V)) = \text{ restriction map from } U \text{ to } V \subseteq U\,. \end{array}$$

(ii) A presheaf $P : O(X)^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf if and only if,
for any open cover $(U_i)_{i \in I}$ of some U, the map

$$P(U) \longrightarrow \mathrm{Eq}\left(\prod_{i \in I} P(U_i) \rightrightarrows \prod_{i_1, i_2 \in I} P(U_{i_1} \cap U_{i_2}) \right)$$

is one-to-one.

(iii) The category of sheaves is the full subcategory

$$\mathrm{Sh}(X) \hookrightarrow \mathrm{Psh}(X)$$

on sheaves.

The sheafification functor

Proposition: The canonical embedding functor

$$\mathrm{Sh}(X) \xhookrightarrow{j_*} \mathrm{Psh}(X)$$

has a left adjoint

$$\begin{array}{rcl} \mathrm{Psh}(X) & \xrightarrow{j^*} & \mathrm{Sh}(X)\,, \\ P & \longmapsto & j^*P\,, \end{array}$$

characterized by the property that any morphism

$$P \longrightarrow F$$

from a presheaf P to a sheaf F uniquely factorises as

$$P \longrightarrow j^*P \longrightarrow F\,.$$

Remark: The sheafification j^*P of P can be constructed by the formula

$$j^*P(U) = \varinjlim_{\mathcal{U}=(U_i)} \varinjlim_{\mathcal{V}=(V_{i_1,i_2,j})} \mathrm{Eq}\left(\prod_i P(U_i) \rightrightarrows \prod_{i_1,i_2,j} P(V_{i_1,i_2,j})\right)$$

where

- the functor $\varinjlim_{\mathcal{U}}$ is indexed by the filtering ordered set of coverings (U_i) of U,
- for any such covering $\mathcal{U} = (U_i)$, $\varinjlim_{\mathcal{V}}$ is indexed by the filtering ordered set of coverings $(V_{i_1,i_2,j})_j$ of the intersections $U_{i_1} \cap U_{i_2}$.

Exactness properties

Proposition:

(i) The category $\mathrm{Psh}(X)$ has arbitrary limits and colimits and they are component-wise, i.e.

$$\Big(\varprojlim_{D} P_d\Big)(U) = \varprojlim_{D} P_d(U)\,,$$

$$\Big(\varinjlim_{D} P_d\Big)(U) = \varinjlim_{D} P_d(U)\,.$$

(ii) The category $\mathrm{Sh}(X)$ has arbitrary limits and colimits with

$$\Big(\varprojlim_{D} F_d\Big)(U) = \varprojlim_{D} F_d(U)\,,$$

$$\varinjlim_{D} F_d = j^*\Big(\varinjlim_{D} j_* F_d\Big)\,.$$

(iii) The functor

$$j_* : \mathrm{Sh}(X) \longrightarrow \mathrm{Psh}(X)$$

respects arbitrary limits, while its left adjoint

$$j^* : \mathrm{Psh}(X) \longrightarrow \mathrm{Sh}(X)$$

respects arbitrary colimits and finite limits.

Remarks:

(i) For any pair of adjoint functors

$$\left(\mathcal{C} \xrightarrow{F} \mathcal{D},\ \mathcal{D} \xrightarrow{G} \mathcal{C}\right),$$

F respects arbitrary colimits, and
G respects arbitrary limits.

(ii) A functor

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

is called right-exact [resp. left-exact]
if it respects finite colimits
[resp. finite limits].

It is called exact if it respects
both finite limits and finite colimits.

Ex:

$$\begin{array}{llcll} j^* : & \mathrm{Psh}(X) & \longrightarrow & \mathrm{Sh}(X) & \text{is exact,} \\ j_* : & \mathrm{Sh}(X) & \longrightarrow & \mathrm{Psh}(X) & \text{is left-exact.} \end{array}$$

Corollary:

(i) A group object [resp. ring object, resp. module object over a ring object $\mathcal{O}$] of $\mathrm{Sh}(X)$ is a sheaf of sets

$$U \longmapsto \mathcal{G}(U) \qquad [\text{resp.} \quad \mathcal{O}(U)\,,\ \text{resp. } \mathcal{M}(U)\ ]$$

endowed with a structure of group [resp. ring, module over the ring $\mathcal{O}(U)$] on each

$$\mathcal{G}(U) \qquad [\text{resp.} \quad \mathcal{O}(U)\,,\ \text{resp. } \mathcal{M}(U)\ ]$$

such that all restriction maps

$$\mathcal{G}(U) \longrightarrow \mathcal{G}(V) \qquad [\text{resp.} \quad \mathcal{O}(U) \to \mathcal{O}(V)\,,\ \text{resp. } \mathcal{M}(U) \to \mathcal{M}(V)\ ]$$

are groups [resp. ring, resp. module] morphisms.

(ii) A morphism of group objects [resp. ring objects, resp. module objects over some ring object $\mathcal{O}$] is a morphism of sheaves

$$\mathcal{G}_1 \longrightarrow \mathcal{G}_2 \qquad [\text{resp.} \quad \mathcal{O}_1 \to \mathcal{O}_2\,,\ \text{resp. } \mathcal{M}_1 \to \mathcal{M}_2\ ]$$

such that all maps

$$\mathcal{G}_1(U) \longrightarrow \mathcal{G}_2(U) \qquad [\text{resp.} \quad \mathcal{O}_1(U) \to \mathcal{O}_2(U)\,,\ \text{resp. } \mathcal{M}_1(U) \to \mathcal{M}_2(U)\ ]$$

are group [resp. ring, resp. module] morphisms.

The abelian categories of Modules

Definition:
Let $(X, \mathcal{O}_X) =$ ringed space
$= $ topological space X
$+$ ring object $\mathcal{O}_X$ of $\mathrm{Sh}(X)$.

Then module objects over $\mathcal{O}_X$ in $\mathrm{Sh}(X)$
are called $\mathcal{O}_X$-Modules, and their category is denoted

$$\mathcal{M}od_{\mathcal{O}_X} .$$

Proposition:
For any ringed space,

$$\mathcal{M}od_{\mathcal{O}_X}$$

is an abelian category
with arbitrary limits and colimits.

Definition:

(i) A category $\mathcal{A}$ is called additive if

- it has arbitrary finite products and coproducts, in particular a terminal object 1 and an initial object 0,
- the canonical morphism $0 \rightarrow 1$ is an isomorphism,
- for any object M, the morphism

$$M \amalg M \longrightarrow M \times M$$

defined by the matrix

$$\begin{pmatrix} \mathrm{id}_M & M \rightarrow 1 = 0 \rightarrow M \\ \\ M \rightarrow 1 = 0 \rightarrow M & \mathrm{id}_M \end{pmatrix}$$

is an isomorphism,

- for any objects M and N, the morphism

$$M \times M = M \amalg M \xrightarrow{(\mathrm{id}_M, \mathrm{id}_M)} M$$

defines by composition a law

$$\mathrm{Hom}(N, M) \times \mathrm{Hom}(N, M) \longrightarrow \mathrm{Hom}(N, M)$$

which makes $\mathrm{Hom}(N, M)$ an abelian group whose 0 element is

$$N \longrightarrow 1 = 0 \longrightarrow M\,.$$

(ii) A category $\mathcal{A}$ is abelian if

- it is additive,
- it has arbitrarily finite limits and colimits or, equivalently, any morphism

$$M_1 \xrightarrow{u} M_2$$

has a kernel

$$\mathrm{Ker}(u) = M_1 \times_{M_2} 0$$

and a cokernel

$$\mathrm{Coker}(u) = M_2 \amalg_{M_1} 0\,,$$

- for any such $u : M_1 \to M_2$, the canonical morphism

$$\mathrm{Coker}(\mathrm{Ker}(u) \longrightarrow M_1) \longrightarrow \mathrm{Ker}(M_2 \longrightarrow \mathrm{Coker}(u)) = \mathrm{Im}(u)$$

is an isomorphism.

Remark:

- A functor between additive categories

$$F : \mathcal{A} \longrightarrow \mathcal{A}'$$

is called additive if it respects finite products (or, what is the same, coproducts) or, equivalently, if all maps

$$\mathrm{Hom}(N, M) \longrightarrow \mathrm{Hom}(F(N), F(M))$$

are morphisms of abelian groups.

- Any functor between additive categories which has an adjoint is additive.

Change of structure ring-sheaf

Proposition:
Let X = topological space,
$(\mathcal{O}_1 \to \mathcal{O}_2)$ = morphism of sheaves of rings on X.
Then the forgetful functor

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}_2} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_1}\,, \\ \mathcal{M} & \longmapsto & \mathcal{M}\,, \end{array}$$

has a left adjoint denoted

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}_1} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_2}\,, \\ \mathcal{M} & \longmapsto & \mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{M}\,. \end{array}$$

Remarks:

(i) For any object $\mathcal{M}$ of $\mathcal{M}od_{\mathcal{O}_1}$,

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{M}$$

is constructed as the sheafification of the presheaf

$$U \longmapsto \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{M}(U)\,.$$

(ii) The forgetful functor respects arbitrary limits and colimits while its left adjoint

$$\mathcal{M} \longmapsto \mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{M}$$

respects arbitrary colimits.

Exponentials (or "inner $\mathcal{H}om$") and tensor products

Definition: For any open embedding $U \overset{i}{\hookrightarrow} X$, the inclusion $O(U) \hookrightarrow O(X)$ induces a functor $[O(X)^{\mathrm{op}}, \mathrm{Set}] \to [O(U)^{\mathrm{op}}, \mathrm{Set}]$ which restricts to a functor called the restriction functor

$$\begin{array}{rcl} i^* : \mathrm{Sh}(X) & \longrightarrow & \mathrm{Sh}(U)\,, \\ F & \longmapsto & F_{|U}\,. \end{array}$$

Remarks:

(i) Restriction functors respect arbitrary limits and colimits. In particular, they transform any ring object $\mathcal{O}_X$ of $\mathrm{Sh}(X)$ into ring objects $\mathcal{O}_{X|U} = \mathcal{O}_U$ and induce additive exact functors

$$\mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_U}\,.$$

(ii) For any sheaves F_1 and F_2 on X, the presheaf

$$U \longmapsto \mathrm{Hom}(F_{1|U}, F_{2|U})$$

is a sheaf denoted $F_2^{F_1}$ or $\mathcal{H}om(F_1, F_2)$. It is characterised by the property that, for any sheaf G,

$$\mathrm{Hom}(G, \mathcal{H}om(F_1, F_2)) \quad \text{identifies with} \quad \mathrm{Hom}(G \times F_1, F_2)\,.$$

(iii) In the same way, for any $\mathcal{O}_X$-Modules $\mathcal{M}_1, \mathcal{M}_2$, the presheaf

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}_{1|U}, \mathcal{M}_{2|U})$$

is a sheaf denoted $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_1, \mathcal{M}_2)$.

Proposition:

Let $(X, \mathcal{O}_X)$ = commutative ringed space
= topological space X + commutative ring object $\mathcal{O}_X$ of $\mathrm{Sh}(X)$,
$\mathcal{N} = \mathcal{O}_X$-Module.

Then the functor

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}_X} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_X}\,, \\ \mathcal{L} & \longmapsto & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{L}) \end{array}$$

has a left adjoint denoted

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}_X} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_X}\,, \\ \mathcal{M} & \longmapsto & \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}\,. \end{array}$$

Furthermore, $\otimes$ extends as a double functor

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}_X} \times \mathcal{M}od_{\mathcal{O}_X} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_X}\,, \\ (\mathcal{M}, \mathcal{N}) & \longmapsto & \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \end{array}$$

such that the two triple functors

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}_X}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}_X}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}_X} & \longrightarrow & \mathcal{O}_X(X)\text{-modules}\,, \\ (\mathcal{M}, \mathcal{N}, \mathcal{L}) & \longmapsto & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{L})\,, \\ (\mathcal{M}, \mathcal{N}, \mathcal{L}) & \longmapsto & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{L})) \end{array}$$

are isomorphic.

Remarks:

(i) The tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is constructed as the sheafification of the functor

$$U \longmapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)\,.$$

(ii) The two functors $\mathcal{M}od_{\mathcal{O}_X} \times \mathcal{M}od_{\mathcal{O}_X} \to \mathcal{M}od_{\mathcal{O}_X}$

$$\begin{array}{cccc} & (\mathcal{M},\mathcal{N}) & \longmapsto & \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \\ \text{and} & (\mathcal{M},\mathcal{N}) & \longmapsto & \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M} \end{array}$$

are canonically isomorphic.

(iii) The double functor

$$(\mathcal{M},\mathcal{N}) \longmapsto \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$$

respects arbitrary colimits in $\mathcal{M}$ or $\mathcal{N}$,
while the double functor

$$(\mathcal{N},\mathcal{L}) \longmapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{L})$$

respects arbitrarily limits in $\mathcal{L}$
and transforms arbitrary colimits in $\mathcal{N}$ into limits.

Push-forward and pull-back functors

Proposition:

Let $(X \xrightarrow{f} Y) =$ continuous map between topological spaces.

(i) The functor

$$\mathrm{Psh}(X) = [O(X)^{\mathrm{op}}, \mathrm{Set}] \longrightarrow [O(Y)^{\mathrm{op}}, \mathrm{Set}] = \mathrm{Psh}(Y)$$

induced by the order-preserving map $f^{-1} : O(Y) \to O(X)$ restricts to a functor

$$f_* : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y)\,.$$

(ii) This functor f_* has a left adjoint

$$f^{-1} : \mathrm{Sh}(Y) \longrightarrow \mathrm{Sh}(X)$$

which preserves not only arbitrary colimits
but also finite limits.

Remarks:

(i) The functor $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ assigns to each sheaf F on Y the sheaf $f^{-1}(F)$ on X obtained as the sheafification of the presheaf

$$U \longmapsto \varinjlim_{\substack{V \subset Y \\ f^{-1}(V) \supset U}} F(V)$$

(ii) Both functors f_* and f^{-1} are left-exact.

So they transform group objects into group objects, ring objects into ring objects and define additive functors

$$f_* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{f_*\mathcal{O}_X} \quad \text{(which is left-exact)},$$

$$f^{-1} : \mathcal{M}od_{\mathcal{O}_Y} \longrightarrow \mathcal{M}od_{f^{-1}\mathcal{O}_Y} \quad \text{(which is exact)}.$$

Corollary:

Let $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$

$=$ morphism of ringed spaces,

$=$ continuous map $f : X \to Y$

$+$ morphism of sheaves of rings

$\mathcal{O}_Y \to f_* \mathcal{O}_X$ or, equivalently, $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Then:

(i) The composition of the functor

$$f_* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{f_*\mathcal{O}_X}$$

and of the forgetful functor defines a functor

$$f_* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_Y} .$$

(ii) This functor $f_* : \mathcal{M}od_{\mathcal{O}_X} \to \mathcal{M}od_{\mathcal{O}_Y}$ has a left adjoint functor

$$f^* : \mathcal{M}od_{\mathcal{O}_Y} \longrightarrow \mathcal{M}od_{\mathcal{O}_X}$$

constructed as the composition of the functors

$$f^{-1} : \mathcal{M}od_{\mathcal{O}_Y} \longrightarrow \mathcal{M}od_{f^{-1}\mathcal{O}_Y}$$

and

$$\begin{array}{rcl} \mathcal{M}od_{f^{-1}\mathcal{O}_Y} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_X} , \\ \mathcal{M} & \longmapsto & \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{M} . \end{array}$$

Remark:

$$f_* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_Y} \quad \text{respects limits,}$$

$$f^* : \mathcal{M}od_{\mathcal{O}_Y} \longrightarrow \mathcal{M}od_{\mathcal{O}_X} \quad \text{respects colimits.}$$

Extension by 0

Proposition:
Let $(X, \mathcal{O}_X) =$ ringed space,

$(U \overset{i}{\hookrightarrow} X) =$ open subspace endowed with $\mathcal{O}_U = \mathcal{O}_{X|U}$.

Then the restriction functor

$$i^* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_U}$$

has also a left adjoint functor

$$i_! : \mathcal{M}od_{\mathcal{O}_U} \longrightarrow \mathcal{M}od_{\mathcal{O}_X}$$

defined as

$$\mathcal{M} \longmapsto i_!\mathcal{M} = \Big[\underset{\substack{\| \\ \text{open subset} \\ \text{of } X}}{V} \longmapsto \{m \in \mathcal{M}(U \cap V) \mid \operatorname{supp}(m) \text{ is closed in } V\} \Big].$$

Reminder: For $m \in \mathcal{M}(U)$, the support of m is
$\operatorname{supp}(m) =$ smallest closed subset Z of U such that $m = 0$ on $U - Z$.

Remark: For any $x \in X$, the fiber of $i_!\mathcal{M}$ at x is

$$(i_!\mathcal{M})_x = \begin{cases} \mathcal{M}_x & \text{if} \quad x \in U\,, \\ 0 & \text{if} \quad x \notin U\,. \end{cases}$$

Therefore, the functor $i_!$ is exact.

Derived categories of linear sheaves

Derived categories are formed from any abelian categories, in particular from the categories $\mathcal{M}od_{\mathcal{O}_X}$.

Definition: Let $\mathcal{A}$ = additive category. Then:

(i) One denotes $C(\mathcal{A})$ the additive category of complexes

$$\cdots \longrightarrow A^{-1} \xrightarrow{\mathrm{d}} A^0 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} A^k \xrightarrow{\mathrm{d}} A^{k+1} \xrightarrow{\mathrm{d}} \cdots$$

verifying in any degree $\mathrm{d} \circ \mathrm{d} = 0$.

(ii) One denotes $K(\mathcal{A})$ the additive homotopy category of $\mathcal{A}$ defined in the following way:

- the objects of $K(\mathcal{A})$ are the objects of $C(\mathcal{A})$,
- the morphisms of $K(\mathcal{A})$

$$A^\bullet \longrightarrow B^\bullet$$

are the equivalence classes of morphisms $A^\bullet \to B^\bullet$ of $C(\mathcal{A})$ for the homotopy equivalence relation.

Reminder: Two morphisms $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ are homotopic if there exists a family of morphisms

$$h^k : A^k \longrightarrow B^{k-1}$$

such that

$$f^k - g^k = \mathrm{d} \circ h^k + h^{k+1} \circ \mathrm{d}.$$

Definition: Let $\mathcal{A}$ = additive category.

(i) For any $n \in \mathbb{Z}$, one denotes

$$A \longmapsto A[n]$$

the functor of $C(\mathcal{A})$ or $K(\mathcal{A})$ which associates to any object

$$A = (A^\bullet)$$

the object

$$A[n] = (A[n]^\bullet)$$

defined by $A[n]^k = A^{n+k}$ in any degree
and $\mathrm{d}_{A[n]} = (-1)^n \cdot \mathrm{d}_A$ in any degree k.

(ii) For any morphism $u : A \to B$ of $C(\mathcal{A})$, its "cone" $M(u)$ is the object of $C(\mathcal{A})$ defined by

$$M(u)^n = A^{n+1} \oplus B^n$$

and the differentials

$$\begin{pmatrix} -\mathrm{d} & 0 \\ u^{n+1} & \mathrm{d} \end{pmatrix},$$

endowed with the morphisms

$$B \longrightarrow M(u) \longrightarrow A[1]\,.$$

Corollary: Let $\mathcal{A}$ = abelian category.

(i) The formulas

$$A = (A^\bullet) \longmapsto H^n(A) = \mathrm{Ker}(A^n \xrightarrow{\mathrm{d}} A^{n+1})/\mathrm{Im}(A^{n-1} \xrightarrow{\mathrm{d}} A^n)$$

define functors

$$H^n : C(\mathcal{A}) \longrightarrow \mathcal{A}$$

which factorise as

$$H^n : K(\mathcal{A}) \longrightarrow \mathcal{A}\,.$$

(ii) Any short exact sequence of the abelian category $C(\mathcal{A})$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

yields a long exact sequence of cohomology

$$\cdots \longrightarrow H^n(A) \longrightarrow H^n(B) \longrightarrow H^n(C) \longrightarrow H^{n+1}(A) \longrightarrow H^{n+1}(B) \longrightarrow \cdots$$

and any morphism of such short exact sequences of $C(\mathcal{A})$ yields a morphism of the associated long exact sequences of $\mathcal{A}$.

(iii) This applies in particular to the exact sequences of $C(\mathcal{A})$

$$0 \longrightarrow B \longrightarrow M(u) \longrightarrow A[1] \longrightarrow 0$$

associated to morphisms $u : A \to B$ of $C(\mathcal{A})$, yielding long exact sequences

$$\cdots \longrightarrow H^n(A) \longrightarrow H^n(B) \longrightarrow H^n(M(u)) \longrightarrow H^{n+1}(A) \longrightarrow H^{n+1}(B) \longrightarrow \cdots$$

which depend on $u : A \to B$ in a functorial way, and whose connecting homomorphisms $H^n(A) \longrightarrow H^n(B)$ are the $H^n(u)$'s.

Definition: Let $\mathcal{A}$ = abelian category.
A morphism of $C(\mathcal{A})$ or $K(\mathcal{A})$

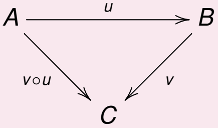
$$A \longrightarrow B$$

is called a quasi-isomorphism if it induces isomorphisms of $\mathcal{A}$

$$H^n(A) \longrightarrow H^n(B) \qquad \text{in all degrees } n.$$

Proposition:

(i) For any commutative triangle of $C(\mathcal{A})$ or $K(\mathcal{A})$



all arrows are quasi-isomorphisms if two of them are.

(ii) In the homotopy category $K(\mathcal{A})$, the collection of quasi-isomorphisms satisfies the Ore condition:
for any morphism $u : A \to B$ [resp. $v : A' \to B$] and any quasi-isomorphism $q : B \to B'$ [resp. $q' : A' \to A$],
there exist a morphism $v : A' \to B$ [resp. $u : A \to B'$] and a quasi-isomorphism $q' : A' \to A$ [resp. $q : B \to B'$] such that $q \circ v = u \circ q'$.

Proof:

(i) is obvious.

(ii) As $\mathcal{A}$ can be replaced by $\mathcal{A}^{\mathrm{op}}$, we only have to consider the case of a morphism $u : A \to B$ and a quasi-isomorphism $q : B' \to B$.

The complex C defined by

$$C^n = B'^{n+1} \oplus B^n \quad \text{and differentials} \quad \begin{pmatrix} -d & 0 \\ q & d \end{pmatrix}$$

is acyclic as q is a quasi-isomorphism.

If A' is the complex defined by

$$A'^n = A^n \oplus C^{n-1} = A^n \oplus (B'^n \oplus B^{n-1}) \quad \text{and differentials} \quad \begin{pmatrix} -d & 0 & 0 \\ 0 & d & 0 \\ u & -q & -d \end{pmatrix},$$

the morphism $A' \to A$ is a quasi-isomorphism as C is acyclic.

Lastly, the two morphisms

$$A' \longrightarrow A \xrightarrow{u} B \quad \text{and} \quad A' \longrightarrow B' \xrightarrow{q} B$$

defined as $(a, b', b) \mapsto u(a)$ and $(a, b', b) \mapsto q(b')$ are related by the homotopy $h = (h^n)$ defined as

$$\begin{array}{rclcl} h^n & : & A'^n = A^n \oplus (B'^n \oplus B^{n-1}) & \longrightarrow & B^{n-1}, \\ & & (a, b', b) & \longmapsto & b \end{array}$$

because $d \circ h^n(a, b', b) = d(b)$

and $h^{n+1} \circ d(a, b', b) = u(a) - q(b') - d(b)$.

Definition: Let $\mathcal{A} =$ abelian category.
The derived category of $\mathcal{A}$
is the additive category $D(\mathcal{A})$ deduced from $K(\mathcal{A})$
by formally inverting quasi-isomorphisms.
In other words, it is characterized up to unique isomorphism
by the following properties:

(1) It is endowed with an additive functor

$$K(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

which transforms quasi-isomorphisms into isomorphisms.

(2) For any additive functor to an additive category

$$K(\mathcal{A}) \longrightarrow \mathcal{D}$$

which transforms quasi-isomorphisms into isomorphisms, there is a unique additive functor

$$D(\mathcal{A}) \longrightarrow \mathcal{D}$$

which factorises $K(\mathcal{A}) \to \mathcal{D}$ as

$$K(\mathcal{A}) \longrightarrow D(\mathcal{A}) \longrightarrow \mathcal{D}\,.$$

Remark:

Thanks to part (ii) of the previous Proposition,
the derived category $D(\mathcal{A})$ can be concretely constructed in the following way:

- Objects of $D(\mathcal{A})$ are the same as the objects of $C(\mathcal{A})$ and $K(\mathcal{A})$.
- Any morphism of $D(\mathcal{A})$ can be formally written as

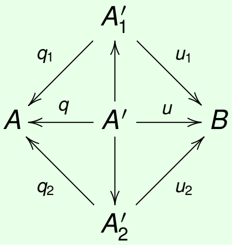
$$u \circ q^{-1} : A \longrightarrow B \qquad \text{[resp.} \quad q^{-1} \circ u : A \longrightarrow B\,]$$

where $q : A' \to A$ [resp. $q : B \to B'$] is a quasi-isomorphism of $K(\mathcal{A})$ and $u : A' \to B$ [resp. $u : A \to B'$] is a morphism of $K(\mathcal{A})$.

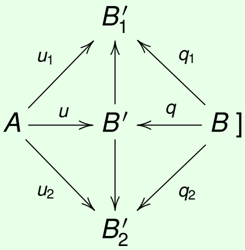
- Two formal writings

$$u_1 \circ q_1^{-1} \quad \text{and} \quad u_2 \circ q_2^{-1} \qquad [\text{resp.} \quad q_1^{-1} \circ u_1 \quad \text{and} \quad q_2^{-1} \circ u_2 \,]$$

define the same morphism of $D(\mathcal{A})$ if and only if there exists a commutative diagram of $K(\mathcal{A})$



[resp.



such that q is a quasi-isomorphism as well as q_1 and q_2.

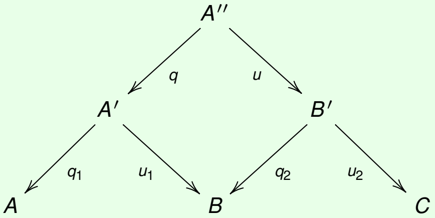
- The composite of two morphisms

$$u_1 \circ q_1^{-1} \quad \text{and} \quad u_2 \circ q_2^{-1} \qquad [\text{resp.} \quad q_1^{-1} \circ u_1 \quad \text{and} \quad q_2^{-1} \circ u_2 \,]$$

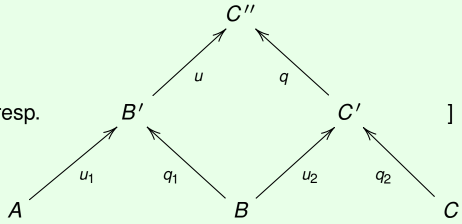
is equal to

$$(u_2 \circ u) \circ (q_1 \circ q)^{-1} \qquad [\text{resp.} \quad (q \circ q_2)^{-1} \circ (u \circ u_1) \,]$$

for any commutative diagram of $K(\mathcal{A})$



[resp.



]

such that q_1, q_2, q are quasi-isomorphisms.

Lemma:

Let $\mathcal{A}$ = abelian category.

The derived category $D(\mathcal{A})$ inherits from $C(\mathcal{A})$ and $K(\mathcal{A})$ functors

$$\begin{array}{rcl} [n] : D(\mathcal{A}) & \longrightarrow & D(\mathcal{A}) \,, \\ A & \longmapsto & A[n] \end{array}$$

and

$$\begin{array}{rcl} H^n : D(\mathcal{A}) & \longrightarrow & \mathcal{A} \,, \\ A & \longmapsto & H^n(A) \end{array}$$

such that

$$[n] \circ [m] = [n+m] \,, \quad \forall\, n, m \,,$$

and

$$H^n \circ [m] = H^{n+m} \,, \quad \forall\, n, m \,.$$

Definition:
Let $\mathcal{A}$ = abelian category.
One denotes

$$D^+(\mathcal{A}), \ D^-(\mathcal{A}) \quad \text{and} \quad D^b(\mathcal{A})$$

the full additive subcategories of $D(\mathcal{A})$
on objects A such that

$$H^n(A) = 0 \quad \text{for} \quad \begin{cases} n \ll 0 & \text{in the case } D^+(\mathcal{A}), \\ n \gg 0 & \text{in the case } D^-(\mathcal{A}), \\ |n| \gg 0 & \text{in the case } D^b(\mathcal{A}). \end{cases}$$

Remarks:

(i) $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$
are equivalent to the full additive subcategories of $D(\mathcal{A})$
on objects $A = (A^\bullet)$ such that

$$A^n = 0 \quad \text{for} \quad \begin{cases} n \ll 0, \\ n \gg 0, \\ |n| \gg 0. \end{cases}$$

(ii) These full subcategories are respected by the functors $[m]$, $m \in \mathbb{Z}$.

Definition:

(i) A triangle of $D(\mathcal{A})$ is a diagram

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

and a morphism of triangles is a commutative diagram:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

(ii) A triangle of $D(\mathcal{A})$ is called "distinguished" if it is isomorphic to a triangle of the form

$$A \xrightarrow{u} B \longrightarrow M(u) \longrightarrow A[1]$$

where $u : A \to B$ is a morphism of $C(\mathcal{A})$ and $M(u)$ is its cone.

Lemma:

Any short exact sequence of the category $C(\mathcal{A})$

$$0 \longrightarrow A \xrightarrow{u} B \longrightarrow C \longrightarrow 0$$

yields a quasi-isomorphism $M(u) \to C$ in $C(\mathcal{A})$ and so defines a distinguished triangle of $D(\mathcal{A})$

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]\,.$$

Proposition:

(i) The notion of distinguished triangle is stable under rotation: that is,

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

is distinguished if and only if

$$B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u} B[1]$$

is distinguished.

(ii) Any distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

yields a long exact sequence of cohomology

$$\cdots \longrightarrow H^n(A) \longrightarrow H^n(B) \longrightarrow H^n(C) \longrightarrow H^{n+1}(A) \longrightarrow \cdots$$

(iii) For any object A of $\mathcal{A}$, the triangle

$$A \xrightarrow{\text{id}} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished.

(iv) Any morphism $A \to B$ of $D(\mathcal{A})$
can be completed in a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]\,.$$

(v) For any distinguished triangles

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]\,,$$

$$A' \longrightarrow B' \longrightarrow C' \longrightarrow A'[1]\,,$$

any commutative diagram of $D(\mathcal{A})$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

can be completed (not uniquely in general) to a morphism of triangles:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

Proof:

(i) Consider a morphism $A \xrightarrow{u} B$ of $C(\mathcal{A})$, its cone $M(u) = C$ defined as

$$C^n = A^{n+1} \oplus B^n \quad \text{with differentials} \quad \begin{pmatrix} -d & 0 \\ u & d \end{pmatrix}$$

and the cone D of $B \to C$ defined as

$$D^n = B^{n+1} \oplus C^n = B^{n+1} \oplus (A^{n+1} \oplus B^n) \quad \text{with differentials} \quad \begin{pmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ \mathrm{id} & u & d \end{pmatrix}.$$

The projections $D^n = B^{n+1} \oplus (A^{n+1} \oplus B^n) \to A^{n+1}$ define a morphism $D \to A[1]$ such that the square

$$\begin{array}{ccc} C & \longrightarrow & D \\ \| & & \downarrow \\ C & \longrightarrow & A[1] \end{array}$$

is commutative.

Furthermore, the square

$$\begin{array}{ccc} D & \longrightarrow & B[1] \\ \downarrow & & \downarrow = \\ A[1] & \xrightarrow{-u} & B[1] \end{array}$$

is commutative up to the homotopy $h = (h^n)$ defined as

$$\begin{array}{rcccl} h^n & : & D^n = B^{n+1} \oplus (A^{n+1} \oplus B^n) & \longrightarrow & B[1]^{n-1} = B^n \,, \\ & & (b, a, b') & \longmapsto & b' \end{array}$$

because $d \circ h^n(b, a, b') = -d(b')$
and $h^n \circ d(b, a, b') = b + u(a) + d(b')$.

(ii) follows from the corresponding statement for cones of $C(\mathcal{A})$

$$A \xrightarrow{u} B \longrightarrow M(u) \longrightarrow A[1].$$

(iii) is a consequence of (i).

(iv) follows from the fact that any morphism of $C(\mathcal{A})$ has a cone.

(v) reduces to the corresponding statement for $K(\mathcal{A})$ which is obvious on the definition of cones.

Remarks:

(i) A morphism of distinguished triangles of $D(\mathcal{A})$

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

is an isomorphism if two of the three arrows

$$a, b, c$$

are isomorphisms of $D(\mathcal{A})$.

(ii) In a distinguished triangle of $D(\mathcal{A})$

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

the objects A, B, C are in the subcategory

$$D^+(\mathcal{A}),\ D^-(\mathcal{A}) \text{ or } D^b(\mathcal{A})$$

if two of them are.

Application: Any ringed space $(X, \mathcal{O}_X)$ defines derived categories

$$D(\mathcal{M}od_{\mathcal{O}_X}),\ D^+(\mathcal{M}od_{\mathcal{O}_X}),\ D^-(\mathcal{M}od_{\mathcal{O}_X}),\ D^b(\mathcal{M}od_{\mathcal{O}_X})$$

endowed with functors $[m]$ and H^n plus a notion of distinguished triangle.

Derived functors

Proposition:
Let $F : \mathcal{A} \to \mathcal{B} =$ exact additive functor between abelian categories.
Then:

(i) The induced functor

$$\begin{array}{rcl} K(\mathcal{A}) & \longrightarrow & K(\mathcal{B})\,, \\ A = (A^n)_{n\in\mathbb{Z}} & \longmapsto & F(A) = F(A^n))_{n\in\mathbb{Z}} \end{array}$$

respects quasi-isomorphisms.

(ii) It induces a functor

$$F : D(\mathcal{A}) \longrightarrow D(\mathcal{B})$$

which

- commutes with the functors $[m]$, $m \in \mathbb{Z}$,
- respects distinguished triangles,
- is endowed with canonical isomorphisms

$$H^n \circ F \xrightarrow{\sim} F \circ H^n$$

of functors $D(\mathcal{A}) \to \mathcal{B}$.

Application:

(i) Any morphism of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

defines a functor

$$f^{-1} : D(\mathcal{M}od_{\mathcal{O}_Y}) \longrightarrow D(\mathcal{M}od_{f^{-1}\mathcal{O}_Y})$$

which commutes with the functors $[m]$, respects distinguished triangles, and commutes with the functors H^m.

(ii) Any open embedding in a ringed space

$$(U, \mathcal{O}_X) \overset{i}{\hookrightarrow} (X, \mathcal{O}_X), \qquad \text{with} \quad \mathcal{O}_U = \mathcal{O}_{X|U},$$

defines two functors

$$\begin{array}{rcccc} i^{-1} = i^* & : & D(\mathcal{M}od_{\mathcal{O}_X}) & \longrightarrow & D(\mathcal{M}od_{\mathcal{O}_U}) \\ i_! & : & D(\mathcal{M}od_{\mathcal{O}_U}) & \longrightarrow & D(\mathcal{M}od_{\mathcal{O}_X}) \end{array}$$

which commute with the functors $[m]$, respect distinguished triangles and commute with the functors H^n.

Furthermore, $i_!$ is left adjoint to i^*.

Remark: These functors f^{-1}, i^* or $i_!$ send the subcategories $D^+(-)$, $D^-(-)$ and $D^b(-)$ to the subcategories $D^+(-)$, $D^-(-)$ and $D^b(-)$.

Lemma:
Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$
$=$ two exact additive functors between abelian categories.

Then
$$G \circ F \cdot \mathcal{A} \longrightarrow \mathcal{C}$$
is an exact additive functor
and the diagram of induced functor

$$\begin{array}{ccc} D(\mathcal{A}) & \xrightarrow{\quad G \circ F \quad} & D(\mathcal{C}) \\ & {}_{F}\searrow \quad \nearrow_{G} & \\ & D(\mathcal{B}) & \end{array}$$

is commutative.

Application: The formation of the functors
$$f^{-1}\,,\ i^{*} \quad \text{or} \quad i_{!}$$
between derived categories of linear sheaves
associated to a morphism of ringed spaces $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$
or to an open embedding $U \overset{i}{\hookrightarrow} X$
is compatible with composition.

Definition:

Let $F : \mathcal{A} \to \mathcal{B}$

$=$ additive functor between abelian categories which is left-exact [resp. right-exact].

A derived functor of F is a functor

$$\begin{array}{ccc} \mathrm{R}F & : & D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}) \\ \text{or} & & D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B}) \\ \text{or} & & D(\mathcal{A}) \longrightarrow D(\mathcal{B}) \end{array} \quad \text{[resp.} \quad \begin{array}{ccc} \mathrm{L}G & : & D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B}) \\ \text{or} & & D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B}) \\ \text{or} & & D(\mathcal{A}) \longrightarrow D(\mathcal{B}) \end{array} \quad \text{]}$$

such that:

(1) $\mathrm{R}F$ [resp. $\mathrm{L}F$] commutes with the functors $[m]$ and respects distinguished triangles

(2) Denoting Q the quotient functors

$$K(\mathcal{A}) \longrightarrow D(\mathcal{A}) \qquad \text{and} \qquad K(\mathcal{B}) \longrightarrow D(\mathcal{B}),$$

$\mathrm{R}F$ [resp. $\mathrm{L}F$] is endowed with a morphism of composite functors

$$Q \circ F \longrightarrow \mathrm{R}F \circ Q \qquad \text{[resp.} \quad \mathrm{L}F \circ Q \longrightarrow Q \circ F \text{]}$$

(3) $\mathrm{R}F$ [resp. $\mathrm{L}F$] is universal with respect to these properties in the sense that for any functor $\mathrm{R}'F$ [resp. $\mathrm{L}'F$] verifying (1) and (2), there is a morphism of functors

$$\mathrm{R}F \longrightarrow \mathrm{R}'F \qquad [\text{resp.} \quad \mathrm{L}'F \longrightarrow \mathrm{L}F\,],$$

unique up to isomorphism, such that

$$Q \circ F \longrightarrow \mathrm{R}'F \circ Q \qquad [\text{resp.} \quad \mathrm{L}'F \circ Q \longrightarrow Q \circ F\,]$$

is isomorphic to

$$Q \circ F \longrightarrow \mathrm{R}F \circ Q \longrightarrow \mathrm{R}'F \circ Q \qquad [\text{resp.} \quad \mathrm{L}'F \circ Q \longrightarrow \mathrm{L}F \circ Q \longrightarrow Q \circ F\,].$$

Remarks:

(i) If $F : \mathcal{A} \to \mathcal{B}$ is left-exact [resp. right-exact] and has a right [resp. left] derived functor

$$\mathrm{R}F \qquad [\text{resp.} \quad \mathrm{L}F\,],$$

the composed functors

$$\mathcal{A} \longrightarrow D^+(\mathcal{A}) \xrightarrow{\mathrm{R}F} D^+(\mathcal{B}) \xrightarrow{H^k} \mathcal{B}$$

$$[\text{resp.} \qquad \mathcal{A} \longrightarrow D^-(\mathcal{A}) \xrightarrow{\mathrm{L}F} D^-(\mathcal{B}) \xrightarrow{H^{-k}} \mathcal{B}\,]$$

are denoted

$$\mathrm{R}^k F \qquad [\text{resp.} \quad \mathrm{L}^k F\,].$$

(ii) In practice, derived functors are always constructed from a full additive subcategory $\mathcal{I}$ of $\mathcal{A}$ which is F-acyclic and big enough in the sense of the following definition.

(iii) In that case, the functors

$$\mathrm{R}^k F \qquad [\text{resp.} \quad \mathrm{L}^k F\,]$$

are 0 for any $k < 0$ and the functor

$$\mathrm{R}^0 F \qquad [\text{resp.} \quad \mathrm{L}^0 F\,]$$

identifies with F.

(iv) As a consequence, any short exact sequence of $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

yields a long exact sequence of $\mathcal{B}$

$$0 \to F(A) \to F(B) \to F(C) \to \mathrm{R}^1F(A) \to \mathrm{R}^1F(B) \to \cdots \mathrm{R}^kF(C) \to \mathrm{R}^{k+1}F(A) \to \cdots$$

[resp.

$$\cdots \to \mathrm{L}^{k+1}F(C) \to \mathrm{L}^kF(A) \to \cdots \to \mathrm{L}^1F(B) \to \mathrm{L}^1F(C) \to F(A) \to F(B) \to F(C) \to 0\,].$$

(v) An object A of $\mathcal{A}$ is called F-acyclic if

$$\mathrm{R}^kF(A) = 0\,, \quad \forall\, k \geq 1\,.$$

(vi) The full additive category of $\mathcal{A}$ on F-acyclic objects is an "F-acyclic category" in the sense of the following definition.

Definition:

Let $F : \mathcal{A} \to \mathcal{B}$

$=$ additive functor between abelian categories which is left-exact [resp. right-exact],

$\mathcal{I} =$ full additive subcategory of $\mathcal{A}$.

Then:

(i) $\mathcal{I}$ is called "F-acyclic" if, for any short exact sequence of $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\,,$$

- the induced short exact sequence of $\mathcal{B}$

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

  is exact if A [resp. C] is an object of $\mathcal{I}$,
- the object C [resp. A] is in $\mathcal{I}$ if A, B [resp. B, C] are in $\mathcal{I}$.

(ii) $\mathcal{I}$ is called "big enough" if, for any object A of $\mathcal{A}$, there is an object I of $\mathcal{I}$ and a monomorphism $A \hookrightarrow I$ [resp. an epimorphism $I \twoheadrightarrow A$].

(iii) $\mathcal{I}$ is called "of codimension $\leq d$" if, for any exact sequence of $\mathcal{A}$ of length d

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_d \longrightarrow 0\,,$$

$$[\text{resp.} \qquad 0 \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_d \qquad],$$

the object A_d [resp. A_0] is in $\mathcal{I}$ if $A_0, \ldots, A_{d-1}$ [resp. $A_1, \ldots, A_d$] are in $\mathcal{I}$.

Proposition:

Let $F : \mathcal{A} \to \mathcal{B}$

$=$ additive functor between abelian categories which is left-exact [resp. right-exact],

$\mathcal{I} =$ full additive subcategory of $\mathcal{A}$ which is F-acyclic.

Then:

(i) If $\mathcal{I}$ is big enough,
there exist for any object A of $C^+(\mathcal{A})$ [resp. $C^-(\mathcal{A})$]
an object I of $C^+(\mathcal{I})$ [resp. $C^-(\mathcal{I})$]
and a quasi-isomorphism in $C^+(\mathcal{A})$

$$A \longrightarrow I \qquad [\text{resp.} \quad I \longrightarrow A\,].$$

Furthermore, F transforms any quasi-isomorphism

$$I_1 \longrightarrow I_2$$

between objects of $C^+(\mathcal{I})$ [resp. $C^-(\mathcal{I})$]
into a quasi-isomorphism of $C^+(\mathcal{B})$ [resp. $C^-(\mathcal{B})$]

$$F(I_1) \longrightarrow F(I_2)\,.$$

(ii) If $\mathcal{I}$ is big enough and of codimension $\leq d$,
there exist for any object A of $C(\mathcal{A})$ or $C^b(\mathcal{A})$
an object I of $C(\mathcal{I})$ or $C^b(\mathcal{I})$
and a quasi-isomorphism in $C(\mathcal{A})$

$$A \longrightarrow I \qquad [\text{resp.} \quad I \longrightarrow A\,].$$

Furthermore, F transforms any quasi-isomorphism

$$I_1 \longrightarrow I_2$$

between objects of $C(\mathcal{I})$
into a quasi-isomorphism of $C(\mathcal{B})$.

Remark:
Any such quasi-isomorphism

$$A \longrightarrow I \qquad [\text{resp.} \quad I \longrightarrow A\,]$$

is called a “resolution” of A
by a complex of the F-acyclic category $\mathcal{I}$.

Sketch of proof of the proposition:
As one can replace $\mathcal{A}$ by $\mathcal{A}^{\mathrm{op}}$, it is enough to consider the case where F is left-exact.

Existence of resolutions:
Let's consider a complex $A = (A^\bullet)$ of $C(\mathcal{A})$.

If $\mathcal{I}$ is big enough and $A^n = 0$ for $n \ll 0$ [resp. and $\mathcal{I}$ has codimension $\leq d$], one can construct a double complex $(I^{n,k})_{n \in \mathbb{Z}, k \in \mathbb{N}}$ of objects of $\mathcal{I}$ inserted in a commutative diagram

$$
\begin{array}{ccccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & \\
0 & \longrightarrow & A^{n+1} & \longrightarrow & I^{n+1,0} & \longrightarrow & I^{n+1,1} & \longrightarrow & \cdots & \longrightarrow & I^{n+1,k} & \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & \\
0 & \longrightarrow & A^{n} & \longrightarrow & I^{n,0} & \longrightarrow & I^{n,1} & \longrightarrow & \cdots & \longrightarrow & I^{n,k} & \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & \\
 & & \vdots & & \vdots & & \vdots & & & & \vdots &
\end{array}
$$

and such that

$$\begin{cases} \bullet \quad \text{each horizontal sequence is exact,} \\ \bullet \quad \text{if } A^n \text{ is } 0, \text{ all } I^{n,k}\text{'s are } 0, \\ \bullet \quad \text{if } \mathcal{I} \text{ has codimension} \leq d, \text{ then } I^{n,k} = 0 \text{ if } k > d. \end{cases}$$

Then there is a quasi-isomorphism

$$A \longrightarrow I$$

to the complex $I = (I^\bullet)$ defined by

$$I^n = \bigoplus_{m+k=n} I^{m,k}, \quad \forall\, n.$$

Preservation of quasi-isomorphisms

A morphism of $C(\mathcal{I})$

$$u : I_1 \longrightarrow I_2$$

is a quasi-isomorphism if and only if the complex $M(u)$ is an exact sequence.

So we are reduced to proving that F transforms any long exact sequence of objects of $\mathcal{I}$

$$\longrightarrow I^{n-1} \xrightarrow{\mathrm{d}^{n-1}} I^n \xrightarrow{\mathrm{d}^n} I^{n+1} \longrightarrow \cdots$$

into a long exact sequence of $\mathcal{B}$ if $\mathcal{I}$ is F-acyclic
and $I^n = 0$ for $n \ll 0$ [resp. and $\mathcal{I}$ has codimension $\leq d$].

In both cases, our long exact sequence decomposes into short exact sequences

$$0 \longrightarrow \mathrm{Im}(\mathrm{d}^{n-1}) \longrightarrow I^n \longrightarrow \mathrm{Im}(\mathrm{d}^n) \longrightarrow 0$$

whose objects I^n and $\mathrm{Im}(\mathrm{d}^n)$ are all in $\mathcal{I}$.

The conclusion follows.

Corollary:

Let $F : \mathcal{A} \to \mathcal{B}$

$=$ additive functor between abelian categories which is left-exact [resp. right-exact],

$\mathcal{I} =$ full additive subcategory of $\mathcal{A}$ which is F-acyclic.

Then:

(i) If $\mathcal{I}$ is big enough, $D^+(\mathcal{A})$ [resp. $D^-(\mathcal{A})$] is equivalent to the category $D^+(\mathcal{I})$ [resp. $D^-(\mathcal{I})$] deduced from $K^+(\mathcal{I})$ [resp. $K^-(\mathcal{I})$] by formally inverting quasi-isomorphisms.

Furthermore, F has a right [resp. left] derived functor

$$\mathrm{R}F : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}) \qquad \text{[resp.} \quad \mathrm{L}F : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B}) \text{]}$$

whose restriction to $D^+(\mathcal{I})$ [resp. $D^-(\mathcal{I})$] is defined by the commutative square

$$\begin{array}{ccc} K^+(\mathcal{I}) & \xrightarrow{F} & K^+(\mathcal{B}) \\ \downarrow & & \downarrow \\ D^+(\mathcal{I}) & \xrightarrow{\mathrm{R}F} & D^+(\mathcal{B}) \end{array} \qquad \text{[resp.} \quad \begin{array}{ccc} K^-(\mathcal{I}) & \xrightarrow{F} & K^-(\mathcal{B}) \\ \downarrow & & \downarrow \\ D^-(\mathcal{I}) & \xrightarrow{\mathrm{L}F} & D^-(\mathcal{B}) \end{array} \quad \text{].}$$

(ii) If $\mathcal{I}$ is big enough and has codimension $\leq d$, $D^b(\mathcal{A})$ and $D(\mathcal{A})$ are equivalent to the categories $D^b(\mathcal{I})$ and $D(\mathcal{I})$ deduced from $K^b(\mathcal{I})$ and $K(\mathcal{I})$ by formally inverting quasi-isomorphisms.

Furthermore, F has a right [resp. left] derived functor

$$\mathrm{R}F \quad : \quad \begin{array}{l} D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B}), \\ D(\mathcal{A}) \longrightarrow D(\mathcal{B}) \end{array} \quad \text{[resp.} \quad \mathrm{L}F \quad : \quad \begin{array}{l} D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B}), \\ D(\mathcal{A}) \longrightarrow D(\mathcal{B}) \end{array} \text{]}$$

whose restriction to $D^b(\mathcal{I})$ or $D(\mathcal{I})$ are defined by the commutative squares:

$$\begin{array}{ccc} K^b(\mathcal{I}) & \xrightarrow{F} & K^b(\mathcal{B}) \\ \downarrow & & \downarrow \\ D^b(\mathcal{I}) & \xrightarrow[\text{[resp. } \mathrm{L}F\text{]}]{\mathrm{R}F} & D^b(\mathcal{B}) \end{array} \quad \text{and} \quad \begin{array}{ccc} K(\mathcal{I}) & \xrightarrow{F} & K(\mathcal{B}) \\ \downarrow & & \downarrow \\ D(\mathcal{I}) & \xrightarrow[\text{[resp. } \mathrm{L}F\text{]}]{\mathrm{R}F} & D(\mathcal{B}) \end{array}$$

Remark:

- If $\mathcal{A}$ contains a full additive subcategory $\mathcal{I}$ which is F-acyclic and big enough, then the full subcategory of F-acyclic objects (which is automatically F-acyclic itself) contains $\mathcal{I}$ and is a fortiori big enough.
- In that case, the subcategory of F-acyclic objects has codimension $\leq d$ if and only if the derived functors

$$\mathrm{R}^k F \qquad [\text{resp.} \quad \mathrm{L}^k F\,]$$

are 0 in all degrees $k > d$.

Indeed, any exact sequence of $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

whose middle object B is F-acyclic yields isomorphisms

$$\mathrm{R}^k F(C) \xrightarrow{\sim} \mathrm{R}^{k+1} F(A) \qquad [\text{resp.} \quad \mathrm{L}^{k+1}(C) \xrightarrow{\sim} \mathrm{L}^k F(A)\,]$$

in all degrees $k \geq 1$.
- We say F has cohomological dimension $\leq d$ if this condition is verified.

Corollary:

Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$
$=$ additive functors between abelian categories
which are left-exact [resp. right-exact],
$\mathcal{I}, \mathcal{J}$ $=$ full additive subcategories of $\mathcal{A}$ and $\mathcal{B}$
such that $\mathcal{I}$ is F-acyclic, $\mathcal{J}$ is G-acyclic
and F sends $\mathcal{I}$ to $\mathcal{J}$.

Then:

(i) If $\mathcal{I}$ and $\mathcal{J}$ are big enough

$$\mathrm{R}(G \circ F) : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{C}) \qquad \text{[resp.} \quad \mathrm{L}(G \circ F) : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{C}) \text{]}$$

is isomorphic to the composed morphism

$$\mathrm{R}G \circ \mathrm{R}F : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}) \longrightarrow D^+(\mathcal{C})$$

$$\text{[resp.} \qquad \mathrm{L}G \circ \mathrm{L}F : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B}) \longrightarrow D^-(\mathcal{C}) \text{]}$$

and its restriction to $D^+(\mathcal{I})$ [resp. $D^+(\mathcal{I})$] is defined by the commutative square

$$\begin{array}{ccccc} K^+(\mathcal{I}) & \xrightarrow{F} & K^+(\mathcal{J}) & \xrightarrow{G} & K^+(\mathcal{B}) \\ \downarrow & & & & \downarrow \\ D^+(\mathcal{I}) & & \xrightarrow{\mathrm{R}(G\circ F)} & & D^+(\mathcal{B}) \end{array}$$

[resp.

$$\begin{array}{ccccc} K^-(\mathcal{I}) & \xrightarrow{F} & K^-(\mathcal{J}) & \xrightarrow{G} & K^-(\mathcal{B}) \\ \downarrow & & & & \downarrow \\ D^-(\mathcal{I}) & & \xrightarrow{\mathrm{L}(G\circ F)} & & D^-(\mathcal{B}) \end{array}$$

].

(ii) If, furthermore, F has cohomological dimension $\leq d$ and G has cohomological dimension $\leq d'$, then $G \circ F$ has cohomological dimension $\leq d + d'$ and has derived functors

$$\mathrm{R}(G \circ F) \qquad [\text{resp.} \quad \mathrm{L}(G \circ F)] \quad : \quad \begin{array}{l} D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{C})\,, \\ D(\mathcal{A}) \longrightarrow D(\mathcal{C}) \end{array}$$

which are isomorphic to the composites

$$\mathrm{R}G \circ \mathrm{R}F \qquad [\text{resp.} \quad \mathrm{L}G \circ \mathrm{L}F] \quad : \quad \begin{array}{l} D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B}) \longrightarrow D^b(\mathcal{C})\,, \\ D(\mathcal{A}) \longrightarrow D(\mathcal{B}) \longrightarrow D(\mathcal{C})\,. \end{array}$$

Proof:

(i) is obvious.

(ii) Under these hypotheses, the full additive subcategory $\mathcal{I}'$ of $\mathcal{A}$ on the objects A which are F-acyclic and $G \circ F$-acyclic and whose transform $F(A)$ is G-acyclic, contains $\mathcal{I}$ and it has codimension $\leq d + d'$.

Indeed, for any exact sequence

$$I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{d+d'} \longrightarrow 0 \qquad [\text{resp.} \quad 0 \longrightarrow I_{d+d'} \longrightarrow \cdots \longrightarrow I_1 \longrightarrow I_0\,]$$

with $I_0, \ldots, I_{d+d'-1}$ in $\mathcal{I}'$, $I_{d+d'}$ belongs to $\mathcal{I}'$.

Application to linear sheaves

Proposition: Let $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) =$ morphism of ringed spaces.

(i) Let $\mathcal{F}b_{\mathcal{O}_X} =$ full additive subcategory of $\mathcal{M}od_{\mathcal{O}_X}$ on the sheaves $\mathcal{M}$ which are "flabby" in the sense that, for any $U \subset X$ open,

$$\mathcal{M}(X) \longrightarrow \mathcal{M}(U)$$

is surjective.

Then $\mathcal{F}b_{\mathcal{O}_X}$ is f_*-acyclic and big enough.
Furthermore, f_* sends $\mathcal{F}b_{\mathcal{O}_X}$ into $\mathcal{F}b_{\mathcal{O}_Y}$.

(ii) Let $\mathcal{P}f_{\mathcal{O}_Y} =$ full additive subcategory of $\mathcal{M}od_{\mathcal{O}_Y}$ on the sheaves $\mathcal{N}$ whose fibers

$$\mathcal{N}_y = \varinjlim_{V \ni y} \mathcal{N}(V)\,, \quad y \in Y\,,$$

are projective modules (= direct summands of free modules) over the fiber rings

$$\mathcal{O}_{Y,y} = \varinjlim_{V \ni y} \mathcal{O}_Y(V)\,.$$

Then $\mathcal{P}f_{\mathcal{O}_Y}$ is f^*-acyclic and big enough.
Furthermore, f^* sends $\mathcal{P}f_{\mathcal{O}_Y}$ into $\mathcal{P}f_{\mathcal{O}_X}$.

Sketch of proof:

(i) It is obvious that $f_*(\mathcal{M})$ is flabby on Y if $\mathcal{M}$ is flabby on X.
Any $\mathcal{O}_X$-Module $\mathcal{M}$ on X has a canonical embedding

$$\mathcal{M} \hookrightarrow \mathcal{M}'$$

into the flabby $\mathcal{O}_X$-Module

$$\mathcal{M}' : U \longmapsto \prod_{x \in U} \mathcal{M}_x .$$

So we are reduced to proving that, for any short exact sequence of $\mathcal{M}od_{\mathcal{O}_X}$

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0 ,$$

the induced sequence

$$0 \longrightarrow f_*\mathcal{M}_1 \longrightarrow f_*\mathcal{M}_2 \longrightarrow f_*\mathcal{M}_3 \longrightarrow 0$$

is exact if $\mathcal{M}_1$ is flabby, and $\mathcal{M}_3$ is flabby if $\mathcal{M}_1, \mathcal{M}_2$ are flabby.
These two statements follow from:

Lemma: For any exact sequence of $\mathcal{M}od_{\mathcal{O}_X}$

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

such that $\mathcal{M}_1$ is flabby, and any open subset $U \subset X$, the sequence

$$0 \longrightarrow \mathcal{M}_1(U) \longrightarrow \mathcal{M}_2(U) \longrightarrow \mathcal{M}_3(U) \longrightarrow 0$$

is exact.

(ii) Any $\mathcal{O}_Y$-Module $\mathcal{N}$ has a canonical epimorphism

$$\mathcal{N}' \twoheadrightarrow \mathcal{N}$$

from the $\mathcal{O}_X$-Module

$$\mathcal{N}' = \bigoplus_{V \overset{i}{\hookrightarrow} Y} \bigoplus_{n \in \mathcal{N}(V)} i_! \mathcal{O}_V$$

whose fibers are the free modules

$$\mathcal{N}'_y = \bigoplus_{y \in V} \bigoplus_{n \in \mathcal{N}(V)} \mathcal{O}_{Y,y} \, .$$

As for any $x \in X$ with $f(x) = y$

$$(f^* \mathcal{N})_x \quad \text{identifies with} \quad \mathcal{O}_{x,X} \otimes_{\mathcal{O}_{Y,y}} \mathcal{N}_y \, ,$$

f^* sends $\mathcal{P}f(\mathcal{O}_Y)$ into $\mathcal{P}f(\mathcal{O}_X)$.

The remaining statements follow from:

Lemma:

(i) A sequence of $\mathcal{M}od_{\mathcal{O}_Y}$

$$0 \longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{N}_2 \longrightarrow \mathcal{N}_3 \longrightarrow 0$$

is exact if and only if, for any $y \in Y$,

$$0 \longrightarrow \mathcal{N}_{1,y} \longrightarrow \mathcal{N}_{2,y} \longrightarrow \mathcal{N}_{3,y} \longrightarrow 0$$

is an exact sequence of $\mathcal{O}_{Y,y}$-modules.

(ii) If $\mathcal{N}_{3,y}$ is projective, a sequence

$$0 \longrightarrow \mathcal{N}_{1,y} \longrightarrow \mathcal{N}_{2,y} \longrightarrow \mathcal{N}_{3,y} \longrightarrow 0$$

is exact if and only if it is split, yielding $\mathcal{N}_{2,y} \cong \mathcal{N}_{1,y} \oplus \mathcal{N}_{3,y}$.

(iii) Additive functors always respect split exact sequences.

Corollary: Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ = morphism of ringed spaces.

(i) The left-exact functor

$$f_* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_Y}$$

has a right derived functor

$$\mathrm{R}f_* : D^+(\mathcal{M}od_{\mathcal{O}_X}) \longrightarrow D^+(\mathcal{M}od_{\mathcal{O}_Y})$$

whose restriction to the equivalent category $D^+(\mathcal{F}b_{\mathcal{O}_X})$ deduced from $K^+(\mathcal{F}b_{\mathcal{O}_X})$ by formally inverting quasi-isomorphisms is defined by the commutative square:

$$\begin{array}{ccc} K^+(\mathcal{F}b_{\mathcal{O}_X}) & \xrightarrow{f_*} & K^+(\mathcal{F}b_{\mathcal{O}_Y}) \\ \downarrow & & \downarrow \\ D^+(\mathcal{F}b_{\mathcal{O}_X}) & \xrightarrow{\mathrm{R}f_*} & D^+(\mathcal{F}b_{\mathcal{O}_Y}) \end{array}$$

Furthermore, if f_* has finite cohomological dimension, it has right derived functors

$$\begin{array}{rcrcl} \mathrm{R}f_* & : & D(\mathcal{M}_{\mathcal{O}_X}) & \longrightarrow & D(\mathcal{M}od_{\mathcal{O}_Y}), \\ & & D^b(\mathcal{M}od_{\mathcal{O}_X}) & \longrightarrow & D^b(\mathcal{M}od_{\mathcal{O}_Y}), \\ & & D^-(\mathcal{M}od_{\mathcal{O}_X}) & \longrightarrow & D^-(\mathcal{M}od_{\mathcal{O}_Y}). \end{array}$$

(ii) The right-exact functor

$$f^* : \mathcal{M}od_{\mathcal{O}_Y} \longrightarrow \mathcal{M}od_{\mathcal{O}_X}$$

has a left derived functor

$$\mathrm{L}f^* : D^-(\mathcal{M}od_{\mathcal{O}_Y}) \longrightarrow D^-(\mathcal{M}od_{\mathcal{O}_X})$$

whose restriction to the equivalent category $D^-(\mathcal{P}f_{\mathcal{O}_Y})$ deduced from $K^-(\mathcal{P}f_{\mathcal{O}_Y})$ by formally inverting quasi-isomorphisms is defined by the commutative square:

$$\begin{array}{ccc} K^-(\mathcal{P}f_{\mathcal{O}_Y}) & \xrightarrow{f^*} & K^-(\mathcal{P}f_{\mathcal{O}_X}) \\ \downarrow & & \downarrow \\ D^-(\mathcal{P}f_{\mathcal{O}_Y}) & \xrightarrow{\mathrm{L}f^*} & D^-(\mathcal{P}f_{\mathcal{O}_Y}) \end{array}$$

Furthermore, if f^* has cohomological dimension $\leq d$ (or, equivalently, if for any $x \in X$ with $y = f(x)$, the functor $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \bullet$ has cohomological dimension $\leq d$), it has left derived functors

$$\begin{array}{rrcl} \mathrm{L}f^* : & D(\mathcal{M}od_{\mathcal{O}_Y}) & \longrightarrow & D(\mathcal{M}od_{\mathcal{O}_X}), \\ & D^b(\mathcal{M}od_{\mathcal{O}_Y}) & \longrightarrow & D^b(\mathcal{M}od_{\mathcal{O}_X}), \\ & D^+(\mathcal{M}od_{\mathcal{O}_Y}) & \longrightarrow & D^+(\mathcal{M}od_{\mathcal{O}_X}). \end{array}$$

Remarks:

(i) For any composed morphism

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) \xrightarrow{g} (Z, \mathcal{O}_Z),$$

the functors

$$\mathrm{R}(g \circ f)_* \quad \text{and} \quad \mathrm{R}g_* \circ \mathrm{R}f_*$$

[resp. $\mathrm{L}(g \circ f)^*$ and $\mathrm{L}f^* \circ \mathrm{L}g^*$]

are canonically isomorphic.

(ii) If f_* has finite cohomological dimension
[resp. if f^* has finite cohomological dimension,
resp. if both f_* and f^* have finite cohomological dimension],
the functors

$$\mathrm{L}f^* : D^-(\mathcal{M}od_{\mathcal{O}_Y}) \to D^-(\mathcal{M}od_{\mathcal{O}_X}) \text{ and } \mathrm{R}f_* : D^-(\mathcal{M}od_{\mathcal{O}_X}) \to D^-(\mathcal{M}od_{\mathcal{O}_Y}),$$

[resp.

$$\mathrm{L}f^* : D^+(\mathcal{M}od_{\mathcal{O}_Y}) \to D^+(\mathcal{M}od_{\mathcal{O}_X}) \text{ and } \mathrm{R}f_* : D^+(\mathcal{M}od_{\mathcal{O}_X}) \to D^+(\mathcal{M}od_{\mathcal{O}_Y}),$$

resp.

$$\mathrm{L}f^* : D(\mathcal{M}od_{\mathcal{O}_Y}) \to D(\mathcal{M}od_{\mathcal{O}_X}) \text{ and } \mathrm{R}f_* : D(\mathcal{M}od_{\mathcal{O}_X}) \to D(\mathcal{M}od_{\mathcal{O}_Y}) \,]$$

are adjoint.

(iii) For any commutative triangle in the category of ringed spaces

$$\begin{array}{ccc} (X_1, \mathcal{O}_{X_1}) & \xrightarrow{\quad f \quad} & (X_2, \mathcal{O}_{X_2}) \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & (S, \mathcal{O}_S) & \end{array}$$

and any $\mathcal{O}_S$-Module $\mathcal{M}$,
f defines a morphism

$$p_2^* \mathcal{M} \longrightarrow f_* \circ f^* \circ p_2^* \mathcal{M} \longrightarrow \mathrm{R} f_* \circ f^* \circ p_2^* \mathcal{M}$$

and, taking its transform by $\mathrm{R} p_{2,*}$,

$$\mathrm{R} p_{2,*} \circ p_2^* \mathcal{M} \longrightarrow \mathrm{R} p_{2,*} \circ \mathrm{R} f_* \circ f^* \circ p_2^* \mathcal{M} \cong \mathrm{R} p_{1,*} \circ p_1^* \mathcal{M} \, .$$

This induces morphisms of $\mathcal{O}_S$-Modules

$$\mathrm{R}^k p_{2,*} \circ p_2^* \mathcal{M} \longrightarrow \mathrm{R}^k p_{1,*} \circ p_1^* \mathcal{M}$$

which depend functorialy on f.

In other words, sheaf-cohomology defines contravariant functors.

Computation of cohomology by soft sheaves

Definition:
Let $X =$ topological space which is locally compact (in particular Hausdorff).

(i) A sheaf $\mathcal{M}$ on X
is called "soft"
if, for any compact subspace $K \overset{i}{\hookrightarrow} X$,
the restriction map

$$\Gamma(X, \mathcal{M}) = \mathcal{M}(X) \longrightarrow i^* \mathcal{M}(K) = \Gamma(K, \mathcal{M})$$

is surjective.

(ii) If $\mathcal{O}_X$ is a sheaf of rings on X,
let's denote $\mathcal{S}f_{\mathcal{O}_X}$
the full additive subcategory of $\mathcal{M}od_{\mathcal{O}_X}$
on $\mathcal{O}_X$-Modules $\mathcal{M}$ which are soft.

Remark:
The restriction of a soft sheaf on X
to any open subspace $U \subset X$ is soft.

Lemma:
Let X = locally compact topological space,

$(K \overset{i}{\hookrightarrow} X)$ = compact subspace,

$\mathcal{M}$ = sheaf on X.

Then:

(i) The natural map

$$\varinjlim_{U \supset K} \mathcal{M}(U) \longrightarrow \Gamma(K, \mathcal{M}) = i^* \mathcal{M}(K)$$

is one-to-one.

(ii) If K is written as a union of two compact subspaces

$$K = K_1 \cup K_2 \,,$$

the natural map

$$\Gamma(K, \mathcal{M}) \longrightarrow \Gamma(K_1, \mathcal{M}) \times_{\Gamma(K_1 \cap K_2, \mathcal{M})} \Gamma(K_2, \mathcal{M})$$

is one-to-one.

Remark:
(i) implies that, on a locally compact topological space X, any flabby sheaf is soft.

Proof of the lemma:

(i) If $m_1 \in \mathcal{M}(U_1)$ and $m_2 \in \mathcal{M}(U_2)$ are sections of $\mathcal{M}$ on $U_1 \supset K$ and $U_2 \supset K$ which have the same image in $\Gamma(K, \mathcal{M})$, then for any $x \in K$, they coincide on some open neighborhood U_x of x and so they coincide on $\bigcup\limits_{x} U_x$.

In the other direction, let $m \in \Gamma(K, \mathcal{M})$.

The compact subset K can be covered by open subsets $U_1, \ldots, U_n$ such that each $\overline{U}_i$ is compact and m lifts to some $m_i \in \Gamma(\overline{U}_i, \mathcal{M})$.

For any indices i, j, there is a closed subset $Z_{i,j} \subset \overline{U}_i \cap \overline{U}_j$ such that $Z_{i,j} \cap K = \emptyset$ and m_i, m_j coincide on $(\overline{U}_i \cap \overline{U}_j) - Z_{i,j}$.

Then the m_i's define a section of $\mathcal{M}$ on $\bigcup\limits_{1 \le i \le n} U_i - \bigcup\limits_{i \neq j} Z_{i,j}$ which lifts m.

(ii) We may suppose that $K = X$. The map is obviously injective.

Conversely, consider elements $m_1 \in \Gamma(K_1, \mathcal{M})$, $m_2 \in \Gamma(K_2, \mathcal{M})$ which coincide on $K_1 \cap K_2$.

There are open neighborhoods $U_1 \supset K_1$, $U_2 \supset K_2$ and $K_1 \cap K_2 \subset U \subset U_1 \cap U_2$ such that m_1, m_2 lift to $m_1' \in \mathcal{M}(U_1)$, $m_2' \in \mathcal{M}(U_2)$ and m_1', m_2' coincide on U with a section $m' \in \mathcal{M}(U)$. Then we may write

$$K = X = (U_1 - U_1 \cap K_2) \cup (U_2 - U_2 \cap K_1) \cup U$$

with $(U_1 - U_1 \cap K_2) \cap (U_2 - U_2 \cap K_1) = \emptyset$

and the sections $m_1' \in \mathcal{M}(U_1 - U_1 \cap K_2)$, $m_2' \in \mathcal{M}(U_2 - U_2 \cap K_1)$, $m' \in \mathcal{M}(U)$ define a section of $\mathcal{M}(X) = \Gamma(K, \mathcal{M})$ as wanted.

Corollary:
Let $(X, \mathcal{O}_X) =$ differential manifold,
$\mathcal{M} = \mathcal{O}_X$-Module on X.

Then $\mathcal{M}$ is a soft sheaf.

Proof:
Let $K =$ compact subspace of X,
$m =$ section of $\mathcal{M}$ on K.

Then m can be lifted to a section $m \in \mathcal{M}(U)$ for some open neighborhood $K \subset U$.
There exists an open neighborhood V of K such that

$$K \subset V \subset \overline{V} \subset U .$$

There exists C^∞ functions $\varphi, \psi : X \to \mathbb{R}_+$ such that $\varphi + \psi = 1$ and

$$\operatorname{supp}(\varphi) \subset U , \quad \operatorname{supp}(\psi) \subset X - \overline{V} .$$

The section

$$\varphi \cdot m \in \mathcal{M}(U)$$

coincides with m on V and a fortiori on K.
Furthermore, its restriction to the open subset

$$U - \operatorname{supp}(\varphi)$$

is 0 and it can be extended by 0 on $X - \operatorname{supp}(\varphi)$ to define a section

$$\varphi \cdot m \in \mathcal{M}(X) .$$

Proposition:
Let $(X, \mathcal{O}_X) =$ locally compact ringed space.
Consider a short exact sequence of $\mathcal{O}_X$-Modules

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0\,.$$

Then:

(i) If $\mathcal{M}_1$ is soft and $K \overset{i}{\hookrightarrow} X$ is a compact subspace, the sequence

$$0 \longrightarrow \Gamma(K, \mathcal{M}_1) \longrightarrow \Gamma(K, \mathcal{M}_2) \longrightarrow \Gamma(K, \mathcal{M}_3) \longrightarrow 0$$

is exact.

(ii) If $\mathcal{M}_1$ and $\mathcal{M}_2$ are soft, $\mathcal{M}_3$ is soft.

(iii) If X is countable at infinity,
the sequence

$$0 \longrightarrow \mathcal{M}_1(X) \longrightarrow \mathcal{M}_2(X) \longrightarrow \mathcal{M}_3(X) \longrightarrow 0$$

is exact.

Proof of the proposition:

(i) Let $m \in \Gamma(K, \mathcal{M}_3)$.

We may cover the compact subspace K by open subspaces of K

$$K = K_1 \cup \cdots \cup K_n$$

such each $\overline{K}_i$ is compact and the restriction of m in $\Gamma(\overline{K}_i, \mathcal{M}_3)$ lifts to some $m_i \in \Gamma(\overline{K}_i, \mathcal{M}_2)$.

Let's prove by induction on k that, writing $K'_k = \overline{K}_1 \cup \cdots \cup \overline{K}_k$, the restriction of m to $\Gamma(K'_k, \mathcal{M}_3)$ lifts to some $m'_k \in \Gamma(K'_k, \mathcal{M}_2)$.

Suppose it is proven for rank k.

Then the difference $m'_k - m_{k+1}$ is well-defined as a section in $\Gamma(K'_k \cap \overline{K}_{k+1}, \mathcal{M}_1)$ and extends to a global section

$$m''_{k+1} \in \Gamma(X, \mathcal{M}_1)$$

as $\mathcal{M}_1$ is soft. Then the sections

$$m'_k \in \Gamma(K'_k, \mathcal{M}_2) \quad \text{and} \quad m_{k+1} + m''_{k+1} \in \Gamma(\overline{K}_{k+1}, \mathcal{M}_2)$$

coincide in $\Gamma(K'_k \cap \overline{K}_{k+1}, \mathcal{M}_2)$ and define a lift

$$m'_{k+1} \in \Gamma(K'_{k+1}, \mathcal{M}_2)$$

of the restriction of m in $\Gamma(K'_{k+1}, \mathcal{M}_3)$.

(ii) According to (i), any element $m_3 \in \Gamma(K, \mathcal{M}_3)$ lifts to some $m_2 \in \Gamma(K, \mathcal{M}_2)$ as $\mathcal{M}_1$ is soft, and m_2 extends to some $m'_2 \in \Gamma(X, \mathcal{M}_2)$ as $\mathcal{M}_2$ is soft.

The image m'_3 of m'_2 in $\Gamma(X, \mathcal{M}_3)$ is an extension of m_3.

(iii) As X is countable at infinity, it can be written as a union

$$X = \bigcup_n U_n$$

of a sequence of open subsets U_n such that each $\overline{U}_n$ is compact and

$$U_n \subset \overline{U}_n \subset U_{n+1} \quad \text{for any } n\,.$$

Let $m \in \mathcal{M}_3(X)$.
For any n, one can choose a lift m'_n of m in $\Gamma(\overline{U}_n, \mathcal{M}_2)$.
Let's construct by induction on n a sequence of lifts

$$m_n \in \Gamma(\overline{U}_n, \mathcal{M}_2) \qquad \text{of } m$$

such that, for any n, m_n is the restriction of m_{n+1} in $\Gamma(\overline{U}_n, \mathcal{M}_2)$.
Suppose $m_1, \ldots, m_n$ are constructed.
The difference $m_n - m'_{n+1}$ is well defined as an element of $\Gamma(\overline{U}_n, \mathcal{M}_1)$.
It extends to an element

$$m''_{n+1} \in \Gamma(X, \mathcal{M}_1)\,.$$

Then

$$m_{n+1} = m'_{n+1} + m''_{n+1}$$

is well defined in $\Gamma(\overline{U}_{n+1}, \mathcal{M}_2)$.
It is a lift of m and extends $m_n \in \Gamma(\overline{U}_n, \mathcal{M}_2)$.
Lastly, the family (m_n) defines a section of $\mathcal{M}_2$ on X which lifts m.

Corollary:

Let $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$
$=$ morphism of ringed spaces.

Suppose X is locally compact and countable at infinity.

Then the full subcategory $\mathcal{S}f_{\mathcal{O}_X}$ of soft $\mathcal{O}_X$-Modules is f_*-acyclic.

Remark:

If $(X, \mathcal{O}_X)$ is a differential manifold which is countable at infinity, we even see that the functor

$$f_* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_Y}$$

is exact.

The sheaf theoretic De Rham theorem

Corollary:
Let $(X, \mathcal{O}_X) =$ differential manifold
which is countable at infinity.
Then the cohomology spaces

$$H^k_{dR}(X)$$

of the De Rham complex

$$0 \longrightarrow \Omega^0_X(X) \longrightarrow \Omega^1_X(X) \longrightarrow \cdots \longrightarrow \Omega^k_X(X) \longrightarrow \cdots$$

identify with the cohomology spaces

$$H^k(X, \mathbb{R}) = \mathrm{R}^k p_* \mathbb{R}_X$$

of the constant sheaf $\mathbb{R}_X = p^{-1}\mathbb{R}$ on X relatively to the projection

$$p : X \longrightarrow \{\bullet\}.$$

Proof: According to Poincaré's lemma, the sequence of $\mathbb{R}_X$-Modules on the topological space X

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \Omega^0_X \longrightarrow \Omega^1_X \longrightarrow \cdots \longrightarrow \Omega^k_X \longrightarrow \cdots$$

is exact.
As the sheaves Ω^k_X are soft, they are p_*-acyclic
and $\Omega^\bullet_X(X) = p_*\Omega^\bullet_X$ computes the cohomology of $\mathbb{R}_X$.

Additive bifunctors

Definition:
Let $\mathcal{A}, \mathcal{B}, \mathcal{C} =$ additive categories. A functor

$$F : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$$

is called additive if, for any object A of $\mathcal{A}$ or B of $\mathcal{B}$, the functor

$$F(A, \bullet) : \mathcal{B} \longrightarrow \mathcal{C} \qquad \text{or} \qquad F(\bullet, B) : \mathcal{A} \longrightarrow \mathcal{C}$$

is additive.

Examples:

- For any additive category $\mathcal{A}$,

$$\begin{array}{rcccl} \mathrm{Hom} & : & \mathcal{A}^{\mathrm{op}} \times \mathcal{A} & \longrightarrow & \mathrm{Ab}\,, \\ & & (X, Y) & \longmapsto & \mathrm{Hom}(X, Y)\,. \end{array}$$

- For any commutative ringed space $(X, \mathcal{O}_X)$,

$$\begin{array}{rcccl} \otimes & : & \mathcal{M}od_{\mathcal{O}_X} \times \mathcal{M}od_{\mathcal{O}_X} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_X}\,, \\ & & (\mathcal{M}, \mathcal{N}) & \longmapsto & \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}\,, \end{array}$$

  and

$$\begin{array}{rcccl} \mathcal{H}om & : & \mathcal{M}od_{\mathcal{O}_X}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}_X} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_X}\,, \\ & & (\mathcal{N}, \mathcal{L}) & \longmapsto & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{L})\,. \end{array}$$

Lemma:
Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$
$=$ additive bifunctor between additive categories.
Then:

(i) F defines additive bifunctors

$$\begin{array}{rcl} C^+(\mathcal{A}) \times C^+(\mathcal{B}) & \longrightarrow & C^+(\mathcal{C})\,, \\ C^-(\mathcal{A}) \times C^-(\mathcal{B}) & \longrightarrow & C^-(\mathcal{C})\,, \\ C^b(\mathcal{A}) \times C^b(\mathcal{B}) & \longrightarrow & C^b(\mathcal{C}) \end{array}$$

and even

$$C(\mathcal{A}) \times C(\mathcal{B}) \longrightarrow C(\mathcal{C})$$

if $\mathcal{C}$ has countable direct sums.
They associate to complexes $(A^\bullet)$ and $(B^\bullet)$ of $\mathcal{A}$ and $\mathcal{B}$ the complex $(C^\bullet)$ defined by

$$C^n = \bigoplus_{n_1+n_2=n} F(A^{n_1}, B^{n_2})$$

and whose differentials $d_C^n : C^n \to C^{n+1}$ are the sums of the

$$F(d_A^{n_1}, \mathrm{id}_{B^{n_2}}) : F(A^{n_1}, B^{n_2}) \longrightarrow F(A^{n_1+1}, B^{n_2})$$

and

$$(-1)^{n_1} \cdot F(\mathrm{id}_{A^{n_1}}, d_B^{n_2}) : F(A^{n_1}, B^{n_2}) \longrightarrow F(A^{n_1}, B^{n_2+1})\,.$$

(ii) These functors induce additive functors

$$\begin{aligned} K^+(\mathcal{A}) \times K^+(\mathcal{B}) &\longrightarrow K^+(\mathcal{C}), \\ K^-(\mathcal{A}) \times K^-(\mathcal{B}) &\longrightarrow K^-(\mathcal{C}), \\ K^b(\mathcal{A}) \times K^b(\mathcal{B}) &\longrightarrow K^b(\mathcal{C}) \end{aligned}$$

and even $K(\mathcal{A}) \times K(\mathcal{B}) \longrightarrow K(\mathcal{C})$ if $\mathcal{C}$ has countable direct sum.

Definition:

Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$

$=$ additive bifunctor between abelian categories.

Then F is called left-exact [resp. right-exact]

if, for any object A of $\mathcal{A}$ or B of $\mathcal{B}$, the functor

$$F(A, \bullet) : \mathcal{B} \longrightarrow \mathcal{C} \qquad \text{or} \qquad F(\bullet, B) : \mathcal{A} \longrightarrow \mathcal{C}$$

is left-exact [resp. right-exact].

Examples:

- For any abelian category $\mathcal{A}$, the additive bifunctor

$$\begin{array}{rccl} \mathrm{Hom} & : & \mathcal{A}^{\mathrm{op}} \times \mathcal{A} & \longrightarrow & \mathrm{Ab}\,, \\ & & (X, Y) & \longmapsto & \mathrm{Hom}(X, Y) \end{array}$$

  is left-exact.

- For any commutative ringed space $(X, \mathcal{O}_X)$, the additive bifunctor

$$\begin{array}{rccl} \mathcal{H}om & : & \mathcal{M}od_{\mathcal{O}_X}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}_X} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_X}\,, \\ & & (\mathcal{N}, \mathcal{L}) & \longmapsto & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{L}) \end{array}$$

  is left-exact, while the additive bifunctor

$$\begin{array}{rccl} \otimes & : & \mathcal{M}od_{\mathcal{O}_X} \times \mathcal{M}od_{\mathcal{O}_X} & \longrightarrow & \mathcal{M}od_{\mathcal{O}_X}\,, \\ & & (\mathcal{M}, \mathcal{N}) & \longmapsto & \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \end{array}$$

  is right-exact.

Derived bifunctors

Definition:

Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$

$=$ additive bifunctors between abelian categories which is left-exact [resp. right-exact].

A derived functor of F is an additive bifunctor

$\mathrm{R}F$:	$D^+(\mathcal{A}) \times D^+(\mathcal{B}) \to D^+(\mathcal{C})$	[resp.	$\mathrm{L}G$:	$D^-(\mathcal{A}) \times D^-(\mathcal{B}) \to D^-(\mathcal{C})$
or	$D(\mathcal{A}) \times D^+(\mathcal{B}) \to D(\mathcal{C})$		or	$D(\mathcal{A}) \times D^-(\mathcal{B}) \to D(\mathcal{C})$
or	$D(\mathcal{A}) \times D(\mathcal{B}) \to D(\mathcal{C})$		or	$D(\mathcal{A}) \times D(\mathcal{B}) \to D(\mathcal{C})$]

such that:

(1) $\mathrm{R}F$ [resp. $\mathrm{L}F$] transforms the functors $[m]$ of $D(\mathcal{A})$ or $D(\mathcal{B})$ into the functors $[m]$ of $D(\mathcal{C})$ and the distinguished triangles of $D(\mathcal{A})$ or $D(\mathcal{B})$ into distinguished triangles of $D(\mathcal{C})$.

(2) Denoting Q the quotient functors

$$K(\mathcal{A}) \longrightarrow D(\mathcal{A}), \; K(\mathcal{B}) \longrightarrow D(\mathcal{B}) \quad \text{and} \quad K(\mathcal{C}) \longrightarrow D(\mathcal{C}),$$

$\mathrm{R}F$ [resp. $\mathrm{L}F$] is endowed with a morphism of composite functors

$$Q \circ F \longrightarrow \mathrm{R}F \circ (Q \times Q) \qquad [\text{resp.} \quad \mathrm{L}F \circ (Q \times Q) \longrightarrow Q \circ F\,].$$

(3) $\mathrm{R}F$ [resp. $\mathrm{L}F$] is universal with respect to these properties.

Remarks:

(i) If $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is left-exact [resp. right-exact] and has a derived functor $\mathrm{R}F$ [resp. $\mathrm{L}F$], the composed functors

$$\mathcal{A} \times \mathcal{B} \xrightarrow{\mathrm{R}F \circ (Q \times Q)} D^{+}(\mathcal{C}) \xrightarrow{H^{k}} \mathcal{C}$$

$$[\text{resp.} \quad \mathcal{A} \times \mathcal{B} \xrightarrow{\mathrm{L}F \circ (Q \times Q)} D^{-}(\mathcal{C}) \xrightarrow{H^{-k}} \mathcal{C} \ ]$$

are denoted

$$\mathrm{R}^{k} F \qquad [\text{resp.} \quad \mathrm{L}^{k} F \ ].$$

(ii) In practice, derived functors $\mathrm{R}F$ [resp. $\mathrm{L}F$] are always constructed through the following proposition and corollary.
Then $\mathrm{R}^{k} F$ [resp. $\mathrm{L}^{k} F$] is 0 for any $k < 0$ and $\mathrm{R}^{\circ} F$ [resp. $\mathrm{L}^{\circ} F$] identifies with F.

(iii) Therefore, any object A of $\mathcal{A}$ and any short exact sequence of $\mathcal{B}$

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

yield a long exact sequence of $\mathcal{C}$

$$0 \longrightarrow F(A, B') \longrightarrow F(A, B) \longrightarrow F(A, B'') \longrightarrow \mathrm{R}^{1} F(A, B') \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathrm{R}^{k} F(A, B'') \longrightarrow \mathrm{R}^{k+1} F(A, B') \longrightarrow \cdots$$

[resp.

$$\cdots \longrightarrow \mathrm{L}^{k+1} F(A, B'') \longrightarrow \mathrm{L}^{k} F(A, B') \longrightarrow \cdots$$
$$\longrightarrow \mathrm{L}^{1} F(A, B'') \longrightarrow F(A, B') \longrightarrow F(A, B) \longrightarrow F(A, B'') \longrightarrow 0 \ ].$$

(iv) Same for any object of $\mathcal{B}$ and any short exact sequence of $\mathcal{A}$.

(v) An object A of $\mathcal{A}$ or B of $\mathcal{B}$ is called "F-acyclic" if, for any object B' of $\mathcal{B}$ or A' of $\mathcal{A}$,

$$\mathrm{R}^k F(A, B') = 0\,,\ \forall\, k \geq 1\,, \quad \text{or} \quad \mathrm{R}^k F(A', B) = 0\,,\ \forall k \geq 1$$

$$[\text{resp.} \qquad \mathrm{L}^k F(A, B') = 0\,,\ \forall\, k \geq 1\,, \quad \text{or} \quad \mathrm{L}^k F(A', B) = 0\,,\ \forall k \geq 1\ ].$$

Proposition:

Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$

$=$ additive bifunctor between abelian categories which is left-exact [resp. right-exact],

$\mathcal{I} =$ full additive subcategory of $\mathcal{B}$ which is $F(A, \bullet)$-acyclic for any object A of $\mathcal{A}$.

(i) For any object A of $C^+(\mathcal{A})$ [resp. $C^-(\mathcal{A})$] and any quasi-isomorphism

$$I_1 \longrightarrow I_2 \quad \text{of} \quad C^+(\mathcal{I}) \qquad \text{[resp. } C^-(\mathcal{I}) \text{]},$$

the morphism of $C^+(\mathcal{C})$

$$F(A, I_1) \longrightarrow F(A, I_2)$$

is a quasi-isomorphism.

(ii) Furthermore, if $\mathcal{C}$ has countable direct sums and the functor $\varinjlim_{\mathbb{N}}$ is exact in $\mathcal{C}$, the same result holds for any object A of $C(\mathcal{A})$.

(iii) Furthermore, if these conditions are verified and $\mathcal{I}$ has codimension $\leq d$, the same result holds for any object A of $C(\mathcal{A})$ and any quasi-isomorphism

$$I_1 \longrightarrow I_2 \quad \text{of} \quad C(\mathcal{I}).$$

Sketch of proof of the proposition:
It is enough to consider the case where F is left-exact.
Replacing the morphism $I_1 \to I_2$ of $C^+(\mathcal{I})$ or $C(\mathcal{I})$ by its cone, we are reduced to the case of an object I of $C^+(\mathcal{I})$ or $C(\mathcal{I})$ which is quasi-isomorphic to 0, in other words is a long exact sequence.
If $I = (I^\bullet)$ is bounded below or if $\mathcal{I}$ has codimension $\leq d$, the long exact sequence I decomposes into short exact sequences

$$0 \longrightarrow \mathrm{Im}(I^{n-1}) \longrightarrow I^n \longrightarrow \mathrm{Im}(I^n) \longrightarrow 0$$

whose three objects are in $\mathcal{I}$.
It follows that for any object A of $\mathcal{A}$, the long exact sequence

$$F(A, I^\bullet)$$

is exact.
Using the five lemma, we derive that for any $A \in C^b(\mathcal{A})$, the complex

$$F(A, I)$$

is quasi-isomorphic to 0.
If I is an object of $C^+(\mathcal{I})$, the result generalises to any object A of $C^+(\mathcal{A})$ as, for any rank k, it reduces to the previous case.
Lastly, the result generalises from $C^b(\mathcal{A})$ to $C(\mathcal{A})$ if $\mathcal{C}$ has countable direct sums and the functor $\varinjlim_{\mathbb{N}}$ is exact in $\mathcal{C}$.

Corollary:

Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$
$=$ additive bifunctor between abelian categories
which is left-exact [resp. right-exact],
$\mathcal{I} =$ full additive subcategory of $\mathcal{B}$
which is $F(A, \bullet)$-acyclic for any object A of $\mathcal{A}$
and such that $F(\bullet, I)$ is exact for any object I of $\mathcal{I}$.

Then:

(i) If $\mathcal{I}$ is big enough,
F has a right [resp. left] derived functor

$$\mathrm{R}F : D^+(\mathcal{A}) \times D^+(\mathcal{B}) \longrightarrow D^+(\mathcal{C}) \qquad \text{[resp.} \quad D^-(\mathcal{A}) \times D^-(\mathcal{B}) \longrightarrow D^-(\mathcal{C}) \text{]}$$

whose restriction to $D^+(\mathcal{A}) \times D^+(\mathcal{I})$ [resp. $D^-(\mathcal{A}) \times D^-(\mathcal{I})$] is defined by the commutative square

$$\begin{array}{ccc} K^+(\mathcal{A}) \times K^+(\mathcal{I}) & \xrightarrow{F} & K^+(\mathcal{C}) \\ \downarrow & & \downarrow \\ D^+(\mathcal{A}) \times D^+(\mathcal{I}) & \xrightarrow{\mathrm{R}F} & D^+(\mathcal{C}) \end{array} \qquad \text{[resp.} \quad \begin{array}{ccc} K^-(\mathcal{A}) \times K^-(\mathcal{I}) & \xrightarrow{F} & K^-(\mathcal{C}) \\ \downarrow & & \downarrow \\ D^-(\mathcal{A}) \times D^-(\mathcal{I}) & \xrightarrow{\mathrm{L}F} & D^-(\mathcal{C}) \end{array} \text{].}$$

(ii) If $\mathcal{I}$ is big enough, $\mathcal{C}$ has countable direct sums and the functor $\varinjlim_{\mathbb{N}}$ is exact in $\mathcal{C}$, F has a right [resp. left] derived functor

$$\mathrm{R}F : D(\mathcal{A}) \times D^+(\mathcal{B}) \longrightarrow D(\mathcal{C}) \qquad [\text{resp.} \quad \mathrm{L}F : D(\mathcal{A}) \times D^-(\mathcal{B}) \longrightarrow D(\mathcal{C})\ ]$$

whose restriction to $D(\mathcal{A}) \times D^+(\mathcal{I})$ [resp. $D(\mathcal{A}) \times D^-(\mathcal{I})$] is induced by the functor

$$K(A) \times K^+(\mathcal{I}) \xrightarrow{F} K(\mathcal{C}) \qquad [\text{resp.} \quad K(A) \times K^-(\mathcal{I}) \xrightarrow{F} K(\mathcal{C})\ ].$$

(iii) If these conditions are verified and $\mathcal{I}$ has codimension $\leq d$, F has a right [resp. left] derived functor

$$\mathrm{R}F : D(\mathcal{A}) \times D(\mathcal{B}) \longrightarrow D(\mathcal{C}) \qquad [\text{resp.} \quad \mathrm{L}F : D(\mathcal{A}) \times D(\mathcal{B}) \longrightarrow D(\mathcal{C})\ ]$$

whose restriction to $D(\mathcal{A}) \times D(\mathcal{I})$ is induced by the functor

$$K(A) \times K(\mathcal{I}) \longrightarrow K(\mathcal{C})\,.$$

Furthermore, $\mathrm{R}F$ [resp. $\mathrm{L}F$] restricts to derive functors

$$\begin{array}{rcl} D^b(\mathcal{A}) \times D^b(\mathcal{B}) & \longrightarrow & D^b(\mathcal{C})\,, \\ D^+(\mathcal{A}) \times D^+(\mathcal{B}) & \longrightarrow & D^+(\mathcal{C})\,, \\ D^-(\mathcal{A}) \times D^-(\mathcal{B}) & \longrightarrow & D^-(\mathcal{C})\,. \end{array}$$

Remark:

- If $\mathcal{B}$ contains a full additive subcategory $\mathcal{I}$ which verifies the conditions of (i), then the full subcategory of F-acyclic objects of $\mathcal{B}$ contains $\mathcal{I}$ and is a fortiori big enough.
- In that case, the subcategory of F-acyclic objects of $\mathcal{B}$ has codimension $\leq d$ if and only if the derived functors

$$\mathrm{R}^k F \qquad [\text{resp.} \quad \mathrm{L}^k F\,]$$

  are 0 in all degrees $k > d$.
- If this condition is verified, we say F has cohomological dimension $\leq d$.

Corollary:

Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$
$=$ left-exact [resp. right-exact] additive bifunctor,

$G : \mathcal{B}' \to \mathcal{B}$
$=$ left-exact [resp. right-exact] additive functor,

$F' = F(\bullet, G(\bullet)) : \mathcal{A} \times \mathcal{B}' \to \mathcal{C}$,

$\mathcal{I} =$ full additive subcategory of $\mathcal{B}$
which is $F(A, \bullet)$-acyclic for any object A of $\mathcal{A}$
and such that $F(\bullet, I)$ is exact for any object I of $\mathcal{I}$,

$\mathcal{I}' =$ full additive subcategory of $\mathcal{B}'$
such that $\mathcal{I}'$ is G-acyclic and G sends $\mathcal{I}'$ to $\mathcal{I}$.

Then:

(i) If $\mathcal{I}$ and $\mathcal{I}'$ are big enough,
F' has a derived functor $\mathrm{R}F'$ [resp. $\mathrm{L}F'$] isomorphic to

$$\mathrm{R}F(\bullet, \mathrm{R}G(\bullet)) \qquad \text{[resp.} \quad \mathrm{L}F(\bullet, \mathrm{L}G(\bullet))\ \text{]}.$$

Its restriction to $D^+(\mathcal{A}) \times D^+(\mathcal{I}')$ [resp. $D^-(\mathcal{A}) \times D^-(\mathcal{I}')$] is defined by the commutative square

$$\begin{array}{ccccc} K^+(\mathcal{A}) \times K^+(\mathcal{I}') & \xrightarrow{\mathrm{id}\times G} & K^+(\mathcal{A}) \times K^+(\mathcal{I}) & \xrightarrow{F} & K^+(\mathcal{C}) \\ \downarrow & & & & \downarrow \\ D^+(\mathcal{A}) \times D^+(\mathcal{I}') & & \xrightarrow{\mathrm{R}F'} & & D^+(\mathcal{C}) \end{array}$$

[resp.

$$\begin{array}{ccccc} K^-(\mathcal{A}) \times K^-(\mathcal{I}') & \xrightarrow{\mathrm{id}\times G} & K^-(\mathcal{A}) \times K^-(\mathcal{I}) & \xrightarrow{F} & K^-(\mathcal{C}) \\ \downarrow & & & & \downarrow \\ D^-(\mathcal{A}) \times D^-(\mathcal{I}') & & \xrightarrow{\mathrm{L}F'} & & D^-(\mathcal{C})\ ]. \end{array}$$

(ii) If furthermore $\mathcal{C}$ has countable direct sums and the functor $\varinjlim_{\mathbb{N}}$ is exact in $\mathcal{C}$,

F' has a derived functor

$$\mathrm{R}F' : D(\mathcal{A}) \times D^+(\mathcal{B}') \longrightarrow D(\mathcal{C}) \qquad \text{[resp.} \quad \mathrm{L}F' : D(\mathcal{A}) \times D^-(\mathcal{B}') \longrightarrow D(\mathcal{C}) \text{]}$$

isomorphic to $\mathrm{R}F(\bullet, \mathrm{R}G(\bullet))$ [resp. $\mathrm{L}F(\bullet, \mathrm{L}G(\bullet))$].
Its restriction to $D(\mathcal{A}) \times D^+(\mathcal{I}')$ [resp. $D(\mathcal{A}) \times D^-(\mathcal{I}')$] is defined by the commutative square

$$\begin{array}{ccccc} K(\mathcal{A}) \times K^+(\mathcal{I}') & \xrightarrow{\mathrm{id} \times G} & K(\mathcal{A}) \times K^+(\mathcal{I}) & \xrightarrow{F} & K(\mathcal{C}) \\ \downarrow & & & & \downarrow \\ D(\mathcal{A}) \times D^+(\mathcal{I}') & & \xrightarrow{\mathrm{R}F'} & & D(\mathcal{C}) \end{array}$$

[resp.

$$\begin{array}{ccccc} K(\mathcal{A}) \times K^-(\mathcal{I}') & \xrightarrow{\mathrm{id} \times G} & K(\mathcal{A}) \times K^-(\mathcal{I}) & \xrightarrow{F} & K(\mathcal{C}) \\ \downarrow & & & & \downarrow \\ D(\mathcal{A}) \times D^-(\mathcal{I}') & & \xrightarrow{\mathrm{L}F'} & & D(\mathcal{C}) \text{].} \end{array}$$

(iii) If the previous conditions are verified,
F has cohomological dimension $\leq d$
and G has cohomological dimension $\leq d'$,
then F' has cohomological dimension $\leq d + d'$
and has a right [resp. left] derived functor

$$\begin{array}{rcrcl} \mathrm{R}F' \quad [\text{resp. } \mathrm{L}F'] & : & D(\mathcal{A}) \times D(\mathcal{B}') & \longrightarrow & D(\mathcal{C})\,, \\ & & D^b(\mathcal{A}) \times D^b(\mathcal{B}') & \longrightarrow & D^b(\mathcal{C}) \end{array}$$

which is isomorphic to the composite

$$D(\mathcal{A}) \times D(\mathcal{B}') \xrightarrow[\text{[resp. } \mathrm{id}\ \times\ \mathrm{L}G]]{\mathrm{id} \times \mathrm{R}G} D(\mathcal{A}) \times D(\mathcal{B}) \xrightarrow[\text{[resp. } \mathrm{L}F]]{\mathrm{R}F} D(\mathcal{C})\,.$$

Injective objects

Definition:

Let $\mathcal{A}$ = abelian category.
An object I of $\mathcal{A}$ is called injective if the functor

$$\begin{array}{ccc} \mathcal{A}^{\mathrm{op}} & \longrightarrow & \mathrm{Ab}\,, \\ X & \longmapsto & \mathrm{Hom}(X, I) \end{array}$$

is exact.

Remark:

An object P of $\mathcal{A}$ is called projective
if it is injective in $\mathcal{A}^{\mathrm{op}}$
i.e. if the functor

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathrm{Ab}\,, \\ Y & \longmapsto & \mathrm{Hom}(P, Y) \end{array}$$

is exact.

Lemma:

Let $\mathcal{A}$ = abelian category,
 I = injective object of $\mathcal{A}$.

Then any monomorphism

$$I \overset{i}{\hookrightarrow} A$$

is a retract in the sense there exists a morphism

$$r : A \longrightarrow I \qquad \text{such that} \qquad r \circ i = \mathrm{id}_A \,.$$

In other words, I is a direct summand of A.

Proof:

As I is injective, the sequence

$$0 \longrightarrow \mathrm{Hom}(A/I, I) \longrightarrow \mathrm{Hom}(A, I) \longrightarrow \mathrm{Hom}(I, I) \longrightarrow 0$$

is exact.

So the element $\mathrm{id}_I \in \mathrm{Hom}(I, I)$ lifts to an element

$$r \in \mathrm{Hom}(A, I) \,.$$

Corollary:

Let $\mathcal{A} =$ abelian category,

$\mathcal{I} =$ full additive subcategory of $\mathcal{A}$ on injective objects.

Then $\mathcal{I}$ is F-acyclic for any left-exact additive functor $F : \mathcal{A} \to \mathcal{B}$ to an abelian category.

In particular, if $\mathcal{A}$ has "enough injectives" in the sense that $\mathcal{I}$ is big enough, then any such functor $F : \mathcal{A} \to \mathcal{B}$ has a right derived functor

$$\mathrm{R}F : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$

whose restriction to $D^+(\mathcal{I})$ is induced by

$$K^+(\mathcal{I}) \xrightarrow{F} K^+(\mathcal{B})\,.$$

Proof:

- Any exact sequence of $\mathcal{A}$

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

  such that M_1 is injective is split;
  so it is preserved by any additive functor $F : \mathcal{A} \to \mathcal{B}$.
- If furthermore M_2 is also injective, M_3 is injective as it is a direct summand of M_2.

Application to linear sheaves

Proposition:
For any (commutative) ringed space $(X, \mathcal{O}_X)$,
the abelian category of $\mathcal{O}_X$-Modules

$$\mathcal{M}od_{\mathcal{O}_X}$$

has enough injectives.

Remark:
For any injective $\mathcal{O}_X$-Module $\mathcal{M}$ and any $\mathcal{O}_X$-Module $\mathcal{N}$,
the $\mathcal{O}_X$-Module $\mathcal{H}om(\mathcal{N}, \mathcal{M})$ is flabby. In particular, $\mathcal{M}$ is flabby.
Indeed, for any open subset $(U \overset{i}{\hookrightarrow} X)$, the monomorphism of $\mathcal{O}_X$-Modules

$$i_! i^* \mathcal{N} \hookrightarrow \mathcal{N}$$

induces a surjective map:

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{N}, \mathcal{M}) & \longrightarrow\!\!\!\!\!\rightarrow & \mathrm{Hom}(i_! i^* \mathcal{N}, \mathcal{M}) \\ \| & & \| \\ \Gamma(X, \mathcal{H}om(\mathcal{N}, \mathcal{M})) & & \mathrm{Hom}(i^* \mathcal{N}, i^* \mathcal{M}) \\ & & \| \\ & & \Gamma(U, \mathcal{H}om(\mathcal{N}, \mathcal{M})) \end{array}$$

Sketch of proof of the proposition:

(1) The case when $X = \{\bullet\}$ and $\mathcal{O}_X = R$ is a commutative ring:

- First, $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module as multiplication by any integer m is surjective in $\mathbb{Q}/\mathbb{Z}$.
- The canonical isomorphism

$$\mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) = \mathrm{Hom}_{\mathbb{Z}}(N \otimes_R M, \mathbb{Q}/\mathbb{Z})$$

  shows that, for any free R-module M, the R-module

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$$

  is injective.
- For any R-module M, the canonical morphism

$$M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

  is injective.
  So, for any free R-module M' endowed with an epimorphism

$$M' \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}),$$

  there is an induced embedding

$$M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{Q}/\mathbb{Z})$$

  into the injective R-module $\mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{Q}/\mathbb{Z})$.

(2) The general case of $(X, \mathcal{O}_X)$:

Let X' = set X with the discrete topology,
$(p : X' \to X)$ = canonical continuous map.

The functor $p^{-1} : \mathrm{Sh}(X) \to \mathrm{Sh}(X')$ associates to any sheaf $\mathcal{M}$ on X the family of its fibers $\mathcal{M}_x$ at the points $x \in X$.

If $\mathcal{M}$ is an object of $\mathcal{M}od_{\mathcal{O}_X}$, choose at any $x \in X$ an embedding

$$\mathcal{M}_x \hookrightarrow \mathcal{M}'_x$$

of $\mathcal{M}_x$ into an injective $\mathcal{O}_{X,x}$-module $\mathcal{M}'_x$.

It can be seen as an embedding

$$p^{-1}\mathcal{M} \hookrightarrow \mathcal{M}'$$

into an injective $p^{-1}\mathcal{O}_X$-Module.

It induces an embedding

$$\mathcal{M} \hookrightarrow p_*\mathcal{M}'$$

into the $\mathcal{O}_X$-Module $p_*\mathcal{M}'$.

Lastly, $p_*\mathcal{M}'$ is injective
according to the following lemma:

Lemma:

Let $(\mathcal{A} \xrightarrow{F} \mathcal{B}, \mathcal{B} \xrightarrow{G} \mathcal{A})$

$=$ pair of adjoint (additive) functors between abelian categories $\mathcal{A}, \mathcal{B}$.

Suppose the left adjoint F is exact.

Then the right adjoint G transforms

injective objects of $\mathcal{B}$ into injective objects of $\mathcal{A}$.

Proof:

Consider an injective object I of $\mathcal{B}$ and a short exact sequence

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

of $\mathcal{A}$.

Then the sequence

$$0 \longrightarrow \mathrm{Hom}(A_3, G(I)) \longrightarrow \mathrm{Hom}(A_2, G(I)) \longrightarrow \mathrm{Hom}(A_1(G(I)) \longrightarrow 0$$

identifies with the sequence

$$0 \longrightarrow \mathrm{Hom}(F(A_3), I) \longrightarrow \mathrm{Hom}(F(A_2), I) \longrightarrow \mathrm{Hom}(F(A_1), I) \longrightarrow 0$$

which is exact.

Corollary:

Let $(X, \mathcal{O}_X)$ = commutative ringed space,

$\mathcal{I}nj_{\mathcal{O}_X}$ = full additive subcategory of $\mathcal{M}od_{\mathcal{O}_X}$ on injective objects.

Then:

(i) The left-exact functor

$$\mathrm{Hom} : \mathcal{M}od_{\mathcal{O}_X}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathrm{Ab}$$

has a right derived functor

$$\begin{array}{rrcl} \mathrm{RHom} \quad : & D(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathcal{O}_X}) & \longrightarrow & D(\mathrm{Ab})\,, \\ & D^-(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathcal{O}_X}) & \longrightarrow & D^+(\mathrm{Ab}) \end{array}$$

whose restriction to the equivalent subcategory $D(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^+(\mathcal{I}nj_{\mathcal{O}_X})$ is defined by the commutative square:

$$\begin{array}{ccc} K(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times K^+(\mathcal{I}nj_{\mathcal{O}_X}) & \xrightarrow{\mathrm{Hom}} & K(\mathrm{Ab}) \\ \downarrow & & \downarrow \\ D(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^+(\mathcal{I}nj_{\mathcal{O}_X}) & \xrightarrow{\mathrm{RHom}} & D(\mathrm{Ab}) \end{array}$$

(ii) The left-exact functor

$$\mathcal{H}om : \mathcal{M}od_{\mathcal{O}_X}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_X}$$

has a right derived functor

$$\begin{array}{rcrcl} \mathrm{R}\mathcal{H}om & : & D(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^{+}(\mathcal{M}od_{\mathcal{O}_X}) & \longrightarrow & D(\mathcal{M}od_{\mathcal{O}_X}), \\ & & D^{-}(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^{+}(\mathcal{M}od_{\mathcal{O}_X}) & \longrightarrow & D^{+}(\mathcal{M}od_{\mathcal{O}_X}) \end{array}$$

whose restriction to the equivalent subcategory $D(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^{+}(\mathcal{I}nj_{\mathcal{O}_X})$ is defined by the commutative square:

$$\begin{array}{ccc} K(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times K^{+}(\mathcal{I}nj_{\mathcal{O}_X}) & \xrightarrow{\mathcal{H}om} & K(\mathcal{M}od_{\mathcal{O}_X}) \\ \downarrow & & \downarrow \\ D(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^{+}(\mathcal{I}nj_{\mathcal{O}_X}) & \xrightarrow{\mathrm{R}\mathcal{H}om} & D(\mathcal{M}od_{\mathcal{O}_X}) \end{array}$$

Remarks:

(i) Let $p : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$
$=$ canonical projection to the point space $S = \{\bullet\}$
endowed with $\mathcal{O}_S = \mathbb{Z}$.

Then the functors

$$\mathrm{RHom} \qquad \text{and} \qquad \mathrm{R}p_* \circ \mathrm{R}\mathcal{H}om$$

from $D^-(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathcal{O}_X})$ [resp. $D(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathcal{O}_X})$] to $D^+(\mathrm{Ab})$ [resp. $D(\mathrm{Ab})$] are canonically isomorphic [resp. if p has finite cohomological dimension].

(ii) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$
$=$ morphism of commutative ringed spaces
such that $f^* : \mathcal{M}od_{\mathcal{O}_Y} \to \mathcal{M}od_{\mathcal{O}_X}$ is exact.

Then the functors

$$\mathrm{RHom}(f^*(\bullet), \bullet) \qquad \text{and} \qquad \mathrm{RHom}(\bullet, \mathrm{R}f_*(\bullet))$$

from $D(\mathcal{M}od_{\mathcal{O}_Y})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathcal{O}_X})$ to $D(\mathrm{Ab})$ are canonically isomorphic, as well as the functors

$$\mathrm{R}f_* \circ \mathrm{R}\mathcal{H}om(f^*(\bullet), \bullet) \qquad \text{and} \qquad \mathrm{R}\mathcal{H}om(\bullet, \mathrm{R}f_*(\bullet))$$

from $D^-(\mathcal{M}od_{\mathcal{O}_Y})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathcal{O}_X})$ [resp. $D(\mathcal{M}od_{\mathcal{O}_Y})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathcal{O}_X})$] to $D^+(\mathcal{M}od_{\mathcal{O}_Y})$ [resp. to $D(\mathcal{M}od_{\mathcal{O}_Y})$ if f_* has finite cohomological dimension].

Proposition: Let $(X, \mathcal{O}_X) =$ commutative ringed space.
Then the full additive subcategory $\mathcal{P}f_{\mathcal{O}_X}$ of $\mathcal{M}od_{\mathcal{O}_X}$ is $(\mathcal{M} \otimes_{\mathcal{O}_X} \bullet)$-acyclic for any object $\mathcal{M}$ of $\mathcal{M}od_{\mathcal{O}_X}$ and such that the functor $\bullet \otimes_{\mathcal{O}_X} \mathcal{P}$ is exact for any object $\mathcal{P}$ of $\mathcal{P}f_{\mathcal{O}_X}$.

Proof: The objects of $\mathcal{P}f_{\mathcal{O}_X}$ are $\mathcal{O}_X$-Modules $\mathcal{P}$ such that, for any $x \in X$, the fiber $\mathcal{P}_x$ is a projective $\mathcal{O}_{X,x}$-module (or, equivalently, a direct summand of a free module).
The conclusion comes from the following facts:

- For any $\mathcal{O}_X$-Modules $\mathcal{M}_1, \mathcal{M}_2$, the fiber $(\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2)_x$ at $x \in X$ identifies with $\mathcal{M}_{1,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{M}_{2,x}$.
- A sequence of $\mathcal{O}_X$-Modules
$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$
is exact if an only if, for any $x \in X$, the sequence
$$0 \longrightarrow \mathcal{M}_{1,x} \longrightarrow \mathcal{M}_{2,x} \longrightarrow \mathcal{M}_{3,x} \longrightarrow 0$$
of $\mathcal{O}_{X,x}$-modules is exact.
- If M is a projective module over a commutative ring R, the functor $\bullet \otimes_R M$ is exact and any exact sequence
$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M \longrightarrow 0$$
is split.

Corollary:
Let $(X, \mathcal{O}_X) =$ commutative ringed space.
Then:

(i) The right-exact functor

$$\otimes : \mathcal{M}od_{\mathcal{O}_X} \times \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_X}$$

has a left-exact functor

$$\overset{\mathrm{L}}{\otimes} : \begin{array}{rcl} D(\mathcal{M}od_{\mathcal{O}_X}) \times D^{-}(\mathcal{M}od_{\mathcal{O}_X}) & \longrightarrow & D(\mathcal{M}od_{\mathcal{O}_X}), \\ D^{-}(\mathcal{M}od_{\mathcal{O}_X}) \times D^{-}(\mathcal{M}od_{\mathcal{O}_X}) & \longrightarrow & D^{-}(\mathcal{M}od_{\mathcal{O}_X}) \end{array}$$

whose restriction to the equivalent subcategory $D(\mathcal{M}od_{\mathcal{O}_X}) \times D^{-}(\mathcal{P}f_{\mathcal{O}_X})$ is defined by the commutative square:

$$\begin{array}{ccc} K(\mathcal{M}od_{\mathcal{O}_X}) \times K^{-}(\mathcal{P}f_{\mathcal{O}_X}) & \xrightarrow{\ \otimes\ } & K(\mathcal{M}od_{\mathcal{O}_X}) \\ \downarrow & & \downarrow \\ D(\mathcal{M}od_{\mathcal{O}_X}) \times D^{-}(\mathcal{P}f_{\mathcal{O}_X}) & \xrightarrow{\ \overset{\mathrm{L}}{\otimes}\ } & D(\mathcal{M}od_{\mathcal{O}_X}) \end{array}$$

(ii) If $\otimes$ has cohomological dimension $\leq d$, it even has a left derived functor

$$\overset{\mathrm{L}}{\otimes} : D(\mathcal{M}od_{\mathcal{O}_X}) \times D(\mathcal{M}od_{\mathcal{O}_X}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}_X}).$$

Remarks:

(i) An $\mathcal{O}_X$-Module $\mathcal{M}$ is called “flat” if it is $\otimes$-acyclic, i.e. verifies the equivalent conditions:

(1) The functor $\bullet \otimes_{\mathcal{O}_X} \mathcal{M} : \mathcal{M}od_{\mathcal{O}_X} \to \mathcal{M}od_{\mathcal{O}_X}$ is exact.

(2) For any short exact sequence of $\mathcal{O}_X$-Modules

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M} \longrightarrow 0$$

and any $\mathcal{O}_X$-Module $\mathcal{N}$, the sequence

$$0 \longrightarrow \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}_1 \longrightarrow \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}_2 \longrightarrow \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow 0$$

is exact.

(ii) An $\mathcal{O}_X$-Module $\mathcal{M}$ is flat if and only if, for any $x \in X$, the fiber $\mathcal{M}_x$ is flat as a module over $\mathcal{O}_{X,x}$.

(iii) The functor $\otimes$ in $\mathcal{M}od_{\mathcal{O}_X}$ has cohomological dimension $\leq d$ if and only if, for any $x \in X$, the functor $\otimes$ in $\mathrm{Mod}_{\mathcal{O}_{X,x}}$ has cohomological dimension $\leq d$.

(iv) Commutativity: The functors

$$(\mathcal{M}_1, \mathcal{M}_2) \longmapsto \mathcal{M}_1 \overset{\mathrm{L}}{\otimes} \mathcal{M}_2 \quad \text{and} \quad (\mathcal{M}_1, \mathcal{M}_2) \longmapsto \mathcal{M}_2 \overset{\mathrm{L}}{\otimes} \mathcal{M}_1$$

from $D^-(\mathcal{M}od_{\mathcal{O}_X}) \times D^-(\mathcal{M}od_{\mathcal{O}_X})$ to $D^-(\mathcal{M}od_{\mathcal{O}_X})$
[resp. from $D(\mathcal{M}od_{\mathcal{O}_X}) \times D(\mathcal{M}od_{\mathcal{O}_X})$ to $D(\mathcal{M}od_{\mathcal{O}_X})$ if $\otimes$ has finite cohomological dimension on $\mathcal{M}od_{\mathcal{O}_X}$] are canonically isomorphic.

(v) Associativity: The functors

$$(\bullet \overset{\mathrm{L}}{\otimes} \bullet) \overset{\mathrm{L}}{\otimes} \bullet \quad \text{and} \quad \bullet \overset{\mathrm{L}}{\otimes} (\bullet \overset{\mathrm{L}}{\otimes} \bullet)$$

from $D(\mathcal{M}od_{\mathcal{O}_X}) \times D^-(\mathcal{M}od_{\mathcal{O}_X}) \times D^-(\mathcal{M}od_{\mathcal{O}_X})$
[resp. $D(\mathcal{M}od_{\mathcal{O}_X}) \times D(\mathcal{M}od_{\mathcal{O}_X}) \times D(\mathcal{M}od_{\mathcal{O}_X})$ if $\otimes$ has finite cohomological dimension on $\mathcal{M}od_{\mathcal{O}_X}$] to $D(\mathcal{M}od_{\mathcal{O}_X})$ are canonically isomorphic.

(vi) Compatibility with pull back: For any morphism of commutative ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, the functors

$$\mathrm{L}f^*(\bullet \overset{\mathrm{L}}{\otimes} \bullet) \quad \text{and} \quad \mathrm{L}f^*(\bullet) \overset{\mathrm{L}}{\otimes} \mathrm{L}f^*(\bullet)$$

from $D^-(\mathcal{M}od_{\mathcal{O}_Y}) \times D^-(\mathcal{M}od_{\mathcal{O}_Y})$ to $D^-(\mathcal{M}od_{\mathcal{O}_X})$
[resp. from $D(\mathcal{M}od_{\mathcal{O}_Y}) \times D^-(\mathcal{M}od_{\mathcal{O}_Y})$ to $D^-(\mathcal{M}od_{\mathcal{O}_X})$ if f^* has finite cohomological dimension, resp. from $D(\mathcal{M}od_{\mathcal{O}_Y}) \times D(\mathcal{M}od_{\mathcal{O}_Y})$ to $D(\mathcal{M}od_{\mathcal{O}_X})$ if $\otimes$ has finite cohomological dimension on $\mathcal{M}od_{\mathcal{O}_X}$ and on $\mathcal{M}od_{\mathcal{O}_Y}$] are canonically isomorphic.

(vii) If $\mathcal{M}$ is a flat $\mathcal{O}_X$-Module and $\mathcal{N}$ an injective $\mathcal{O}_X$-Module, then $\mathcal{H}om(\mathcal{M}, \mathcal{N})$ is an injective $\mathcal{O}_X$-Module.
This follows from the identification between the functors

$$\mathrm{Hom}(\bullet, \mathcal{H}om(\mathcal{M}, \mathcal{N})) \qquad \text{and} \qquad \mathrm{Hom}(\bullet \otimes \mathcal{M}, \mathcal{N})$$

from $\mathcal{M}od_{\mathcal{O}_X}$ to Ab.

(viii) The previous remark implies that the pairs of functors

$$\begin{array}{rll} & \mathrm{R}\mathcal{H}om(\bullet, \mathrm{R}\mathcal{H}om(\bullet, \bullet)) & \text{and} \quad \mathrm{R}\mathcal{H}om(\bullet \overset{\mathrm{L}}{\otimes} \bullet, \bullet) \\ \text{or} & \mathrm{RHom}(\bullet, \mathrm{R}\mathcal{H}om(\bullet, \bullet)) & \text{and} \quad \mathrm{RHom}(\bullet \overset{\mathrm{L}}{\otimes} \bullet, \bullet) \\ \text{or} & \mathrm{Hom}(\bullet, \mathrm{R}\mathcal{H}om(\bullet, \bullet)) & \text{and} \quad \mathrm{Hom}(\bullet \overset{\mathrm{L}}{\otimes} \bullet, \bullet) \end{array}$$

from $D(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^-(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathcal{O}_X})$ to $D(\mathcal{M}od_{\mathcal{O}_X})$, $D(\mathrm{Ab})$ or Ab are canonically isomorphic.

(ix) For any object $\mathcal{L}$ of $D^+(\mathcal{M}od_{\mathcal{O}_X})$, there is a canonical morphism from the identity functor $\mathrm{id} : \mathcal{M} \mapsto \mathcal{M}$ of $D(\mathcal{M}od_{\mathcal{O}_X})$ to the functor

$$\mathcal{M} \longmapsto \mathrm{R}\mathcal{H}om(\mathrm{R}\mathcal{H}om(\mathcal{M}, \mathcal{L}), \mathcal{L}) \,.$$

Geometric categories

Definition:
Let Sp = category of (commutative) ringed spaces $(X, \mathcal{O}_X)$.
A subcategory $\mathcal{G}$ of Sp is called "geometric" if:

- If $(X, \mathcal{O}_X)$ is an object of $\mathcal{G}$,
  then any open subspace $(U, \mathcal{O}_{X|U})$ is in $\mathcal{G}$
  and the associated open embedding $(U, \mathcal{O}_{X|U}) \to (X, \mathcal{O}_X)$ is in $\mathcal{G}$.
- If $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ is a morphism of $\mathcal{G}$,
  then for any open subspace V of Y, the induced morphism of Sp

$$(f^{-1}(V), \mathcal{O}_{X|f^{-1}(V)}) \xrightarrow{f} (V, \mathcal{O}_{Y|V}) \quad \text{is in} \quad \mathcal{G}\,.$$

- Conversely, if $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ are 2 objects of $\mathcal{G}$
  related by a morphism $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ of Sp
  such that there exists an open cover $Y = \bigcup\limits_{i \in I} V_i$ of Y
  for which the induced morphisms $(f^{-1}(V_i), \mathcal{O}_{X|f^{-1}(V_i)}) \to (V_i, \mathcal{O}_{Y|V_i})$ are in $\mathcal{G}$,
  then f is a morphism of $\mathcal{G}$.

Examples:

- The choice of any commutative ring R defines an embedding

$$\mathrm{Top} \hookrightarrow \mathrm{Sp}$$

by endowing any topological space X with the "constant" structure ring

$$\mathrm{R}_X = p_X^{-1}\mathrm{R}$$

if p_X denotes the canonical projection $p_X : X \to \{\bullet\}$.
- The category of (countable at infinity) differential manifolds.
- The category of (countable at infinity) analytic manifolds.
- The category of schemes.

Definition: Let $\mathcal{G}$ = geometric subcategory of $\mathbb{S}\mathrm{p}$.

(i) A property (P) of objects of $\mathcal{G}$ (which is stable by isomorphisms) is called "local" if

- any open subspace of an object of $\mathcal{G}$ verifying (P) also verifies (P),
- conversely, if an object of $\mathcal{G}$ has an open cover by open subspaces which verify (P), then it verifies (P).

(ii) A property (P) of morphisms $X \xrightarrow{f} S$ of $\mathcal{G}$ (which is stable by composition with isomorphisms) is called "local on the base" if, for any morphism $X \xrightarrow{f} S$ of $\mathcal{G}$:

- if f verifies (P), then for any open subspace V of S, the induced morphism $f^{-1}(V) \xrightarrow{f} V$ verifies (P),
- conversely, if there exists an open cover $S = \bigcup\limits_{i \in I} V_i$ such that each $f^{-1}(V_i) \xrightarrow{f} V_i$ verifies (P), then f verifies (P).

(iii) Such a property is called "local on the source" if, furthermore, for any morphism $X \xrightarrow{f} S$ of $\mathcal{G}$:

- if f verifies (P), then for any open subspace U of X, the induced morphism $U \xrightarrow{f} S$ verifies (P),
- conversely, if there exists an open cover $X = \bigcup\limits_{i \in I} U_i$ such that each $U_i \to S$ verifies (P), then f verifies (P).

(iv) A morphism of $\mathcal{G}$

$$X \longrightarrow S$$

is called "squarable" if, for any morphism of $\mathcal{G}$

$$S' \longrightarrow S\,,$$

the fiber product

$$X \times_S S' \longrightarrow S'$$

is representable in $\mathcal{G}$.

(v) A (stable) property of morphisms of $\mathcal{G}$ is called "universal" if any morphism $X \to S$ of $\mathcal{G}$ verifying (P) is squarable and all induced morphisms

$$X \times_S S' \longrightarrow S'$$

also verify (P).

Remarks:

- A fiber product $X \times_S S'$ in the category $\mathcal{G}$ is not necessarily a fiber product in the category Sp of (commutative) ringed spaces.
- A squarable morphism $X \xrightarrow{f} S$ of $\mathcal{G}$ is said to verify "universally" some property (P) if, for any morphism $S' \to S$, the induced morphism

$$X \times_S S' \longrightarrow S'$$

verifies (P).

Examples:

(i) The property for an object $(X, \mathcal{O}_X)$ of Sp to be

$$\begin{cases} \bullet & \text{a topological space endowed with a constant structure sheaf } R_X, \\ \bullet & \text{locally ringed,} \\ \bullet & \text{a differential [resp. analytic] manifold,} \\ \bullet & \text{a scheme,} \\ \bullet & \text{such that the functor } \bullet \otimes_{\mathcal{O}_X} \bullet \text{ has cohomological dimension } \leq d, \end{cases}$$

is local.

(ii) The property for a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of Sp to be

$$\begin{cases} \bullet & \text{an open embedding,} \\ \bullet & \text{“closed” in the sense that the image of any closed subset of } X \\ & \text{is a closed subset of } Y, \\ \bullet & \text{such that the functor } f_* \text{ has cohomological dimension } \leq d, \end{cases}$$

is local on the base.

(iii) The property for a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of Sp to be

- a morphism of locally ringed spaces,
- a morphism of differential [resp. analytic] manifolds,
- a morphism of schemes,
- such that the functor f^* has cohomological dimension $\leq d$,
- flat in the sense that $\mathcal{O}_X$ is flat over $f^{-1}\mathcal{O}_Y$ (or, equivalently, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$ for any $x \in X$),
- a submersion of differential [resp. analytic] manifolds in the sense it is locally diffeomorphic to the projection
  $$\mathbb{R}^d \times Y \longrightarrow Y \qquad [\text{resp. } \mathbb{C}^d \times Y \longrightarrow Y\,],$$

is local on the source.

The property for a morphism $X \to Y$ of Top to be "smooth" of relative dimension d, in the sense it is locally homeomorphic to the projection

$$\mathbb{R}^d \times Y \longrightarrow Y$$

is local on the source.

(iv) In the category of differential [resp. analytic] manifolds, submersions are squarable.

In the category Top of topological spaces [resp. Sch of schemes], all morphisms are squarable.

(v) In any geometric category $\mathcal{G}$, the property to be an open immersion is universal.

In the category of differential [resp. analytic] manifolds, the property to be a submersion is universal.

In the category Top of topological spaces, the property to be "smooth" of relative dimension d is universal.

In the category Top or the category Sch of schemes, the property for a morphism $X \to Y$ to be

- "separated" ($=$ relatively Hausdorff) in the sense that the diagonal morphism $X \to X \times_Y X$ is closed,
- "proper" ($=$ relatively compact) in the sense that it is separated and universally closed (i.e. $X \times_Y Y' \to Y'$ is closed for any $Y' \to Y$)

is universal.

The base change morphisms

Lemma:

(i) For any commutative square of Sp

$$\begin{array}{ccc} (X', \mathcal{O}_{X'}) & \xrightarrow{f} & (X, \mathcal{O}_X) \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ (S', \mathcal{O}_{S'}) & \xrightarrow{s} & (S, \mathcal{O}_S) \end{array}$$

there is a canonical morphism of functors

$$s^* \circ p_* \longrightarrow p'_* \circ f^*$$

from $\mathcal{M}od_{\mathcal{O}_X}$ to $\mathcal{M}od_{\mathcal{O}_{S'}}$.

(ii) If furthermore p_*, p'_* [resp. s^*, f^*] have finite cohomological dimension, there is a canonical morphism of functors

$$\mathrm{L}s^* \circ \mathrm{R}p_* \longrightarrow \mathrm{R}p'_* \circ \mathrm{L}f^*$$

from $D^-(\mathcal{M}od_{\mathcal{O}_X})$ to $D^-(\mathcal{M}od_{\mathcal{O}_{S'}})$
[resp. from $D^+(\mathcal{M}od_{\mathcal{O}_X})$ to $D^+(\mathcal{M}od_{\mathcal{O}_{S'}})$].

Proof: This is a consequence of adjointness.

(i) For any object $\mathcal{M}$ of $\mathcal{M}od_{\mathcal{O}_X}$, the identity morphism

$$f^*\mathcal{M} \longrightarrow f^*\mathcal{M}$$

corresponds to a morphism

$$\mathcal{M} \longrightarrow f_* \circ f^*\mathcal{M}$$

which yields

$$p_*\mathcal{M} \longrightarrow p_* \circ f_* \circ f^*\mathcal{M} = s_* \circ p'_* \circ f^*\mathcal{M}$$

which corresponds to a morphism

$$s^* \circ p_*\mathcal{M} \longrightarrow p'_* \circ f^*\mathcal{M}\,.$$

(ii) For any object $\mathcal{M}$ of $D^-(\mathcal{M}od_{\mathcal{O}_X})$ [resp. $D^+(\mathcal{M}od_{\mathcal{O}_X})$], the identity morphism

$$\mathrm{L}f^*\mathcal{M} \longrightarrow \mathrm{L}f^*\mathcal{M}$$

corresponds by adjointness to a morphism

$$\mathcal{M} \longrightarrow \mathrm{R}f_* \circ \mathrm{L}f^*\mathcal{M}$$

which yields

$$\mathrm{R}p_*\mathcal{M} \longrightarrow \mathrm{R}p_* \circ \mathrm{R}f_* \circ \mathrm{L}f^*\mathcal{M} = \mathrm{R}s_* \circ \mathrm{R}p'_* \circ \mathrm{L}f^*\mathcal{M}$$

and again by adjointness

$$\mathrm{L}s^* \circ \mathrm{R}p_*\mathcal{M} \longrightarrow \mathrm{R}p'_* \circ \mathrm{L}f^*\mathcal{M}\,.$$

Compatibility with base change

Definition:

Let $\mathcal{G} =$ geometric subcategory of Sp.

(i) A morphism of $\mathcal{G}$

$$(X, \mathcal{O}_X) \xrightarrow{p} (S, \mathcal{O}_S)$$

is called "cohomologically proper" (of dimension $\leq d$) if

- it is squarable,
- for any cartesian square of $\mathcal{G}$

$$\begin{array}{ccc} (X', \mathcal{O}_{X'}) & \xrightarrow{f} & (X, \mathcal{O}_X) \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ (S', \mathcal{O}_{S'}) & \xrightarrow{s} & (S, \mathcal{O}_S) \end{array}$$

p'_* has finite cohomological dimension $(\leq d)$ and the morphisms

$$s^* \circ p_*(\mathcal{M}) \longrightarrow p'_* \circ f^*(\mathcal{M})$$

[resp.

$$\mathrm{L}s^* \circ \mathrm{R}p_*(\mathcal{M}) \longrightarrow \mathrm{R}p'_* \circ \mathrm{L}f^*(\mathcal{M}) \;]$$

are isomorphisms for any object $\mathcal{M}$ of $\mathcal{M}od_{\mathcal{O}_X}$ [resp. $D^-(\mathcal{M}od_{\mathcal{O}_X})$].

(ii) A morphism of $\mathcal{G}$

$$(Y, \mathcal{O}_Y) \xrightarrow{s} (S, \mathcal{O}_S)$$

is called "cohomologically smooth" (of dimension $\leq d$) if

- it is squarable,
- for any cartesian square of $\mathcal{G}$

$$\begin{array}{ccc} (X', \mathcal{O}_{X'}) & \xrightarrow{f} & (X, \mathcal{O}_X) \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ (Y, \mathcal{O}_Y) & \xrightarrow{s} & (S, \mathcal{O}_S) \end{array}$$

f^* has finite cohomological dimension $(\leq d)$ and the morphisms

$$s^* \circ p_*(\mathcal{M}) \longrightarrow p'_* \circ f^*(\mathcal{M})$$

[resp.

$$\mathrm{L}s^* \circ \mathrm{R}p_*(\mathcal{M}) \longrightarrow \mathrm{R}p'_* \circ \mathrm{L}f^*(\mathcal{M}) \ ]$$

are isomorphisms for any object $\mathcal{M}$ of $\mathcal{M}od_{\mathcal{O}_X}$ [resp. $D^+(\mathcal{M}od_{\mathcal{O}_X})$].

Remarks:

(i) For squarable morphisms of $\mathcal{G}$,
the property to be "cohomologically proper of dimension $\leq d$"
is universal and local on the base.

(ii) For squarable morphisms of $\mathcal{G}$,
the property to be "cohomologically smooth of dimension $\leq d$"
is universal and local on the source.

(iii) We are going to prove that
in the category Top
embedded in Sp by the choice of a coefficient ring R:

- any proper continuous map
$$X \longrightarrow S$$
whose fibers have cohomological dimension $\leq d$
is cohomologically proper of dimension $\leq d$,
- any continuous map
$$Y \longrightarrow S$$
which is "smooth" of relative dimension d
is cohomologically smooth of dimension $\leq d$.

Cohomological properness for proper maps of topological spaces

Lemma:
Let $X \xrightarrow{p} S$

$=$ continuous map between topological spaces which is proper.

Then, for any point s of S, the fiber $X_s = p^{-1}(s)$ is Hausdorff and compact.

Proof:

- The diagonal embedding $X \to X \times_S X$ is closed, so each $X_s \hookrightarrow X_s \times X_s$ is closed, which means that X_s is Hausdorff.
- Let $s \in S$ and consider an open cover $X_s = \bigcup_{i \in I} U_i$ of X_s. Let $\mathcal{P}(I)$ be endowed with the topology for which a subset $P \subset \mathcal{P}(I)$ is open if, for any element $J \in P$ there exists a finite subset $\{i_1, \ldots, i_k\} = J_0 \subset J$ such that $J' \supseteq J_0 \Rightarrow J' \in P$. The projection $X_s \times \mathcal{P}(I) \to \mathcal{P}(I)$ is closed and its fiber over the element $I \in \mathcal{P}(I)$ is covered by the family of open subsets $U_i \times \{J \in \mathcal{P}(I) \mid i \in J\}$. So there exists a finite subset $\{i_1, \ldots, i_k\} = J_0$ of I such that, for any $x \in X_s$,

  $$J \supseteq J_0 \Rightarrow \exists\, i \in I,\ x \in U_i \quad \text{and} \quad J \ni i\,.$$

  Taking $J = J_0$, it means $X_s = U_{i_1} \cup \cdots \cup U_{i_k}$ as wanted.

Theorem:

Let $R =$ coefficient (commutative) ring,

$\mathrm{Top} =$ category of topological spaces X endowed with the constant structure sheaf R_X,

$(X \xrightarrow{p} S) =$ proper morphism of Top.

Then:

(i) For any cartesian square of Top

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{s} & S \end{array}$$

and any object $\mathcal{M}$ of $\mathcal{M}od_{R_X}$ [resp. $D^+(\mathcal{M}od_{R_X})$], the canonical morphism

$$\begin{array}{lccc} & s^* \circ p_*(\mathcal{M}) & \longrightarrow & p'_* \circ f^*(\mathcal{M}) \\ \text{[resp.} & s^* \circ \mathrm{R}p_*(\mathcal{M}) & \longrightarrow & \mathrm{R}p'_* \circ f^*(\mathcal{M}) \text{]} \end{array}$$

is an isomorphism.

(ii) If the fibers X_s of $X \to S$ all have cohomological dimension $\leq d$, p_* and the p'_* all have cohomological dimension $\leq d$, p is cohomologically proper of dimension $\leq d$ and (i) even holds for any object $\mathcal{M}$ of $D(\mathcal{M}od_{R_X})$.

Remark: All morphisms $s^* = s^{-1}$ and $f^* = f^{-1}$ are exact.

Proof:

(i) It is enough to prove the assertions when $S' = \{\bullet\}$
so that s is a point of S and $X' = p^{-1}(s) = X_s$.
If $\mathcal{M}$ is an object of $\mathcal{M}od_{R_S}$, denote $\mathcal{M}_s$ its pull-back on X_s. Then

$$s^* \circ p_* \mathcal{M} = (p_* \mathcal{M})_s = \varinjlim_{V \ni s} \Gamma(p^{-1}(V), \mathcal{M})$$

while

$$p'_* \circ f^* \mathcal{M} = \Gamma(X_s, \mathcal{M}_s)\,.$$

- **Injectivity of** $\varinjlim_{V \ni s} \Gamma(p^{-1}(V), \mathcal{M}) \to \Gamma(X_s, \mathcal{M}_s)$**:**

  Let $V =$ open neighborhood of s in S,
  $m =$ section of $\mathcal{M}$ on $p^{-1}(V)$ whose image in $\Gamma(X_s, \mathcal{M}_s) = 0$.
  For any $x \in X_s$, there exists an open neighborhood $U_x \subset p^{-1}(V)$ of x in X such that $m = 0$ on U_x.
  Then m is 0 on the open subset $U = \bigcup_{x \in X_s} U_x$ which covers the fiber X_s.

  As $X \xrightarrow{p} S$ is closed, there exists an open neighborhood $V' \subset V$ of s in S such that $p^{-1}(V') \subset U$ and the image of m in

  $$\Gamma(p^{-1}(V), \mathcal{M})$$

  is 0.

- **Surjectivity of** $\varinjlim_{V \ni s} \Gamma(p^{-1}(V), \mathcal{M}) \to \Gamma(X_s, \mathcal{M}_s)$**:**

  Let m be a section of $\mathcal{M}_s$ on X_s.

  For any $x \in X_s$, there is an open subset $U_x \ni x$ of X such that $m \in \Gamma(X_s \cap U_x, \mathcal{M}_s)$ lifts to some $m_x \in \Gamma(U_x, \mathcal{M})$. As $X \to S$ is separated, there are for any $y \in X_s - (X_s \cap U_x)$ open subsets $V'_y \ni x$, $V''_y \ni y$ of X such that $V'_y \cap V''_y = \emptyset$. The compact set $X_s - (X_s \cap U_x)$ can be covered by finitely many V''_y and so one can find an open subset

  $$V_x \ni x \quad \text{such that} \quad \overline{V}_x \cap X_s \subset U_x \cap X_s \,.$$

  The compact fiber X_s can be covered by finitely many open subsets V_{x_i}, $1 \leq i \leq k$.

  For any i, $\overline{V}_{x_i} - (\overline{V}_{x_i} \cap U_{x_i})$ is a closed subset of X whose intersection with X_s is $\emptyset$.

  As $X \xrightarrow{p} S$ is closed, there is an open subset $U \ni s$ of S such that $\overline{V}_{x_i} \cap p^{-1}(U) \subset U_{x_i} \cap p^{-1}(U)$, $1 \leq i \leq k$, and also $p^{-1}(U) \subset \bigcup_{1 \leq i \leq k} V_{x_i}$.

  For any $i \neq j$, the support of the section

  $$m_{x_i} - m_{x_j} \in \Gamma(p^{-1}(U) \cap \overline{V}_{x_i} \cap \overline{V}_{x_j}, \mathcal{M})$$

  is a closed subset of $p^{-1}(U)$ whose intersection with X_s is $\emptyset$.

  So there is an open subset $U' \ni s$ of $U \subset S$ such that, for any $i \neq j$, m_{x_i} and m_{x_j} coincide on $p^{-1}(U') \cap \overline{V}_{x_i} \cap \overline{V}_{x_j}$.

  They define an element of $\Gamma(p^{-1}(U'), \mathcal{M})$ which lifts m.

- **Restrictions of flabby sheaves to fibers are soft:**

  Let's prove that if $\mathcal{M}$ is flabby, $\mathcal{M}_s$ is soft.

  It is enough to prove that any section $m \in \Gamma(K, \mathcal{M}_s)$ on a compact subset K of X_s lifts to $\Gamma(U, \mathcal{M})$ for some open subset U of X containing K.

  For any $x \in K$, there is an open subset $U_x \ni x$ of X such that $m \in \Gamma(K \cap U_x, \mathcal{M}_s)$ lifts to some $m_x \in \Gamma(U_x, \mathcal{M})$.

  Then, for any such x, one can find an open subset

  $$V_x \ni x \quad \text{such that} \quad \overline{V}_x \cap X_s \subset U_x \cap X_s \, .$$

  The compact set K can be covered by finitely many open subsets V_{x_i}, $1 \leq i \leq k$.

  As $X \xrightarrow{p} S$ is closed, there is an open subset $V \ni s$ of S such that $\overline{V}_{x_i} \cap p^{-1}(V) \subset U_{x_i} \cap p^{-1}(V)$, $1 \leq i \leq k$.

  For any $i \neq j$, the support $Z_{i,j}$ of the section

  $$m_{x_i} - m_{x_j} \in \Gamma(p^{-1}(V) \cap \overline{V}_{x_i} \cap \overline{V}_{x_j}, \mathcal{M})$$

  is a closed subset of $p^{-1}(V)$ whose intersection with K is $\emptyset$.

  Then $U = p^{-1}(V) \cap \big(\bigcup_{1 \leq i \leq k} V_{x_i} \big) - \bigcup_{i \neq j} Z_{i,j}$ is an open subset of X which contains K and the section $m \in \Gamma(K, \mathcal{M}_s)$ lifts to $\Gamma(U, \mathcal{M})$.

  So, if $\mathcal{M}$ is flabby, $\mathcal{M}_s$ is acyclic relatively to the functor $\mathrm{R}\Gamma(X_s, \bullet)$.

  It is enough for proving that $s^* \circ \mathrm{R}p_* \to \mathrm{R}'p_* \circ f^*$ is an isomorphism.

(ii) follows from (i).

Sheaf cohomology of the interval $[0,1]$

Proposition:

Let $p:[0,1]\to\{\bullet\}$ be the projection,
$R=$ coefficient (commutative) ring,
$\mathcal{M}=p^{-1}R$-Module on $[0,1]$.

Then:

(i) We always have $R^j p_*\mathcal{M}=0, \forall j>1$.

(ii) If $p_*\mathcal{M}\to\mathcal{M}_t$ is surjective at any point $t\in[0,1]$, we even have

$$R^j p_*\mathcal{M}=0\,,\qquad \forall j\geq 1\,.$$

(iii) If M is an R-module, the natural morphism

$$M\longrightarrow Rp_*\circ p^{-1}M$$

is an isomorphism.

Corollary:
Let $(Y, \mathcal{O}_Y)$ = (commutative) ringed space,
$X = Y \times [0,1]^d$ for some $d \geq 1$
endowed with $p : Y \times [0,1]^d \to Y$
and $\mathcal{O}_X = p^{-1}\mathcal{O}_Y$.

Then:

(i) The functor $p_* : \mathcal{M}od_{\mathcal{O}_X} \to \mathcal{M}od_{\mathcal{O}_Y}$ has cohomological dimension $\leq d$.

(ii) For any object of $D(\mathcal{M}od_{\mathcal{O}_Y})$, the canonical morphism

$$\mathcal{M} \longrightarrow \mathrm{R}p_* \circ p^{-1}\mathcal{M}$$

is an isomorphism.

Proof of the corollary:
It is enough to consider the case when $d = 1$.
As $p : Y \times [0,1] \to Y$ is proper, we are reduced to the case when Y is a point $\{\bullet\}$ endowed with a commutative coefficient ring $R = \mathcal{O}_Y$.
So we are reduced to the proposition.

Proof of the proposition:

(i), (ii) Consider $j \geq 1$.
For any $0 \leq t \leq t' \leq 1$, consider the embedding

$$i_{t,t'} : [t,t'] \hookrightarrow [0,1]$$

and the induced map

$$i_{t,t'}^* : H^j([0,1],\mathcal{M}) \longrightarrow H^j([0,1],(i_{t,t'})_*\mathcal{M}) = H^j([t,t'],\mathcal{M}) \,.$$

For $m \in H^j([0,1],\mathcal{M})$, let

$$J_m = \{t \in [0,1], i_{0,t}^*(m) = 0\} \,.$$

- First, we have $0 \in J_m$.
- Secondly, we have for $t < 1$

$$H^j([0,t],\mathcal{M}) = \varinjlim_{t'>t} H^j([0,t'],\mathcal{M}) \,.$$

  This implies that if $t < 1$ belongs to J_m,
  there exists $t' > t$ belonging to J_m.

- For $0 \le t \le t' \le 1$, the short exact sequence of sheaves on $[0,1]$

$$0 \longrightarrow (i_{0,t'})_* i_{0,t'}^* \mathcal{M} \longrightarrow (i_{0,t})_* i_{0,t}^* \mathcal{M} \oplus (i_{t,t'})_* i_{t,t'}^* \mathcal{M} \longrightarrow (i_{t,t})_* \mathcal{M}_t \longrightarrow 0$$

  induces a long exact sequence of cohomology which yields isomorphisms

$$H^j([0,t'],\mathcal{M}) \xrightarrow{\sim} H^j([0,t],\mathcal{M}) \oplus H^j([t,t'],\mathcal{M})$$

  for any $j \geq 2$ and even for $j = 1$ if

$$H^0([0,1],\mathcal{M}) \longrightarrow \mathcal{M}_t \qquad \text{is surjective.}$$

  As $\varinjlim_{t<t'} H^j([t,t'],\mathcal{M}) = 0$, we get that

$$\sup J_m \quad \text{belongs to} \quad J_m\,.$$

- We conclude that $J_m = [0,1]$ which means that $m = 0$ and, as m is arbitrary, $H^j([0,1],\mathcal{M}) = 0$.

(iii) It only remains to prove that

$$M \longrightarrow p_* \circ p^{-1} M$$

is an isomorphism.
It is injective as $[0,1] \xrightarrow{p} \{\bullet\}$ has sections.
Lastly, for any $m \in \Gamma([0,1], p^{-1}M)$, the support of m is both closed and open.
So m is 0 if its image in any fiber $(p^{-1}M)_t = M$ is 0.

Homotopy invariance of sheaf cohomology

Theorem:
Let $(S, \mathcal{O}_S) =$ base (commutative) ringed space,
$(X_1, p_1 : X_1 \to S)$, $(X_2, p_2 : X_2 \to S)$
$=$ two topological spaces endowed with continuous maps to S
and the induced structure sheaves $p_1^{-1}\mathcal{O}_S$, $p_2^{-1}\mathcal{O}_S$.
Suppose we are given two continuous maps

$$f, g : X_1 \rightrightarrows X_2$$

which are compatible with the projections to S and homotopic (relatively to S) in the sense that there exists a xommutative triangle of Top

$$\begin{array}{ccc} X_1 \times [0,1] & \xrightarrow{h} & X_2 \\ \big\downarrow & & \big\downarrow {\scriptstyle p_2} \\ X_1 & \xrightarrow{p_1} & S \end{array}$$

with $f = h(\bullet, 0)$, $g = h(\bullet, 1)$.
Then, for any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathcal{O}_S})$, the morphisms induced by f and g

$$f^*, g^* : \mathrm{R}p_{2,*} \circ p_2^{-1}\mathcal{M} \rightrightarrows \mathrm{R}p_{1,*} \circ p_1^{-1}\mathcal{M}$$

are equal.

Proof:
By functoriality, it is enough to consider the case when

$$X_2 = X_1 \times [0,1]$$

and h is id_{X_2}.
We can also suppose that $S = X_1$, $p_1 = \mathrm{id}_{X_1}$ and p_2 is

$$p : X_1 \times [0,1] \longrightarrow X_1 \, .$$

The conclusion follows from the fact that, for any object

$$\mathcal{M} \quad \text{of} \quad D^+(\mathcal{M}od_{\mathcal{O}_{X_1}}) \, ,$$

the canonical morphism

$$\mathcal{M} \longrightarrow \mathrm{R}p_* \circ p^{-1}\mathcal{M}$$

is an isomorphism whose inverse is the morphism

$$\mathrm{R}p_* \circ p^{-1}\mathcal{M} \longrightarrow \mathcal{M}$$

defined by the section

$$X_1 \longrightarrow X_1 \times [0,1]$$

associated with the choice of any element $t \in [0,1]$.

Cohomological smoothness for smooth maps of topological spaces

Theorem:
Let $R =$ coefficient (commutative) ring,
$\mathrm{Top} =$ category of topological spaces X
endowed with the constant structure sheaf R_X,
$(Y \xrightarrow{s} S) =$ smooth morphism of Top.
Then:

(i) For any cartesian square of Top

$$\begin{array}{ccc} X_Y & \xrightarrow{f} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{s} & S \end{array}$$

and any object $\mathcal{M}$ of $\mathcal{M}od_{R_X}$ [resp. $D^+(\mathcal{M}od_{R_X})$], the canonical morphism

$$s^* \circ p_*(\mathcal{M}) \longrightarrow p'_* \circ f^*(\mathcal{M})$$
$$[\text{resp.} \quad s^* \circ \mathrm{R}p_*(\mathcal{M}) \longrightarrow \mathrm{R}p'_* \circ f^*(\mathcal{M})\ ]$$

is an isomorphism. In other words, s is cohomologically smooth.

(ii) If p_* and p'_* have finite cohomological dimension, (i) even holds for any object $\mathcal{M}$ of $D(\mathcal{M}od_{R_X})$.

Proof:

As the asssertion is local on Y, it is enough to consider the case when

$$Y = S \times \mathbb{R}^d \qquad \text{and so} \qquad X_Y = X \times \mathbb{R}^d .$$

(i) For an object $\mathcal{M}$ of $\mathcal{M}od_{R_X}$ and a degree $k \geq 0$, let's prove that the sheaf morphism

$$s^{-1}\mathrm{R}^k p_* \mathcal{M} \longrightarrow \mathrm{R}^k p'_* f^{-1}\mathcal{M}$$

is an isomorphism.

Let's consider fibers at a point $(t, u) \in S \times \mathbb{R}^d$. The fiber of $s^{-1}\mathrm{R}^k p_* \mathcal{M}$ is

$$\varinjlim_{V \ni t} H^k(p^{-1}(V), \mathcal{M})$$

while the fiber of $\mathrm{R}^k p'_* f^{-1}\mathcal{M}$ is

$$\varinjlim_{V \ni t, U \ni u} H^k(p^{-1}(V) \times U, f^{-1}\mathcal{M}) .$$

But u has a basis of open neighborhoods U in $\mathbb{R}^d$ which are contractible, implying

$$H^k(p^{-1}(V) \times U, f^{-1}\mathcal{M}) = H^k(p'^{-1}(V), \mathcal{M}) .$$

So

$$s^{-1} \circ \mathrm{R}p_* \mathcal{M} \longrightarrow \mathrm{R}p'_* \circ f^{-1}\mathcal{M}$$

is an isomorphism for any object $\mathcal{M}$ of $\mathcal{M}od_{\mathcal{O}_X}$.

This result extends to any object of $D^+(\mathcal{M}od_{\mathcal{O}_X})$ and even of $D(\mathcal{M}od_X)$ if p_* and p'_* both have finite cohomological dimension.

Cohomology with compact support

Lemma:

Let $\mathrm{Top}_{\mathrm{lc}}$ = full subcategory of Top
on spaces X which are Hausdorff and locally compact.

Then:

(i) Any object X of $\mathrm{Top}_{\mathrm{lc}}$ can be written as an open subspace

$$X \hookrightarrow \overline{X}$$

of a topological space $\overline{X}$ which is Hausdorff and compact.

(ii) Any morphism $X \to Y$ of $\mathrm{Top}_{\mathrm{lc}}$ factorises as a composition

$$X \overset{i}{\hookrightarrow} X_1 \xrightarrow{p} Y$$

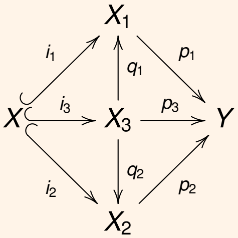
of an open immersion $X \overset{i}{\hookrightarrow} X_1$ into an object X_1 of $\mathrm{Top}_{\mathrm{lc}}$
and a proper continuous map $X_1 \xrightarrow{p} Y$.

(iii) For any two such factorisations

$$X \xhookrightarrow{i_1} X_1 \xrightarrow{p_1} Y\,,$$

$$X \xhookrightarrow{i_2} X_2 \xrightarrow{p_2} Y$$

of a morphism $X \to Y$ of $\mathrm{Top}_{\mathrm{lc}}$, there exists a commutative diagram



such that i_3 is an open immersion just as i_1, i_2,
p_3, q_1, q_2 are proper continuous maps just as p_1, p_2
and $i_3(X) = q_1^{-1}(i_1(X))$, $i_3(X) = q_2^{-1}(i_2(X))$.

Proof:

(i) Let $\overline{X} = X \cup \{\infty\}$ be endowed with the topology such that

$$\begin{cases} \bullet & \text{its restriction to } X \text{ is the topology of } X, \\ \bullet & \text{a subset of } \overline{X} \text{ which contains } \infty \text{ is open} \\ & \text{if and only if its complement is a (closed) compact subset of } X. \end{cases}$$

Then $X \hookrightarrow \overline{X}$ is an open embedding and $\overline{X}$ is Hausdorff and compact.

(ii) Let $X \overset{i}{\hookrightarrow} \overline{X}$ be an open embedding as in (i).

Let X_1 be the closure in $\overline{X} \times Y$ of the graph $X \xhookrightarrow{(\mathrm{id}, f)} X \times Y$ of $f : X \to Y$.
Then X_1 is an object of $\mathrm{Top}_{\mathrm{lc}}$, the projection $X_1 \to Y$ is proper
and $X \hookrightarrow X_1$ is an open immersion.

(iii) Let X_3 be the closure in $X_1 \times_Y X_2$ of the image of $X \xhookrightarrow{(i_1, i_2)} X_1 \times_Y X_2$.
Then X_3 is an object of $\mathrm{Top}_{\mathrm{lc}}$,
its projections on X_1, X_2 and Y are proper continuous maps,
the embedding $i_3 : X \hookrightarrow X_3$ is an open immersion
whose image $i_3(X)$ is the pull-back of $i_1(X)$ or $i_2(X)$.

Theorem:

Let $\mathrm{Top}_{\mathrm{lc}}$ = category of (Hausdorff) locally compact spaces,
R = (commutative) coefficient ring.

Then:

(i) For any morphism $X \xrightarrow{f} Y$ of $\mathrm{Top}_{\mathrm{lc}}$ factorised as

$$X \xhookrightarrow{i} X_1 \xrightarrow{p} Y$$

the composed functor

$$\mathrm{R}f_! = \mathrm{R}p_* \circ i_! : D^+(\mathcal{M}od_{\mathrm{R}_X}) \longrightarrow D^+(\mathcal{M}od_{\mathrm{R}_{X_1}}) \longrightarrow D^+(\mathcal{M}od_Y)$$

doesn't depend, up to canonical isomorphism,
of the choice of the factorisation $X \xhookrightarrow{i} X_1 \xrightarrow{p} Y$ of f.

(ii) For any morphisms of $\mathrm{Top}_{\mathrm{lc}}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z\,,$$

the composed functor $\mathrm{R}(g \circ f)_!$ is canonically isomorphic to $\mathrm{R}g_! \circ \mathrm{R}f_!$.

(iii) For any cartesian square of $\mathrm{Top}_{\mathrm{lc}}$

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y \end{array}$$

the canonical morphism of functors

$$y^* \circ \mathrm{R}f_! \longrightarrow \mathrm{R}f'_! \circ x^*$$

from $D^+(\mathcal{M}od_{\mathrm{R}_X})$ to $D^+(\mathcal{M}od_{\mathrm{R}_{Y'}})$

is an isomorphism.

Remarks:

(i) The functor $\mathrm{R}f_!$ is called the functor of "cohomology with compact support" of X over Y. It can be proven that it is a derived functor of the functor

$$\begin{array}{rcccl} f_! & : & \mathcal{M}od_{\mathrm{R}_X} & \longrightarrow & \mathcal{M}od_{\mathrm{R}_Y}\,, \\ & & \mathcal{M} & \longmapsto & f_!\mathcal{M} \end{array}$$

where, for any open subset V of Y,

$$f_!\mathcal{M}(V) = \{m \in \mathcal{M}(f^{-1}(V)) \mid \mathrm{supp}(m) \text{ is proper over } V\}.$$

(ii) Let

$$\begin{array}{rcl} \mathrm{Top}_{\mathrm{flc}} & = & \text{full subcategory of } \mathrm{Top}_{\mathrm{lc}} \\ & & \text{on spaces } X \text{ which can be written as} \\ & & \text{open subsets } X \hookrightarrow \overline{X} \\ & & \text{of (Hausdorff) compact spaces} \\ & & \text{which have finite cohomological dimension.} \end{array}$$

Then:

- any morphism $f : X \to Y$ of $\mathrm{Top}_{\mathrm{flc}}$ defines a functor
$$\mathrm{R}f_! : D(\mathcal{M}od_{\mathrm{R}_X}) \longrightarrow D(\mathcal{M}od_{\mathrm{R}_Y})$$
isomorphic to the composition $\mathrm{R}p_* \circ i_!$ for any factorisation $X \overset{i}{\hookrightarrow} X_1 \xrightarrow{p} Y$ of f in an open immersion i and a proper continuous map p of finite cohomological dimension,
- each $\mathrm{R}(g \circ f)_!$ is canonically isomorphic to $\mathrm{R}g_! \circ \mathrm{R}f_!$,
- the functors $\mathrm{R}f_!$ commute with base change.

Proof of the theorem:

(i) It is enough to consider two factorisations of f

$$X \overset{i_1}{\hookrightarrow} X_1 \xrightarrow{p_1} Y\,,$$
$$X \overset{i_2}{\hookrightarrow} X_2 \xrightarrow{p_2} Y$$

related by a proper morphism

$$q : X_2 \longrightarrow X_1$$

such that $q \circ i_2 = i_1$, $p_1 \circ q = p_2$ and $q^{-1}(i_1(X)) = i_2(X)$.

As $\mathrm{R}p_{2,*}$ identifies with $\mathrm{R}p_{1,*} \circ \mathrm{R}q_*$, we are reduced to proving that

$$\mathrm{R}q_* \circ (i_2)_! \quad \text{identifies with} \quad (i_1)_!\,.$$

For any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_X})$, the canonical morphism

$$\mathcal{M} \longrightarrow i_2^* \circ (i_2)_! \mathcal{M} = i_1^* \circ \mathrm{R}q_* \circ (i_2)_! \mathcal{M}$$

corresponds to a morphism

$$(i_1)_! \mathcal{M} \longrightarrow \mathrm{R}q_* \circ (i_2)_! \mathcal{M}$$

which reduces to

$$\mathcal{M} \xrightarrow{\mathrm{id}} \mathcal{M}$$

over the open subset $i_1(X)$ of X_1.

As $\mathrm{R}q_*$ is compatible with base change, its fiber at any point of $X_1 - i_1(X)$ is

$$0 \longrightarrow 0\,.$$

So, $(i_1)_! \mathcal{M} \to \mathrm{R}q_* \circ (i_2)_! \mathcal{M}$ is an isomorphism.

(ii) Consider two factorisations

$$\begin{array}{rcl} f & : & X \xrightarrow{i_1} X_1 \xrightarrow{p_1} Y\,, \\ g & : & Y \xrightarrow{j_1} Y_1 \xrightarrow{q_1} Z \end{array}$$

of f, g and a factorisation of $j_1 \circ p_1$

$$\begin{array}{ccc} X_1 & \overset{i_2}{\hookrightarrow} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ Y & \overset{j_1}{\hookrightarrow} & Y_1 \end{array}$$

yielding a commutative diagram

$$\begin{array}{ccccc} X_1 & \overset{i}{\hookrightarrow} & Y \times_{Y_1} X_2 & \overset{j_2}{\hookrightarrow} & X_2 \\ & {\scriptstyle p_1}\searrow & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_2} \\ & & Y & \overset{j_1}{\hookrightarrow} & Y_1 \end{array}$$

with $i_2 = j_2 \circ i$.

We already know that the functors $\mathrm{R}p_{1,*}$ and $\mathrm{R}p_* \circ i_!$ identify.

We are reduced to proving that the functors

$\mathrm{R}p_{2,*} \circ (j_2)_!$ and $(j_1)_! \circ \mathrm{R}p_*$ identify.

For any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_{X_2}})$, the canonical morphism

$$\mathrm{R}p_*\mathcal{M} \xrightarrow{\mathrm{id}} \mathrm{R}p_*\mathcal{M} = \mathrm{R}p_* \circ j_2^* \circ (j_2)_!\mathcal{M} = j_1^* \circ \mathrm{R}p_{2,*} \circ (j_2)_!\mathcal{M}$$

corresponds to a morphism

$$(j_1)_! \circ \mathrm{R}p_*\mathcal{M} \longrightarrow \mathrm{R}p_{2,*} \circ (j_2)_!\mathcal{M}$$

whose restriction to $Y \overset{j_1}{\hookrightarrow} Y_1$ is an isomorphism and whose fiber at any point of $Y_1 - j_1(Y)$ is $0 \to 0$.

So it is an isomorphism.

The Künneth formula

Proposition:
Let $R =$ commutative coefficient ring
such that the functor $\otimes_{\mathrm{R}}$ has finite cohomological dimension.
Then:

(i) For any morphism $f : X \longrightarrow Y$ of $\mathrm{Top}_{\mathrm{lc}}$
and objects $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_X})$, $\mathcal{N}$ of $D^+(\mathcal{M}od_{\mathrm{R}_Y})$,

$$\mathrm{R}f_!(\mathcal{M} \overset{\mathrm{L}}{\otimes} f^{-1}\mathcal{N}) \quad \text{and} \quad \mathrm{R}f_!(\mathcal{M}) \overset{\mathrm{L}}{\otimes} \mathcal{N}$$

are canonically isomorphic.

(ii) For any cartesian square of $\mathrm{Top}_{\mathrm{lc}}$

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{q'} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{q} & S \end{array}$$

with $r = q \circ p' = p \circ q'$,
and objects $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_X})$, $\mathcal{N}$ of $D^+(\mathcal{M}od_{\mathrm{R}_Y})$,

$$\mathrm{R}r_!(q'^{-1}\mathcal{M} \overset{\mathrm{L}}{\otimes} p'^{-1}\mathcal{N}) \quad \text{and} \quad \mathrm{R}p_!\mathcal{M} \overset{\mathrm{L}}{\otimes} \mathrm{R}q_!\mathcal{N}$$

are canonically isomorphic.

Proof:

(i) is obvious if f is an open immersion.
So we can suppose that f is proper and $\mathrm{R}p_! = \mathrm{R}p_*$.
For any objects $\mathcal{M}$ of $\mathcal{M}od_{\mathrm{R}_X}$ and $\mathcal{N}$ of $\mathcal{M}od_{\mathrm{R}_Y}$,
there is a canonical morphism

$$f_*\mathcal{M} \otimes_{\mathrm{R}_Y} \mathcal{N} \longrightarrow f_*(\mathcal{M} \otimes_{\mathrm{R}_X} f^{-1}\mathcal{N})\,.$$

Furthermore, $f^{-1}\mathcal{N}$ is flat if $\mathcal{N}$ is flat.
This yields a canonical morphism

$$\mathrm{R}f_*\mathcal{M} \overset{\mathrm{L}}{\otimes} \mathcal{N} \longrightarrow \mathrm{R}f_*(\mathcal{M} \overset{\mathrm{L}}{\otimes} f^{-1}\mathcal{N})$$

for any objects $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_X})$, $\mathcal{N}$ of $D^+(\mathcal{M}od_{\mathrm{R}_Y})$.
We have to prove this is an isomorphism.
As $\mathrm{R}f_*$ commutes with base change, we can suppose that Y is a point.
If $\mathcal{N}$ is a flat R-module, we have for any R_X-Module $\mathcal{M}$

$$(\mathcal{M} \otimes f^{-1}\mathcal{N})(U) = \mathcal{M}(U) \otimes_{\mathrm{R}} \mathcal{N} \quad \text{for any open subset } U \text{ of } X$$

and $\mathcal{M} \otimes f^{-1}\mathcal{N}$ is f_*-acyclic if $\mathcal{M}$ is f_*-acyclic. The conclusion follows.

(ii) According to (i), we have canonical isomorphisms

$$\begin{aligned} \mathrm{R}r_!(q'^{-1}\mathcal{M} \overset{\mathrm{L}}{\otimes} p'^{-1}\mathcal{N}) &\cong \mathrm{R}p_!\mathrm{R}q'_!(q'^{-1}\mathcal{M} \overset{\mathrm{L}}{\otimes} p'^{-1}\mathcal{N}) \\ &\cong \mathrm{R}p_!(\mathcal{M} \overset{\mathrm{L}}{\otimes} Rq'_! p'^{-1}\mathcal{N}) \\ &\cong \mathrm{R}p_!(\mathcal{M} \overset{\mathrm{L}}{\otimes} p^{-1}\mathrm{R}q_!\mathcal{N}) \\ &\cong \mathrm{R}p_!\mathcal{M} \overset{\mathrm{L}}{\otimes} \mathrm{R}q_!\mathcal{N}\,. \end{aligned}$$

The exceptionnal inverse image functor

Theorem:
Let $R =$ (commutative) coefficient ring,
$f : X \to Y$
$=$ morphism of $\mathrm{Top}_{\mathrm{flc}}$.
Then the functor

$$\mathrm{R}f_! : D^+(\mathcal{M}od_{\mathrm{R}_X}) \longrightarrow D^+(\mathcal{M}od_{\mathrm{R}_Y})$$

has a right adjoint

$$f^! : D^+(\mathcal{M}od_{\mathrm{R}_Y}) \longrightarrow D^+(\mathcal{M}od_{\mathrm{R}_X})$$

and the two functors

$$\begin{array}{rcl} D^+(\mathcal{M}od_{\mathrm{R}_X})^{\mathrm{op}} \times D^+(\mathcal{M}od_{\mathrm{R}_Y}) & \longrightarrow & D^+(\mathrm{Ab})\,, \\ (\mathcal{M}, \mathcal{N}) & \longmapsto & \mathrm{RHom}(\mathcal{M}, f^! \mathcal{N})\,, \\ (\mathcal{M}, \mathcal{N}) & \longmapsto & \mathrm{RHom}(\mathrm{R}f_! \mathcal{M}, \mathcal{N}) \end{array}$$

are canonically isomorphic.

Remark:

- If $i : X \hookrightarrow X_1$ is an open immersion,

  is left adjoint to

  $$i_! : D^+(\mathcal{M}od_{\mathrm{R}_X}) \longrightarrow D^+(\mathcal{M}od_{\mathrm{R}_{X_1}})$$

  $$i^* : D^+(\mathcal{M}od_{\mathrm{R}_{X_1}}) \longrightarrow D^+(\mathcal{M}od_{\mathrm{R}_X})$$

  so that we can take in that case $i^! = i^*$.
- So it is enough to prove the theorem when $f : X \to Y$ is proper and $\mathrm{R}f_! = \mathrm{R}f_*$.

Principle of the construction

- We can suppose that $f : X \to Y$ is proper and $\mathrm{R}f_! = \mathrm{R}f_*$ has dimension $\leq d$.
- For any open embedding $i : U \hookrightarrow X$ and any R_X-Module $\mathcal{M}$ on X, we shall denote

$$\mathcal{M}_U = i_! i^* \mathcal{M} .$$

- For any object $\mathcal{N}$ of $D^+(\mathcal{M}od_{\mathrm{R}_Y})$, we should have

$$\begin{aligned} \mathrm{R}\Gamma(U, f^! \mathcal{N}) &= \mathrm{RHom}(\mathrm{R}_U, f^! \mathcal{N}) \\ &= \mathrm{RHom}(\mathrm{R}f_! \mathrm{R}_U, \mathcal{N}) . \end{aligned}$$

We shall prove there exists a finite resolution

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow S^0 \longrightarrow S^1 \longrightarrow \cdots \longrightarrow S^d \longrightarrow 0$$

of $\mathbb{Z}_X$ by objects S^j of the full additive subcategory $\mathcal{S}_X$ of $\mathcal{M}od_{\mathbb{Z}_X}$ on $\mathbb{Z}_X$-Modules S which are flat and such that S_U is f_*-acyclic for any open subset U of X.
Then we shall prove that for any object S of $\mathcal{S}_X$ and any injective R_Y-Module I,

$$U \longmapsto \mathrm{Hom}_{\mathrm{R}_Y}(f_*(R_U \otimes_{\mathbb{Z}_X} S), I)$$

is an injective R_X-Module (in particular a sheaf) denoted $f^!_S(I)$.
Choosing an injective resolution $\mathcal{N} \to I$ of $\mathcal{N}$ by $I = (I^k)$, we shall define $f^! \mathcal{N}$ as the complex

$$\left(\bigoplus_{k-j=n} f^!_{S^j}(I^k) \right)_{n \in \mathbb{Z}} .$$

Proof of the theorem

Step 1:

Lemma:
Let $S =$ object of $\mathcal{S}_X$. Then:

(i) For any object $\mathcal{M}$ of $\mathcal{M}od_{\mathbb{Z}_X}$, the object $\mathcal{M} \otimes_{\mathbb{Z}_X} S$ of $\mathcal{M}_{\mathbb{Z}_X}$ is f_*-acyclic.

(ii) The functor

$$\begin{array}{ccc} \mathcal{M}od_{\mathbb{Z}_X} & \longrightarrow & \mathcal{M}od_{\mathbb{Z}_Y}, \\ \mathcal{M} & \longmapsto & f_*(\mathcal{M} \otimes_{\mathbb{Z}_X} S) \end{array}$$

is exact.

Proof:

(i) The object $\mathcal{M}$ has a resolution

$$\cdots \longrightarrow \mathcal{M}_{-2} \longrightarrow \mathcal{M}_{-1} \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

where each $\mathcal{M}_i$ is a direct sum of sheaves $\mathbb{Z}_U$.
So $\mathcal{M} \otimes_{\mathbb{Z}_X} S$ has a resolution by the sheaves $\mathcal{M}_i \otimes_{\mathbb{Z}_X} S$ which are direct sums of sheaves S_U and so are f_*-acyclic.
As f_* has cohomological dimension $\leq d$,
it implies that $\mathcal{M} \otimes_{\mathbb{Z}_X} S$ is f_*-acyclic.

(ii) follows from (i).

Corollary:
Let $S =$ object of $\mathcal{S}_X$,
$I =$ object of $\mathcal{M}od_{\mathrm{R}_Y}$.
Then the presheaf on X

$$U \longmapsto f_S^! I(U) = \mathrm{Hom}_{\mathrm{R}_Y}(f_*(\mathrm{R}_U \otimes_{\mathbb{Z}_X} S), I)$$

is a sheaf and an object of $\mathcal{M}od_{\mathrm{R}_X}$.

Proof:
Any open covering of an open subset U of X

$$U = \bigcup_{i \in I} U_i$$

yields an exact sequence of $\mathbb{Z}_X$-Modules

$$\bigoplus_{i,j} \mathrm{R}_{U_i \cap U_j} \longrightarrow \bigoplus_{i} \mathrm{R}_{U_i} \longrightarrow \mathrm{R}_U \longrightarrow 0 .$$

Its transform by the functor $f_*(\bullet \otimes_{\mathbb{Z}_X} S)$ is an exact sequence of R_Y-Modules and, applying the functor $\mathrm{Hom}_{\mathrm{R}_Y}(\bullet, I)$, we get an exact sequence

$$0 \longrightarrow f_S^! I(U) \longrightarrow \prod_i f_S^! I(U_i) \longrightarrow \prod_{i,j} f_S^! I(U_i \cap U_j) .$$

It means that $f_S^! I$ is a sheaf.

Step 2:

Lemma:

Let $S =$ object of $\mathcal{S}_X$,
$I =$ object of $\mathcal{M}od_{\mathrm{R}_Y}$.

Then:

(i) For any object $\mathcal{M}$ of $\mathcal{M}od_{\mathrm{R}_X}$,

$$\mathrm{Hom}_{\mathrm{R}_X}(\mathcal{M}, f_S^! I)$$

identifies with

$$\mathrm{Hom}_{\mathrm{R}_Y}(f_*(\mathcal{M} \otimes_{\mathbb{Z}_X} S), I)\,.$$

(ii) If I is injective, the R_X-Module $f_S^! I$ is injective.

Proof:

(ii) follows from (i) as the functor

$$\mathcal{M} \longmapsto f_*(\mathcal{M} \otimes_{\mathbb{Z}_X} S)$$

is exact.

(i) Any morphism $f_*(\mathcal{M} \otimes_{\mathbb{Z}_X} S) \to I$ and any element $m \in \mathcal{M}(U)$ seen as a morphism $\mathrm{R}_U \to \mathcal{M}$ define a morphism $f_*(\mathrm{R}_U \otimes_{\mathbb{Z}_S} S) \to I$ or, equivalently, an element of $f_S^! I(U)$. This defines a morphism

$$\mathrm{Hom}_{\mathrm{R}_Y}(f_*(\mathcal{M} \otimes_{\mathbb{Z}_X} S), I) \longrightarrow \mathrm{Hom}_{\mathrm{R}_X}(\mathcal{M}, f_S^! I)\,.$$

This morphism is an isomorphism
when $\mathcal{M}$ is a direct sum of sheaves R_U.

The conclusion for an arbitrary $\mathcal{M}$ follows from
the fact that it has a resolution

$$\mathcal{M}_{-1} \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

by R_X-Modules $\mathcal{M}_0$, $\mathcal{M}_{-1}$ which are direct sums of sheaves R_U
and that the two functors

$$\begin{array}{rcl} (\mathcal{M}od_{\mathrm{R}_X})^{\mathrm{op}} & \longrightarrow & \mathrm{Ab}\,, \\ \mathcal{M} & \longmapsto & \mathrm{Hom}_{\mathrm{R}_Y}(f_*(\mathcal{M} \otimes_{\mathbb{Z}_X} S), I)\,, \\ \mathcal{M} & \longmapsto & \mathrm{Hom}_{\mathrm{R}_X}(\mathcal{M}, f_S^! I) \end{array}$$

are left-exact.

Step 3:

Lemma:
The sheaf $\mathbb{Z}_X$ on X has a resolution

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow S^0 \longrightarrow S^1 \longrightarrow \cdots \longrightarrow S^d \longrightarrow 0$$

where each $\mathbb{Z}_X$-Module S^j belongs to the subcategory $\mathcal{S}_X$.

Proof:

- Let S^0 be the sheaf $U \longmapsto \prod_{x\in U} \mathbb{Z}$, and, denoting

$$\begin{aligned} C^0 &= \mathrm{Coker}(\mathbb{Z}_X \to S^0)\,, \\ C^j &= \mathrm{Coker}(S^{j-1} \to S^j) \quad \text{for} \quad 1 \le j \le d-1\,, \end{aligned}$$

$$S^{j+1} : U \longmapsto \prod_{x\in U} C^j_x \quad \text{for} \quad 1 \le j \le d-2, \qquad S^d = C^{d-1}.$$

- For any U, there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_U \longrightarrow S^0_U \longrightarrow S^1_U \longrightarrow \cdots \longrightarrow S^d_U \longrightarrow 0$$

  and each S^j_U, $0 \le j \le d-1$, is flabby and a fortiori f_*-acyclic.
  As f_* has cohomological dimension $\le d$, S^d_U is also f_*-acyclic.
- For $1 \le j \le d$, the fiber of C^j at x is

$$\varinjlim_{U \ni x} \prod_{\substack{x' \in U \\ x' \neq x}} S^{j-1}_y\,.$$

  So we get by induction on j that each S^j and C^j is flat.

Step 4: conclusion of the construction

Definition:
Let $\mathcal{I}nj_{\mathrm{R}_Y}$ = full additive subcategory of $\mathcal{M}od_{\mathrm{R}_Y}$ on injective objects,
and $(0 \longrightarrow S^0 \longrightarrow S^1 \longrightarrow \cdots \longrightarrow S^d \longrightarrow 0)$
$=$ resolution of $\mathbb{Z}_X$ by objects of $\mathcal{S}_X$.

Then the functor

$$f^! : D^+(\mathcal{M}od_{\mathrm{R}_Y}) \longrightarrow D^+(\mathcal{M}od_{\mathrm{R}_X})$$

is defined by its restriction to the equivalent subcategory

$$D^+(\mathcal{I}nj_{\mathrm{R}_Y}) = K^+(\mathcal{I}nj_{\mathrm{R}_Y})$$

as

$$\begin{array}{rcl} K^+(\mathcal{I}nj_{\mathrm{R}_Y}) & \longrightarrow & K^+(\mathcal{M}od_{\mathrm{R}_X}) \longrightarrow D^+(\mathcal{M}od_{\mathrm{R}_X})\,, \\ I = (I^k)_{k\in\mathbb{Z}} & \longmapsto & f^! I = \Big(\bigoplus_{k-j=n} f^!_{S^j} I^k\Big)_{n\in\mathbb{Z}}\,. \end{array}$$

Remark:
There is an equality $D^+(\mathcal{I}nj_{\mathrm{R}_Y}) = K^+(\mathcal{I}nj_{\mathrm{R}_Y})$
as any quasi-isomorphism $I_1 \to I_2$ in $C^+(\mathcal{I}nj_{\mathrm{R}_Y})$
is invertible in $K^+(\mathcal{I}nj_{\mathrm{R}_Y})$.

Lemma:
With this definition, we have for any object I of $K^+(\mathcal{I}nj_{\mathrm{R}_Y})$ and any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_X})$ identifications

$$\begin{array}{rcl} \mathrm{RHom}(\mathcal{M}, f^! I) & \cong & \mathrm{RHom}(\mathrm{R}f_* \mathcal{M}, I)\,, \\ \mathrm{Hom}(\mathcal{M}, f^! I) & \cong & \mathrm{Hom}(\mathrm{R}f_* \mathcal{M}, I)\,. \end{array}$$

Proof:
For any $\mathcal{M}$, the complex associated to the double complex

$$0 \longrightarrow f_*(\mathcal{M} \otimes_{\mathbb{Z}_X} S^0) \longrightarrow f_*(\mathcal{M} \otimes_{\mathbb{Z}_X} S^1) \longrightarrow \cdots \longrightarrow f_*(\mathcal{M} \otimes_{\mathbb{Z}_X} S^d) \longrightarrow 0$$

represents the image

$$\mathrm{R}f_* \mathcal{M} \qquad \text{in} \qquad D^+(\mathcal{M}od_{\mathrm{R}_Y})\,.$$

So the first identification follows from the lemma of Step 2.

The second identification follows from the first one by applying the functor

$$H^0 : D(\mathrm{Ab}) \longrightarrow \mathrm{Ab}\,.$$

Corollary:

(i) For any morphism $f : X \to Y$ of $\mathrm{Top}_{\mathrm{flc}}$ and any object $\mathcal{N}$ of $D^+(\mathcal{M}od_{\mathrm{R}_Y})$, the square

$$\begin{array}{ccc} D^+(\mathcal{M}od_{\mathrm{R}_X}) & \xrightarrow{\mathrm{R}\mathcal{H}om(\bullet, f^!\mathcal{N})} & D(\mathcal{M}od_{\mathrm{R}_X}) \\ \downarrow{\scriptstyle \mathrm{R}f_!} & & \downarrow{\scriptstyle \mathrm{R}f_*} \\ D^+(\mathcal{M}od_{\mathrm{R}_Y}) & \xrightarrow{\mathrm{R}\mathcal{H}om(\bullet, \mathcal{N})} & D(\mathcal{M}od_{\mathrm{R}_Y}) \end{array}$$

is commutative up to canonical isomorphism.

(ii) For any morphisms of $\mathrm{Top}_{\mathrm{flc}}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

$(g \circ f)^!$ is canonically isomorphic to $f^! \circ g^!$.

(iii) For any cartesian square of $\mathrm{Top}_{\mathrm{flc}}$

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y \end{array}$$

there is a canonical isomorphism of functors

$$f^! \circ \mathrm{R}y_* \cong \mathrm{R}x_* \circ f'^!$$

from $D^+(\mathcal{M}od_{\mathrm{R}_{Y'}})$ to $D^+(\mathcal{M}od_{\mathrm{R}_X})$.

Remarks:

(i) For any morphism $f : X \to Y$ of $\mathrm{Top}_{\mathrm{flc}}$
and any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_Y})$, the canonical morphism

$$\mathrm{R}f_! \circ f^! \mathcal{M} \longrightarrow \mathcal{M} \quad \text{associated to} \quad f^! \mathcal{M} \xrightarrow{\mathrm{id}} f^! \mathcal{M}$$

is often denoted Tr and called the "trace" morphism.

It is a sheaf theoretic version of integration.

(ii) For any commutative triangle in the category $\mathrm{Top}_{\mathrm{flc}}$

$$\begin{array}{ccc} X_1 & \xrightarrow{\ f\ } & X_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & S & \end{array}$$

and any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_S})$,
the transform by $\mathrm{R}p_{2,!}$ of the morphism

$$\mathrm{Tr} : \mathrm{R}f_! \circ f^! \circ p_2^! \mathcal{M} \longrightarrow p_2^! \mathcal{M}$$

is a morphsim of $D^+(\mathcal{M}od_{\mathrm{R}_S})$

$$\mathrm{R}p_{1,!} \circ p_1^! \mathcal{M} \longrightarrow \mathrm{R}p_{2,!} \circ p_2^! \mathcal{M}$$

and induces morphisms of $\mathcal{M}od_{\mathrm{R}_S}$

$$\mathrm{R}^k p_{1,!}(p_1^! \mathcal{M}) \longrightarrow \mathrm{R}^k p_{2,!}(p_2^! \mathcal{M}) \,, \quad k \in \mathbb{Z} \,.$$

In other words, cohomology with compact support of coefficients defined by the exceptional inverse image functors is covariant.

Concrete expressions of the exceptional inverse image functors

If $U \overset{i}{\hookrightarrow} X$ is an open immersion of $\mathrm{Top}_{\mathrm{flc}}$, $i^!$ is $i^* = i^{-1}$.
In the case of closed immersions, we have:

Proposition:

Let $j : Z \hookrightarrow X$ be a closed immersion of $\mathrm{Top}_{\mathrm{flc}}$.
Then the functor $j^! : D^+(\mathcal{M}od_{\mathrm{R}_X}) \to D^+(\mathcal{M}od_{\mathrm{R}_Z})$
identifies with the composite $j^{-1} \circ \mathrm{R}\Gamma_Z$ where

$$\mathrm{R}\Gamma_Z : D^+(\mathcal{M}od_{\mathrm{R}_X}) \longrightarrow D^+(\mathcal{M}od_{\mathrm{R}_X})$$

is the derived functor of the left-exact functor

$$\begin{array}{rcrcl} \Gamma_Z & : & \mathcal{M}od_{\mathrm{R}_X} & \longrightarrow & \mathcal{M}od_{\mathrm{R}_X}\,, \\ & & \mathcal{M} & \longmapsto & \Gamma_Z(\mathcal{M}) = \mathcal{M}_Z \end{array}$$

where, for any open subset U of X,

$$\mathcal{M}_Z(U) = \{m \in \mathcal{M}(U) \mid \mathrm{supp}(m) \subset Z \cap U\}.$$

Proof:

For objects $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_X})$, $\mathcal{N}$ of $D^+(\mathcal{M}od_{\mathrm{R}_Z})$, we have

$$\begin{aligned} \mathrm{Hom}(j_*\mathcal{N},\mathcal{M}) &= \mathrm{Hom}(j_*\mathcal{N},\mathrm{R}\Gamma_Z(\mathcal{M})) \\ &= \mathrm{Hom}(j^{-1}\circ j_*\mathcal{N}, j^{-1}\circ \mathrm{R}\Gamma_Z(\mathcal{M})) \\ &= \mathrm{Hom}(\mathcal{N}, j^{-1}\circ \mathrm{R}\Gamma_Z(\mathcal{M}))\,. \end{aligned}$$

This means that $j^{-1}\circ \mathrm{R}\Gamma_Z$ is right adjoint to $j_* = j_!$.

Remark:

If $i : U \hookrightarrow X$ is the open embedding of $U = X - Z$,
any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_X})$ yields a distinguished triangle in $D^+(\mathcal{M}od_{\mathrm{R}_X})$

$$\mathrm{R}\Gamma_Z(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow \mathrm{R}i_*i^*\mathcal{M} \longrightarrow \mathrm{R}\Gamma_Z(\mathcal{M})[1]\,.$$

Indeed, if $\mathcal{M}$ is a complex of injective R_X-Modules,
it yields a short exact sequence

$$0 \longrightarrow \Gamma_Z(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow i_*i^*\mathcal{M} \longrightarrow 0\,.$$

Theorem:

Let $R =$ (commutative) coefficient ring,
$Y =$ topological space,
$X = Y \times \mathbb{R}^d$ endowed with the projection $p : Y \times \mathbb{R}^d \to Y$
and the 0 section: $j : Y \hookrightarrow Y \times \mathbb{R}^d$.

Then:

(i) The composite functor

$$\begin{array}{rcl} D^+(\mathcal{M}od_{\mathrm{R}_Y}) & \longrightarrow & D^+(\mathcal{M}od_{\mathrm{R}_Y}), \\ \mathcal{M} & \longmapsto & j^{-1} \circ \mathrm{R}\Gamma_Y \circ p^{-1} \end{array}$$

identifies with $\mathcal{M} \mapsto \mathcal{M}[-d]$.

(ii) The composite functor

$$\begin{array}{rcl} D^+(\mathcal{M}od_{\mathrm{R}_Y}) & \longrightarrow & D^+(\mathcal{M}od_{\mathrm{R}_Y}), \\ \mathcal{M} & \longmapsto & \mathrm{R}p_! \circ p^{-1}\mathcal{M} \end{array}$$

identifies with $\mathcal{M} \mapsto \mathcal{M}[-d]$.

(iii) If Y is an object of $\mathrm{Top}_{\mathrm{flc}}$, the functor

$$p^! : D^+(\mathcal{M}od_{\mathrm{R}_Y}) \longrightarrow D^+(\mathcal{M}od_{\mathrm{R}_X})$$

identifies with $\mathcal{M} \mapsto f^{-1}\mathcal{M}[d]$.

Proof: We can suppose that $d = 1$.

(ii) As $\mathbb{R}$ is diffeomorphic to $]0,1[$, let's consider the projection

$$q : Y \times [0,1] \longrightarrow Y$$

with its 0 and 1 sections $j_0, j_1 : Y \hookrightarrow Y \times [0,1]$
and the open embedding $i : Y \times]0,1[\hookrightarrow Y \times [0,1]$.
We already know that, for any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_Y})$,

$$\mathrm{R}q_* \circ q^{-1}\mathcal{M} = \mathrm{R}q_! \circ q^{-1}\mathcal{M} \quad \text{identifies with } \mathcal{M}\,.$$

For any object $\mathcal{M}$ of $C^+(\mathcal{M}od_{\mathrm{R}_Y})$, the short sequence of complexes on $Y \times [0,1]$

$$0 \longrightarrow i_! \circ i^{-1} \circ q^{-1}\mathcal{M} \longrightarrow q^{-1}\mathcal{M} \longrightarrow j_{0,*}\mathcal{M} \oplus j_{1,*}\mathcal{M} \longrightarrow 0$$

is exact. Its transform by $\mathrm{R}q_* = \mathrm{R}q_!$ is a distinguished triangle

$$\begin{array}{ccccccc} \mathrm{R}(q\circ i)_! \circ (q \circ i)^{-1}\mathcal{M} & \to & \mathcal{M} & \to & \mathcal{M} \oplus \mathcal{M} & \to & \mathrm{R}(q\circ i)_! \circ (q \circ i)^{-1}\mathcal{M}[1] \\ \| & & & & & & \| \\ \mathrm{R}p_! \circ p^{-1}\mathcal{M} & & & & & & \mathrm{R}p_! \circ p^{-1}\mathcal{M}[1] \end{array}$$

and $\mathrm{R}p_! \circ p^{-1}\mathcal{M}[1]$ is canonically isomorphic to $(\mathcal{M} \oplus \mathcal{M})/\mathcal{M} \cong \mathcal{M}$.

(i) There is a canonical morphism of functors

$$j^{-1} \circ \mathrm{R}\Gamma_Y \circ p^{-1} \longrightarrow \mathrm{R}p_! \circ p^{-1}$$

from $D^+(\mathcal{M}od_{\mathrm{R}_Y})$ to $D^+(\mathcal{M}od_{\mathrm{R}_Y})$.

We have to show that it is an isomorphism.

As both $\mathrm{R}p_!$ and $\mathrm{R}\Gamma_Y$ commute with base changes $Y' \to Y$, we can suppose that $Y = \{\bullet\}$ is the point space and p is

$$\mathbb{R} \longrightarrow \{0\}$$

with the 0 section $j : \{0\} \hookrightarrow \mathbb{R}$.

We have to show that for any R-module M,

$$H^k_{\{0\}}(\mathbb{R}, p^{-1}M) \longrightarrow H^k_c(\mathbb{R}, p^{-1}M)$$

is an isomorphism for any $k \geq 0$.

For any $a > 0$, the morphism of long exact sequences

$$\begin{array}{ccccccccc}
\cdots \to & H^{k-1}(\mathbb{R} - \{0\}, p^{-1}\mathcal{M}) & \to & H^k_{\{0\}}(\mathbb{R}, p^{-1}\mathcal{M}) & \to & H^k(\mathbb{R}, p^{-1}\mathcal{M}) & \to & H^k(\mathbb{R} - \{0\}, p^{-1}\mathcal{M}) & \to \cdots \\
& \downarrow\wr & & \downarrow & & \| & & \downarrow\wr & \\
\cdots \to & H^{k-1}(\mathbb{R} - [-a,a], p^{-1}\mathcal{M}) & \to & H^k([-a,a], p^{-1}\mathcal{M}) & \to & H^k(\mathbb{R}, p^{-1}\mathcal{M}) & \to & H^k(\mathbb{R} - [-a,a], p^{-1}\mathcal{M}) & \to \cdots
\end{array}$$

shows that

$$H^k_{\{0\}}(\mathbb{R}, p^{-1}\mathcal{M}) \longrightarrow H^k([-a,a], p^{-1}\mathcal{M})$$

is an isomorphism for any $k \geq 0$. The conclusion follows from the isomorphisms

$$\varinjlim_{a>0} H^k([-a,a], p^{-1}\mathcal{M}) \xrightarrow{\sim} H^k_c(\mathbb{R}, p^{-1}\mathcal{M}), \quad \forall\, k \geq 0 .$$

(iii) For any object $\mathcal{M}$ of $\mathcal{M}od_{\mathrm{R}_Y}$
and open subsets $U =]a, b[\subset \mathbb{R}$,
$V \subset Y$,

we have canonical isomorphisms

$$\begin{aligned} \mathrm{R}\Gamma(U \times V, p^! \mathcal{M}) &\cong \mathrm{RHom}(\mathrm{R}_{U \times V}, p^! \mathcal{M}) \\ &\cong \mathrm{RHom}(\mathrm{R}p_! \mathrm{R}_{U \times V}, \mathcal{M}) \\ &\cong \mathrm{RHom}(\mathrm{R}_V[-1], \mathcal{M}) \\ &\cong \mathrm{R}\Gamma(V, \mathcal{M}[1]) \,. \end{aligned}$$

This proves that the complex $p^! \mathcal{M}$ is concentrated in degree -1
where it identifies with $p^{-1}\mathcal{M}$.
For an arbitrary object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_Y})$,
it follows that the canonical morphism

$$p^! \mathbb{Z}_Y \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_X} p^{-1}\mathcal{M} \longrightarrow p^! \mathcal{M}$$

corresponding to the morphism

$$\mathrm{R}p_!(p^! \mathbb{Z}_Y \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_X} p^{-1}\mathcal{M}) \cong \mathrm{R}p_! \circ p^! \mathbb{Z}_Y \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_Y} \mathcal{M} \longrightarrow \mathcal{M}$$

is an isomorphism in $D^+(\mathcal{M}od_{\mathrm{R}_X})$.

Corollary:
Let $R =$ commutative coefficient ring.

(i) For any morphism $X \xrightarrow{f} Y$ of $\mathrm{Top}_{\mathrm{flc}}$ which is smooth of dimension d, i.e. locally homeomorphic to $Y \times \mathbb{R}^d \to \mathbb{R}^d$, the functor

$$f^! : D^+(\mathcal{M}od_{\mathrm{R}_Y}) \longrightarrow D^+(\mathcal{M}od_{\mathrm{R}_X})$$

is canonically isomorphic to the functor

$$\mathcal{M} \longmapsto (f^! \mathbb{Z}_Y) \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_X} f^{-1}\mathcal{M}$$

where $f^! \mathbb{Z}_Y$ is concentrated in degree $-d$
and of the form

$$\mathrm{or}_{X/Y}[d]$$

for a $\mathbb{Z}_d$-Module $\mathrm{or}_{X/Y}$
which is locally isomorphic to $\mathbb{Z}_X$
and called the “orientation” sheaf.

(ii) For any commutative triangle of $\mathrm{Top}_{\mathrm{flc}}$

$$\begin{array}{ccc} Z & \overset{j}{\hookrightarrow} & X \\ {\scriptstyle q}\searrow & & \swarrow{\scriptstyle p} \\ & S & \end{array}$$

such that $Z \overset{j}{\hookrightarrow} X$ is a "regular" closed immersion of codimension d,
i.e. is locally homeomorphic to $Z \hookrightarrow Z \times \mathbb{R}^d$,
then for any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_S})$,

$$j^! \circ p^{-1}\mathcal{M}$$

identifies with

$$(j^!\mathbb{Z}_X) \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_X} q^{-1}\mathcal{M}$$

where $j^!\mathbb{Z}_X$ is concentrated in degree d
and of the form

$$\mathrm{or}_{Z/X}[-d]$$

for a $\mathbb{Z}_X$-Module $\mathrm{or}_{Z/X} = j^{-1} \circ \mathrm{R}^d\Gamma_Z\mathbb{Z}_X$
which is locally isomorphic to $\mathbb{Z}_Z$
and called the "orientation" sheaf.

Remarks:

(i) In the situation of (i) we get that for any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_X})$, $\mathrm{R}f_*\mathrm{R}\mathcal{H}om(\mathcal{M}, \mathrm{R}_X \otimes \mathrm{or}_{X/Y}[d])$ identifies with $\mathrm{R}\mathcal{H}om(\mathrm{R}p_!\mathcal{M}, \mathrm{R}_Y)$.

In particular, if Y is a point $\{\bullet\}$ and R is a field, each

$$H^{d-k}(\mathrm{R}\mathcal{H}om(\mathcal{M}, \mathrm{R}_X \otimes \mathrm{or}_{Z/X}))$$

is the dual of

$$H_c^k(\mathcal{M}) .$$

(ii) In the situation of (ii) we get that for any object $\mathcal{M}$ of $D^+(\mathcal{M}od_{\mathrm{R}_Z})$,

$$j_*\mathrm{R}\mathcal{H}om(\mathcal{M}, \mathrm{R}_X \otimes \mathrm{or}_{Z/X}[-d])$$

identifies with

$$\mathrm{R}\mathcal{H}om(j_*\mathcal{M}, \mathrm{R}_X) .$$