Cohomology of toposes

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Chapter IV:

Sheaves on a topological space

Sheaves on a site

Abelian categories

The four operations on $\mathcal{O}$-Modules

Presheaves on a topological space

Definition

Let X be a topological space. A presheaf $\mathcal{F}$ on X consists of the data:

(i) for every open subset U of X, a set $\mathcal{F}(U)$ and

(ii) for every inclusion $V \subseteq U$ of open subsets of X, a function $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ subject to the conditions

- $\rho_{U,U}$ is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$ and
- if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

The maps $\rho_{U,V}$ are called restriction maps, and we sometimes write $s|_V$ instead of $\rho_{U,V}(s)$, if $s \in \mathcal{F}(U)$.

A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ on a topological space X is a collection of maps $\mathcal{F}(U) \to \mathcal{G}(U)$ which is compatible with respect to restriction maps.

Remark

Categorically, a presheaf $\mathcal{F}$ on X is a functor $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$, where $\mathcal{O}(X)$ is the poset category corresponding to the lattice of open sets of the topological space X (with respect to the inclusion relation).

A morphism of presheaves is then just a natural transformation between the corresponding functors.

So we have a category $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on X.

Sheaves on a topological space

Definition

A sheaf $\mathcal{F}$ on a topological space X is a presheaf on X satisfying the additional conditions

(i) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if $s, t \in \mathcal{F}(U)$ are elements such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$;

(ii) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i, with the property that for each $i, j \in I$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ (necessarily unique by (i)) such that $s|_{V_i} = s_i$ for each i.

A morphism of sheaves is defined as a morphism of the underlying presheaves.

Remark

Categorically, a sheaf is a functor $\mathcal{O}(X)^{\text{op}} \to \mathbf{Set}$ which satisfies certain conditions expressible in categorical language entirely in terms of the poset category $\mathcal{O}(X)$ and of the usual notion of covering on it. The category $\mathbf{Sh}(X)$ of sheaves on a topological space X is a full subcategory of the category $[\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$ of presheaves on X.

This paves the way for a significant categorical generalization of the notion of sheaf, leading to the notion of Grothendieck topos.

Categorical reformulations

- The sheaf condition for a presheaf $\mathcal{F}$ on a topological space X can be categorically reformulated as the requirement that the canonical arrow

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$$

given by $s \to (s|_{U_i} \mid i \in I)$ should be the equalizer of the two arrows

$$\prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

given by $(s_i \to (s_i|_{U_i \cap U_j}))$ and $(s_i \to (s_j|_{U_i \cap U_j}))$.

- For any covering family $F = \{U_i \subseteq U \mid i \in I\}$, giving a family of elements $s_i \in \mathcal{F}(U_i)$ such that for any $i, j \in I$ $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ is equivalent to giving a family of elements $\{s_W \in \mathcal{F}(W) \mid W \in S_F\}$ such that for any open set $W' \subseteq W$, $s_W|_{W'} = s_{W'}$, where S_F is the sieve generated by F.

Examples of sheaves

Examples

- the sheaf of continuous real-valued functions on any topological space
- the sheaf of regular functions on a variety
- the sheaf of differentiable functions on a differentiable manifold
- the sheaf of holomorphic functions on a complex manifold

In each of the above examples, the restriction maps of the sheaf are the usual set-theoretic restrictions of functions to a subset.

Remark

Sheaves arising in Mathematics are often equipped with more structure than the mere set-theoretic one; for example, one may wish to consider sheaves of modules (resp. rings, abelian groups, ...) on a topological space X*.*
The natural categorical way of looking at these notions is to consider them as models of certain (geometric) theories in a category $\mathbf{Sh}(X)$ *of sheaves of sets.*

Sheaves of algebraic structures

A sheaf $\mathcal{F}$ of abelian groups on a topological space X is an R is a model of the theory of abelian groups, that is, an abelian group object in the category $\mathbf{Sh}(X)$.

This is equivalent to saying that $\mathcal{F}$ is a sheaf of sets such that, for each open set U of X, $\mathcal{F}(U)$ is an abelian group and for each inclusion $V \subseteq U$ of open sets, $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ is an abelian group homomorphism.

Similarly, a sheaf of rings (resp. of R-modules, where R is a commutative ring with unit) is a sheaf of sets $\mathcal{F}$ whose sets of sections are all rings (resp. R-modules) and whose structure maps between them are ring (resp. R-module) homomorphisms.

Remark

On the other hand, sheaves of more sophisticated structures which cannot be axiomatized by only using equational axioms (such as local rings or fields) cannot be characterized in this way. In general, a sheaf of models of a geometric theory $\mathbb{T}$ on a topological space is a sheaf of sets whose stalks are all models of $\mathbb{T}$.

The sheaf of cross-sections of a bundle

Definition

- For any topological space X, a continuous map $p : Y \to X$ is called a bundle over X. In fact, the category of bundles is the slice category $\mathbf{Top}/X$.
- Given an open subset U of X, a cross-section over U of a bundle $p : Y \to X$ is a continuous map $s : U \to Y$ such that the composite $p \circ s$ is the inclusion $i : U \hookrightarrow X$. Let

$$\Gamma_p U = \{s \mid s : U \to Y \text{ and } p \circ s = i : U \to X\}$$

  denote the set of all such cross-sections over U.
- If $V \subseteq U$, one has a restriction operation $\Gamma_p U \to \Gamma_p V$. The functor $\Gamma_p : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ obtained in this way is a sheaf and is called the sheaf of cross-sections of the bundle p.

The bundle of germs of a presheaf

Definition

- Given any presheaf $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ on a space X, a point $x \in X$, two open neighbourhoods U and V of x, and two elements $s \in \mathcal{F}(U), t \in \mathcal{F}(V)$. We say that s and t have the same germ at x when there is some open set $W \subseteq U \cap V$ with $x \in W$ and $s|_W = t|_W$. This relation 'to have the same germ at x' is an equivalence relation, and the equivalence class of any one such s is called the germ of s at x, in symbols $germ_x(s)$ or s_x.
- Let
$$\mathcal{F}_x = \{germ_x(s) \mid s \in \mathcal{F}(U),\ x \in U \text{ open in } X\}$$
be the stalk of $\mathcal{F}$ at x, that is the set of all germs of $\mathcal{F}$ at x.
- Let $\Gamma_{\mathcal{F}}$ be the disjoint union of the $\mathcal{F}_x$
$$\Lambda_{\mathcal{F}} = \{\langle x, r\rangle \mid x \in X, r \in \mathcal{F}_x\}$$
topologized by taking as a base of open sets all the image sets $\tilde{s}(U)$, where $\tilde{s} : U \to \Lambda_{\mathcal{F}}$ is the map induced by an element $s \in \mathcal{F}(U)$ by taking its germs at points in U.
- With respect to this topology, the natural projection map $\Lambda_{\mathcal{F}} \to X$ becomes a continuous map, called the bundle of germs of the presheaf $\mathcal{F}$.

Sheaves as étale bundles I

Definition

- A bundle $p : E \to X$ is said to be étale (over X) when p is a local homeomorphism in the following sense: for each $e \in E$ there is an open set V, with $e \in V$, such that $p(V)$ is open in X and $p|_V$ is a homeomorphism $V \to p(V)$.
- The full subcategory of **Top**$/X$ on the étale bundles is denoted by **Etale**(X).

Theorem

- *For any topological space X, there is a pair of adjoint functors*

$$\Gamma : \mathbf{Top}/X \to [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}], \quad \Lambda : [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Top}/X,$$

where Γ assigns to each bundle $p : Y \to X$ the sheaf of cross-sections of p, while its left adjoint Λ assigns to each presheaf $\mathcal{F}$ the bundle of germs of $\mathcal{F}$.

- *This adjunction restricts to an equivalence of categories*

$$\mathbf{Sh}(X) \simeq \mathbf{Etale}(X)\ .$$

Sheaves as étale bundles II

This adjunction is naturally presented by speciying its unit and counit:

- The unit $\eta : 1_{[\mathcal{O}(X)^{\mathrm{op}},\mathbf{Set}]} \to \Gamma \circ \Lambda$ acts on a presheaf $\mathcal{F}$ by sending a section $s \in \mathcal{F}(U)$ to the section $\dot{s} \in \Gamma_{\Lambda_{\mathcal{F}}}(U)$;
- The counit $\epsilon : \Lambda \circ \Gamma \to 1_{\mathbf{Top}/X}$ acts on a bundle $p : Y \to X$ by sending any element $(x, germ_x(s))$ of Λ_{Γ_p} to the value $s(x)$.

One then verifies that these natural transformations satisfy the triangular identities:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\Gamma\eta} & \Gamma \circ \Lambda \circ \Gamma \\ & \searrow^{1_\Gamma} & \downarrow{\scriptstyle \epsilon\Gamma} \\ & & \Gamma \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Lambda & \xrightarrow{\eta\Lambda} & \Lambda \circ \Gamma \circ \Lambda \\ & \searrow^{1_\Lambda} & \downarrow{\scriptstyle \Lambda\epsilon} \\ & & \Lambda \end{array}$$

One further proves that if p is étale then ϵ_p is an isomorphism (and conversely), while if $\mathcal{F}$ is a sheaf then $\eta_{\mathcal{F}}$ is an isomorphism (and conversely). It thus follows from general abstract nonsense that the adjunction restricts to a duality between the full subcategories on sheaves and on étale bundles.

The associated sheaf functor

Theorem

Given a presheaf $\mathcal{F}$, there is a sheaf $a(\mathcal{F})$ and a morphism $\theta : \mathcal{F} \to a(\mathcal{F})$, with the property that for any sheaf $\mathcal{G}$, and any morphism $\phi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : a(\mathcal{F}) \to \mathcal{G}$ such that $\psi \circ \theta = \phi$.

The sheaf $a(\mathcal{F})$ is called the sheaf associated to the presheaf $\mathcal{F}$.

Remark

Categorically, this means that the inclusion functor $i : \mathbf{Sh}(X) \to [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$ has a left adjoint $a : [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(X)$.

The left adjoint $a : [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(X)$ is called the associated sheaf functor.

Theorem

The associated sheaf functor a is given by the composite $\Gamma \circ \Lambda$.

Concretely, $a(\mathcal{F})(U)$ is the collection of functions $s : U \to \Lambda_{\mathcal{F}}$ which satisfy the following properties:

- $s(x) \in \mathcal{F}_x$ for each $x \in U$;
- for each $x \in U$ there exist an open set $Z_x \subseteq U$ containing x and a section $\xi^{Z_x} \in \mathcal{F}(Z_x)$ such that $s(y) = (\xi^{Z_x})_y$ for each $y \in Z_x$.

Limits and colimits in $\mathrm{Sh}(X)$

Theorem

(i) The category $\mathbf{Sh}(X)$ *is closed in* $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ *under arbitrary (small) limits.*

(ii) The associated sheaf functor $a : [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(X)$ *(having a right adjoint) preserves all (small) colimits.*

- *Part (i) follows from the fact that limits commute with limits, in light of the characterization of sheaves in terms of limits.*
- *From part (ii) it follows that* $\mathbf{Sh}(X)$ *has all small colimits, which are computed by applying the associated sheaf functor to the colimit of the diagram considered with values in* $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$.

Adjunctions induced by points

Let x be a point of a topological space X.

Definition

Let A be a set. Then the skyscraper sheaf $\text{Sky}_x(A)$ of A at x is the sheaf on X defined as

- $\text{Sky}_x(A)(U) = A$ if $x \in U$
- $\text{Sky}_x(A)(U) = 1 = \{*\}$ if $x \notin U$

and in the obvious way on arrows.

The assignment $A \to \text{Sky}_x(A)$ is clearly functorial.

Theorem

The stalk functor $\text{Stalk}_x : \mathbf{Sh}(X) \to \mathbf{Set}$ *at* x *is left adjoint to the skyscraper functor* $\text{Sky}_x : \mathbf{Set} \to \mathbf{Sh}(X)$.

In fact, as we shall see later in the course, points in topos theory are defined as suitable kinds of functors (more precisely, colimit and finite-limit preserving ones).

Open sets as subterminal objects

Since limits in a category $\mathbf{Sh}(X)$ are computed as in the category of presheaves $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$, a subobject of a sheaf F in $\mathbf{Sh}(X)$ is just a subsheaf, that is a subfunctor which is a sheaf.
Notice that a subfunctor $S \subseteq F$ is a sheaf if and only if for every open covering $\{U_i \subseteq U \mid i \in I\}$ and every element $x \in F(U)$, $x \in S(U)$ if and only if $x|_{U_i} \in S(U_i)$.

Definition

In a category with a terminal object, a subterminal object is an object whose unique arrow to the terminal object is a monomorphism.

Theorem

Let X be a topological space. Then we have a frame isomorphism

$$\mathrm{Sub}_{\mathbf{Sh}(X)}(1) \cong \mathcal{O}(X) .$$

between the subterminal objects of $\mathbf{Sh}(X)$ and the open sets of X.

Direct and inverse images of sheaves

Definition

Let $f : X \to Y$ be a continuous function between topological spaces. The direct image $f_*(P)$ of a sheaf P on X and the inverse image $f^*(Q)$ of a sheaf Q on Y are defined as follows:

- $f_*(P)(V) = P(f^{-1}(V))$ for any open set V of Y; in other words $f^*(P)$ is the sheaf on Y given by the composite $P \circ f^{-1}$.
- f^* acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.

Theorem

The operations $P \mapsto f_*(P)$ and $Q \mapsto f^*(Q)$ define a pair of adjoint functors $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ (where f^* is the left adjoint and f_* is the right adjoint).

Sieves

In order to 'categorify' the notion of sheaf of a topological space, the first step is to introduce an abstract notion of covering (of an object by a family of arrows to it) in a category.

Definition

- Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a presieve P in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c.
- Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a sieve S in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c such that

$$f \in S \Rightarrow f \circ g \in S$$

  whenever this composition makes sense.
- We say that a sieve S is generated by a presieve P on an object c if it is the smallest sieve containing it, that is if it is the collection of arrows to c which factor through an arrow in P.

If S is a sieve on c and $h : d \to c$ is any arrow to c, then

$$h^*(S) := \{g \mid \mathrm{cod}(g) = d,\ h \circ g \in S\}$$

is a sieve on d.

Grothendieck topologies I

Definition

- A Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each object c of $\mathcal{C}$ a collection $J(c)$ of sieves on c in such a way that
  - (i) (maximality axiom) the maximal sieve $M_c = \{f \mid \mathrm{cod}(f) = c\}$ is in $J(c)$;
  - (ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;
  - (iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

  The sieves S which belong to $J(c)$ for some object c of $\mathcal{C}$ are said to be J-covering.
- A site is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a small category and J is a Grothendieck topology on $\mathcal{C}$.

Notice the following basic properties:

- If $R, S \in J(c)$ then $R \cap S \in J(c)$;
- If R and R' are sieves on an object c such that $R' \supseteq R$ then $R \in J(c)$ implies $R' \in J(c)$.

Grothendieck topologies II

The notion of a Grothendieck topology can be put in the following alternative (but equivalent) form:

Definition

A Grothendieck topology on a category $\mathcal{C}$ is an assignment J sending any object c of $\mathcal{C}$ to a collection $J(c)$ of sieves on c in such a way that

(a) the maximal sieve M_c belongs to $J(c)$;

(b) for each pair of sieves S and T on c such that $T \in J(c)$ and $S \supseteq T$, $S \in J(c)$;

(c) if $R \in J(c)$ then for any arrow $g : d \to c$ there exists a sieve $S \in J(d)$ such that for each arrow f in S, $g \circ f \in R$;

(d) if the sieve S generated by a presieve $\{f_i : c_i \to c \mid i \in I\}$ belongs to $J(c)$ and for each $i \in I$ we have a presieve $\{g_{ij} : d_{ij} \to c_i \mid j \in I_i\}$ such that the sieve T_i generated by it belongs to $J(c_i)$, then the sieve R generated by the family of composites $\{ f_i \circ g_{ij} : d_{ij} \to c \mid i \in I, j \in I_i\}$ belongs to $J(c)$.

The sieve R defined in (d) will be called the composite of the sieve S with the sieves T_i for $i \in I$ and denoted by $S * \{T_i \mid i \in I\}$.

Bases for a Grothendieck topology

Definition

A basis (for a Grothendieck topology) on a category $\mathcal{C}$ with pullbacks is a function K assigning to each object c of $\mathcal{C}$ a collection $K(c)$ of presieves on c in such a way that the following properties hold:

(i) $\{1_c : c \to c\} \in K(c)$

(ii) if $\{f_i : c_i \to c \mid i \in I\} \in K(c)$ then for any arrow $g : d \to c$ in $\mathcal{C}$, the family of pullbacks $\{g^*(f_i) : c_i \times_c d \to d \mid i \in I\}$ lies in $K(d)$.

(iii) if $\{f_i : c_i \to c \mid i \in I\} \in K(c)$ and for each $i \in I$ we have a presieve $\{g_{ij} : d_{ij} \to c_i \mid j \in I_i\} \in K(c_i)$ then the family of composites $\{ f_i \circ g_{ij} : d_{ij} \to c \mid i \in I, j \in I_i\}$ belongs to $K(c)$.

N.B. If $\mathcal{C}$ does not have pullbacks then condition (ii) can be replaced by the following requirement: if $\{f_i : c_i \to c \mid i \in I\} \in K(c)$ then for any arrow $g : d \to c$ in $\mathcal{C}$, there is a presieve $\{h_j : d_j \to d \mid j \in J\} \in K(d)$ such that for each $j \in J$, $g \circ h_j$ factors through some f_i.

Every basis K generates a Grothendieck topology J given by:

$$R \in J(c) \text{ if and only if } R \supseteq S \text{ for some } S \in K(c)$$

Grothendieck topology generated by a coverage

As we shall also see when we talk about sheaves, the axioms for Grothendieck topologies do not have all the same *status*: the most important one is the stability axiom. This motivates the following definition.

Definition

A (sifted) coverage on a category $\mathcal{C}$ is a collection of sieves which is stable under pullback.

Fact

The Grothendieck topology generated by a coverage is the smallest collection of sieves containing it which is closed under maximality and transitivity.

Theorem

Let $\mathcal{C}$ be a small category and D a coverage on D. Then the Grothendieck topology G_D generated by D is given by

$$\begin{aligned} G_D(c) \;\; = \;\; & \{S \text{ sieve on } c \mid \text{for any arrow } d \xrightarrow{f} c \text{ and sieve } T \text{ on } d, \\ & [(\text{for any arrow } e \xrightarrow{g} d \text{ and sieve } Z \text{ on } e \\ & (Z \in D(e) \text{ and } Z \subseteq g^*(T)) \text{ implies } g \in T) \text{ and} \\ & (f^*(S) \subseteq T)] \text{ implies } T = M_d\} \end{aligned}$$

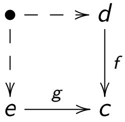
for any object $c \in \mathcal{C}$. □

Examples of Grothendieck topologies I

- For any (small) category $\mathcal{C}$, the trivial topology on $\mathcal{C}$ is the Grothendieck topology in which the only sieve covering an object c is the maximal sieve M_c.
- The dense topology D on a category $\mathcal{C}$ is defined by: for a sieve S,

$$S \in D(c) \quad \text{if and only if} \quad \begin{array}{l} \text{for any } f : d \to c \text{ there exists} \\ g : e \to d \text{ such that } f \circ g \in S \,. \end{array}$$

If $\mathcal{C}$ satisfies the right Ore condition i.e. the property that any two arrows $f : d \to c$ and $g : e \to c$ with a common codomain c can be completed to a commutative square



then the dense topology on $\mathcal{C}$ specializes to the atomic topology on $\mathcal{C}$ i.e. the topology J_{at} defined by: for a sieve S,

$$S \in J_{at}(c) \text{ if and only if } S \neq \emptyset \,.$$

Examples of Grothendieck topologies II

- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathcal{O}(X)}$ on the poset category $\mathcal{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathcal{O}(X))$,

$$S \in J_{\mathcal{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U \,.$$

- More generally, given a frame (or complete Heyting algebra) H, we can define a Grothendieck topology J_H, called the *canonical topology on* H, by:

$$\{a_i \mid i \in I\} \in J_H(a) \text{ if and only if } \bigvee_{i \in I} a_i = a \,.$$

- Given a small category of topological spaces which is closed under finite limits and under taking open subspaces, one may define the open cover topology on it by specifying as basis the collection of open embeddings $\{Y_i \hookrightarrow X \mid i \in I\}$ such that $\bigcup_{i \in I} Y_i = X$.

The Zariski site I

- Given a commutative ring with unit A, we can endow the collection $\mathrm{Spec}(A)$ of its prime ideals with the Zariski topology, whose basis of open sets is given by the subsets

$$\mathrm{Spec}(A)_f := \{P \in \mathrm{Spec}(A) \mid f \notin P\}$$

(for $f \in A$).

- One can prove that $\mathrm{Spec}(A) = \mathrm{Spec}(A)_{f_1} \cup \ldots \cup \mathrm{Spec}(A)_{f_n}$ if and only if $A = (f_1, \ldots, f_n)$.
- We have a structure sheaf $\mathcal{O}$ on $\mathrm{Spec}(A)$ such that $\mathcal{O}(\mathrm{Spec}(A)_f) = A_f$ for each $f \in A$. The fact that it is a sheaf results from the fact that if $A = (f_1, \ldots, f_n)$ then the canonical map

$$A \to \prod_{i \in \{1, \ldots n\}} A_{f_i}$$

is the equalizer of the two canonical maps

$$\prod_{i \in I} A_{f_i} \to \prod_{i,j \in \{1, \ldots, n\}} A_{f_i f_j} .$$

- The stalk $\mathcal{O}_P$ of $\mathcal{O}$ at a prime ideal P is the localization $A_P = \mathrm{colim}_{f \notin P} A_f$.

The Zariski site II

Notice that $\mathrm{Spec}(A)_f$ identifies with $\mathrm{Spec}(A_f)$ under the embedding

$$\mathrm{Spec}(A_f) \hookrightarrow \mathrm{Spec}(A)$$

induced by the canonical homomorphism $A \to A_f$.
This motivates the following definition.

Definition

The Zariski site (over $\mathbb{Z}$) is obtained by equipping the opposite of the category $\mathbf{Rng}_{\mathrm{f.g.}}$ of finitely generated commutative rings with unit with the Grothendieck topology Z given by: for any cosieve S in $\mathbf{Rng}_{\mathrm{f.g.}}$ on an object A, $S \in Z(A)$ if and only if S contains a finite family $\{\xi_i : A \to A_{f_i} \mid 1 \leq i \leq n\}$ of canonical maps $\xi_i : A \to A_{f_i}$ in $\mathbf{Rng}_{\mathrm{f.g.}}$ where $\{f_1, \ldots, f_n\}$ is a set of elements of A which is not contained in any proper ideal of A.

This definition can be generalized to an arbitrary (commutative) base ring k, by considering the category of finitely presented (equivalently, finitely generated) k-algebras and homomorphisms between them. Notice that pushouts exist in this category (whence pullbacks exist in the opposite category) as they are given by tensor products of k-algebras.

Sheaves on a site I

Definition

- A presheaf on a (small) category $\mathcal{C}$ is a functor $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$.
- Let $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a presheaf on $\mathcal{C}$ and S be a sieve on an object c of $\mathcal{C}$.

  A matching family for S of elements of P is a function which assigns to each arrow $f : d \longrightarrow c$ in S an element $x_f \in P(d)$ in such a way that

  $$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \longrightarrow d \,.$$

  An amalgamation for such a family is a single element $x \in P(c)$ such that

  $$P(f)(x) = x_f \quad \text{for all } f \text{ in } S \,.$$

Sheaves on a site II

- Given a site $(\mathcal{C}, J)$, a presheaf on $\mathcal{C}$ is a J-sheaf if every matching family for any J-covering sieve on any object of $\mathcal{C}$ has a unique amalgamation.
- The J-sheaf condition can be expressed as the requirement that for every J-covering sieve S the canonical arrow

$$P(c) \to \prod_{f \in S} P(\mathrm{dom}(f))$$

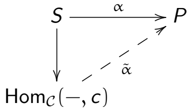
given by $x \to (P(f)(x) \mid f \in S)$ should be the equalizer of the two arrows

$$\prod_{f \in S} P(\mathrm{dom}(f)) \to \prod_{\substack{\text{f,g, } f \in S \\ \mathrm{cod}(g) = \mathrm{dom}(f)}} P(\mathrm{dom}(g))$$

given by $(x_f \to (x_{f \circ g}))$ and $(x_f \to (P(g)(x_f)))$.

The notion of Grothendieck topos

- The J-sheaf condition can also be expressed as the requirement that for every J-covering sieve S (regarded as a subobject of $\text{Hom}_{\mathcal{C}}(-, c)$ in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$), every natural transformation $\alpha : S \to P$ admits a unique extension $\tilde{\alpha}$ along the embedding $S \rightarrowtail \text{Hom}_{\mathcal{C}}(-, c)$:



  (notice that a matching family for R of elements of P is precisely a natural transformation $R \to P$)

- It can also be expressed as the condition

$$P(c) = \varprojlim_{f : d \to c \in S} P(d)$$

  for each J-covering sieve S on an object c.

- The category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on the site $(\mathcal{C}, J)$ is the full subcategory of $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.

- A Grothendieck topos is any category equivalent to the category of sheaves on a site.

Examples of toposes

The following examples show that toposes can be naturally attached to mathematical notions as different as (small) categories, topological spaces, or groups. In fact, as we shall see later in the course, toposes can also be naturally attached to many other kinds of mathematical objects.

Examples

- For any (small) category $\mathcal{C}$, $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is the category of sheaves $\mathbf{Sh}(\mathcal{C}, T)$ where T is the trivial topology on $\mathcal{C}$.
- For any topological space X, $\mathbf{Sh}(\mathcal{O}(X), J_{\mathcal{O}(X)})$ is equivalent to the usual category $\mathbf{Sh}(X)$ of sheaves on X.
- For any (topological) group G, the category $BG = \mathbf{Cont}(G)$ of continuous actions of G on discrete sets is a Grothendieck topos (equivalent, as we shall see, to the category $\mathbf{Sh}(\mathbf{Cont}_{\text{t}}(G), J_{\text{at}})$ of sheaves on the full subcategory $\mathbf{Cont}_{\text{t}}(G)$ on the non-empty transitive actions with respect to the atomic topology).

The sheaf condition for presieves

It is sometimes convenient to check the sheaf condition for the sieve generated by a presieve directly in terms of the presieve.

Definition

A presheaf $F : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ satisfies the sheaf condition with respect to a presieve $P = \{f_i : c_i \to c \mid i \in I\}$ if for any family of elements $\{x_i \in P(c_i) \mid i \in I\}$ such that for any arrows h and k with $f_i \circ h = f_j \circ k$, $F(h)(x_i) = F(k)(x_j)$ there exists a unique element $x \in P(c)$ such that $F(f_i)(x) = x_i$ for all i.

Clearly, F satisfies the sheaf condition with respect to the presieve P if and only if it satisfies it with respect to the sieve generated by P.

The sheaf condition for the presieve P can be expressed as the requirement that the canonical diagram

$$F(c) \longrightarrow \prod_{i \in I} F(c_i) \rightrightarrows \prod_{\substack{h : e \to c_i,\, k : e \to c_j \\ f_i \circ h = f_j \circ k}} F(e)$$

is an equalizer.

N.B. If $\mathcal{C}$ has pullbacks then the product on the right-hand side can be simply indexed by the pairs (i, j) ($e = c_i \times_c c_j$ and h and k being equal to the pullback projections).

Some remarks

The following facts show that the notion of sheaf behaves very naturally with respect to the notions of coverage and of Grothendieck topology:

(i) For any presheaf P, the collection L_P of sieves R such that P satisfies the sheaf axiom with respect to all the pullbacks sieves $f^*(R)$ is a Grothendieck topology, and the largest one for which P is a sheaf.

(ii) By intersecting such topologies, we can deduce that for any given collection of presheaves there is a largest Grothendieck topology for which all of them are sheaves.

(iii) By (i), if a presheaf satisfies the sheaf condition with respect to a coverage then it satisfies the sheaf condition with respect to the Grothendieck topology generated by it.

Subcanonical sites

Definition

A Grothendieck topology J on a (small) category $\mathcal{C}$ is said to be subcanonical if every representable functor $\mathrm{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a J-sheaf.

Fact

For any locally small category $\mathcal{C}$, there exists the largest Grothendieck topology J on $\mathcal{C}$ for which all representables on $\mathcal{C}$ are J-sheaves. It is called the canonical topology on $\mathcal{C}$.

Definition

- A sieve R on an object c of a locally small category $\mathcal{C}$ is said to be effective-epimorphic if it forms a colimit cone under the (large!) diagram consisting of the domains of all the morphisms in R, and all the morphisms over c between them.
- It is said to be universally effective-epimorphic if its pullback along every arrow to c is effective-epimorphic.

The covering sieves for the canonical topology on a locally small category are precisely the universally effective-epimorphic ones. It follows that a Grothendieck topology is subcanonical if and only if it is contained in the canonical topology, that is if and only if all its covering sieves are effective-epimorphic.

Basic properties of Grothendieck toposes

Grothendieck toposes satisfy all the categorical properties that one might hope for:

Theorem

Let $(\mathcal{C}, J)$ be a site. Then

- *the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ has a left adjoint $a : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ (called the associated sheaf functor), which preserves finite limits.*
- *The category $\mathbf{Sh}(\mathcal{C}, J)$ has all (small) limits, which are preserved by the inclusion functor $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$; in particular, limits are computed pointwise and the terminal object $1_{\mathbf{Sh}(\mathcal{C},J)}$ of $\mathbf{Sh}(\mathcal{C}, J)$ is the functor $T : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ sending each object $c \in Ob(\mathcal{C})$ to the singleton $\{*\}$.*
- *The associated sheaf functor $a : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ preserves colimits; in particular, $\mathbf{Sh}(\mathcal{C}, J)$ has all (small) colimits.*
- *The category $\mathbf{Sh}(\mathcal{C}, J)$ has exponentials, which are constructed as in the topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$.*

The associated sheaf functor

Let us start by establishing the following fundamental theorem.

Theorem

For any site $(\mathcal{C}, J)$, the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ has a left adjoint $a : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$, called the associated sheaf functor, which preserves finite limits.

Definition

Let $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ be a presheaf and J a Grothendieck topology on $\mathcal{C}$. Then

- We say that two elements $x, y \in P(c)$ of P are locally equal if there exists a J-covering sieve R on c such that $P(f)(x) = P(f)(y)$ for each $f \in R$.
- Given a sieve S on an object c, a locally matching family for S of elements of P is a function assigning to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that, whenever g is composable with f, $P(g)(x_f)$ and $P(f \circ g)(x)$ are locally equal.

Then $a_J(P)(c)$ consists of equivalence classes of locally matching families for J-covering sieves on c of elements P modulo local equality on a common refinement.

The closure operation on subobjects I

The associated sheaf functor $a_J : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ induces a closure operation $c_J(m)$ on subobjects m of $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ (compatible with pullbacks of subobjects), defined by taking the pullback of the image $a_J(m)$ of $m : A' \rightarrowtail A$ under a_J along the unit η_J of the adjunction between i_J and a_J:

$$\begin{array}{ccc} c_J(A') & \longrightarrow & a_J(A') \\ {\scriptstyle c_J(m)} \downarrow & & \downarrow {\scriptstyle a_J(m)} \\ A & \xrightarrow{\eta_J(A)} & a_J(A) \end{array}$$

Concretely, we have

$$c_J(A')(c) = \{x \in A(c) \mid \{f : d \to c \mid A(f)(x) \in A'(d)\} \in J(c)\} .$$

Remarks

- *If A is a J-sheaf then $a_J(A')$ is isomorphic to $c_J(A')$.*
- *m is c_J-dense (that is, $c_J(m) = 1_A$) if and only if $a_J(m)$ is an isomorphism.*

The closure operation on subobjects II

Proposition

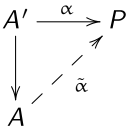
Given a sieve S on an object c, regarded as a subobject $m_S : S \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, c)$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, the following conditions are equivalent:

(a) a_J sends m_S to an isomorphism;

(b) the collection of arrows $a_J(y_{\mathcal{C}}(f))$ for $f \in S$ is jointly epimorphic;

(c) S is J-covering.

We have previously remarked that the sheaf condition for a presheaf P with respect to a sieve S could be reformulated as the requirement that every morphism $S \to P$ admits a unique extension along the canonical embedding $S \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, c)$. In fact, for any c_J-dense subobject $A' \rightarrowtail A$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, if P is a J-sheaf then every morphism $\alpha : A' \to P$ admits a unique extension $\tilde{\alpha} : A \to P$ along the embedding $A' \rightarrowtail A$:



Monomorphisms and epimorphisms in $\mathrm{Sh}(\mathcal{C}, J)$

- Since limits in $\mathbf{Sh}(\mathcal{C}, J)$ are computed as in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, and the latter are computed pointwise, we have that a morphism $\alpha : P \to Q$ in $\mathbf{Sh}(\mathcal{C}, J)$ is a monomorphism if and only if for every $c \in \mathcal{C}$,

$$\alpha(c) : P(c) \to Q(c)$$

  is an injective function.

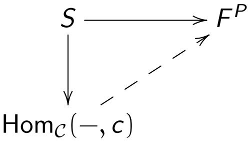
- Since the epimorphisms in $\mathbf{Sh}(\mathcal{C}, J)$ are precisely the morphisms whose image is an isomorphism, we have that a morphism $\alpha : P \to Q$ in $\mathbf{Sh}(\mathcal{C}, J)$ is an epimorphism if and only if it is locally surjective in the sense that for every $c \in \mathcal{C}$ and every $x \in Q(c)$,

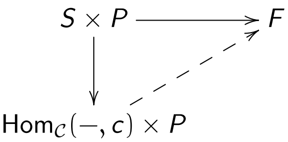
$$\{f : d \to c \mid Q(f)(x) \in \mathrm{Im}(\alpha(d))\} \in J(c) \, .$$

Exponentials in $\mathrm{Sh}(\mathcal{C}, J)$

- We preliminarily remark that *if* exponentials exist in $\mathbf{Sh}(\mathcal{C}, J)$ then they are computed as in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, by using the adjunction between a_J and i_J and the fact that a_J preserves finite products.

- Next, we use the characterization of the J-sheaves on $\mathcal{C}$ as the presheaves P such that for every c_J-dense subobject $A' \rightarrowtail A$, every morphism $A' \to P$ admits a unique extension $A \to P$ along the embedding $A' \rightarrowtail A$ to conclude that if F is a sheaf then F^P is a sheaf for every presheaf P:





Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism. The natural notion of morphism of geometric morphisms if that of geometric transformation.

Definition

(i) Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of a pair of functors $f_* : \mathcal{E} \to \mathcal{F}$ (the direct image of f) and $f^* : \mathcal{F} \to \mathcal{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathcal{E} \to \mathcal{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

- Grothendieck toposes and geometric morphisms between them form a category, denoted by $\mathfrak{BTop}$.
- Given two toposes $\mathcal{E}$ and $\mathcal{F}$, geometric morphisms from $\mathcal{E}$ to $\mathcal{F}$ and geometric transformations between them form a category, denoted by $\mathbf{Geom}(\mathcal{E}, \mathcal{F})$.

Examples of geometric morphisms

- A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$, whose direct image is the functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and whose inverse image is the functor $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$.
- Every Grothendieck topos $\mathcal{E}$ has a unique geometric morphism $\mathcal{E} \to \mathbf{Set}$. The direct image is the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$, sending an object $e \in \mathcal{E}$ to the set $\mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, e)$, while the inverse image functor $\Delta : \mathbf{Set} \to \mathcal{E}$ sends a set S to the coproduct $\bigsqcup_{s \in S} 1_{\mathcal{E}}$.
- For any site $(\mathcal{C}, J)$, the pair of functors formed by the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and the associated sheaf functor $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ yields a geometric morphism $i : \mathbf{Sh}(\mathcal{C}, J) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

Slice toposes

The notion of Grothendieck topos is stable with respect to the slice construction:

Proposition

(i) *For any Grothendieck topos $\mathcal{E}$ and any object P of $\mathcal{E}$, the slice category $\mathcal{E}/P$ is also a Grothendieck topos; more precisely, if $\mathcal{E} = \mathbf{Sh}(\mathcal{C}, J)$ then $\mathcal{E}/P \simeq \mathbf{Sh}(\int P, J_P)$, where J_P is the Grothendieck topology on $\int P$ whose covering sieves are precisely the sieves whose image under the canonical projection functor $\pi_P : \int P \to \mathcal{C}$ is J-covering.*

(ii) *For any Grothendieck topos $\mathcal{E}$ and any morphism $f : P \to Q$ in $\mathcal{E}$, the pullback functor $f^* : \mathcal{E}/Q \to \mathcal{E}/P$ has both a left adjoint (namely, the functor Σ_f given by composition with f) and a right adjoint π_f. It is therefore the inverse image of a geometric morphism $\mathcal{E}/P \to \mathcal{E}/Q$.*

A general hom-tensor adjunction I

Theorem

Let $\mathcal{C}$ be a small category, $\mathcal{E}$ be a locally small cocomplete category and $A : \mathcal{C} \to \mathcal{E}$ a functor. Then we have an adjunction

$$L_A : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \rightleftarrows \mathcal{E} : R_A$$

where the right adjoint $R_A : \mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is defined for each $e \in Ob(\mathcal{E})$ and $c \in Ob(\mathcal{C})$ by:

$$R_A(e)(c) = \mathrm{Hom}_{\mathcal{E}}(A(c), e)$$

and the left adjoint $L_A : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$ is defined by

$$L_A(P) = \mathrm{colim}(A \circ \pi_P),$$

where π_P is the canonical projection functor $\int P \to \mathcal{C}$ from the category of elements $\int P$ of P to $\mathcal{C}$.

A general hom-tensor adjunction II

Remarks

- *The functor L_A can be considered as a generalized tensor product, since, by the construction of colimits in terms of coproducts and coequalizers, we have the following coequalizer diagram:*

$$\coprod_{\substack{c \in \mathcal{C}, p \in P(c) \\ u : c' \to c}} A(c') \overset{\theta}{\underset{\tau}{\rightrightarrows}} \coprod_{c \in \mathcal{C},\, p \in P(c)} A(c) \xrightarrow{\phi} L_A(P),$$

where

$$\theta(c, p, u, x) = (c', P(u)(p), x)$$

and

$$\tau(c, p, u, x) = (c, p, A(u)(x)) \ .$$

For this reason, we shall also denote L_A by

$$- \otimes_{\mathcal{C}} A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E} \ .$$

- *We can rewrite the above coequalizer as follows:*

$$\coprod_{c, c' \in \mathcal{C}} P(c) \times \text{Hom}_{\mathcal{C}}(c', c) \times A(c') \overset{\theta}{\underset{\tau}{\rightrightarrows}} \coprod_{c \in \mathcal{C}} P(c) \times A(c) \xrightarrow{\phi} P \otimes_{\mathcal{C}} A \ .$$

From this we see that this definition is symmetric in P and A, that is

$$P \otimes_{\mathcal{C}} A \cong A \otimes_{\mathcal{C}^{\text{op}}} P \ .$$

A couple of corollaries

Corollary

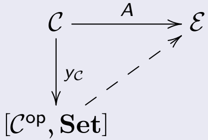
Every presheaf is a colimit of representables. More precisely, for any presheaf $P : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$, we have

$$P \cong \mathrm{colim}(y_{\mathcal{C}} \circ \pi_P),$$

where $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ is a Yoneda embedding and π_P is the canonical projection $\int P \to \mathcal{C}$.

Corollary

For any small category $\mathcal{C}$, the topos $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ is the free cocompletion of $\mathcal{C}$ (via the Yoneda embedding $y_{\mathcal{C}}$); that is, any functor $A : \mathcal{C} \to \mathcal{E}$ to a cocomplete category $\mathcal{E}$ extends, uniquely up to isomorphism, to a colimit-preserving functor $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}] \to \mathcal{E}$ along $y_{\mathcal{C}}$:



Geometric morphisms as flat functors I

Definition

- A functor $A : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a locally small topos $\mathcal{E}$ with small colimits is said to be flat if the functor $- \otimes_{\mathcal{C}} A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}$ preserves finite limits.
- The full subcategory of $[\mathcal{C}, \mathcal{E}]$ on the flat functors will be denoted by $\mathbf{Flat}(\mathcal{C}, \mathcal{E})$.

Proposition

- *For any small category $\mathcal{C}$, a functor $P : \mathcal{C} \to \mathbf{Set}$ is filtering if and only if its category of elements $\int P$ is a filtered category (equivalently, if it is a filtered colimit of representables).*
- *For any small cartesian category $\mathcal{C}$, a functor $\mathcal{C} \to \mathcal{E}$ is flat if and only if it preserves finite limits.*

Theorem

Let $\mathcal{C}$ be a small category and $\mathcal{E}$ be a Grothendieck topos. Then we have an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, [\mathcal{C}^{\text{op}}, \mathbf{Set}]) \simeq \mathbf{Flat}(\mathcal{C}, \mathcal{E})$$

(natural in $\mathcal{E}$), which sends

- *a flat functor $A : \mathcal{C} \to \mathcal{E}$ to the geometric morphism $\mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ determined by the functors R_A and $- \otimes_{\mathcal{C}} A$, and*
- *a geometric morphism $f : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ to the flat functor given by the composite $f^* \circ y_{\mathcal{C}}$ of $f^* : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}$ with the Yoneda embedding $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$.*

Geometric morphisms as flat functors II

Definition

If $(\mathcal{C}, J)$ is a site, a functor $F : \mathcal{C} \to \mathcal{E}$ to a Grothendieck topos is said to be *J-continuous* if it sends J-covering sieves to epimorphic families.

The full subcategory of $\mathbf{Flat}(\mathcal{C}, \mathcal{E})$ on the J-continuous flat functors will be denoted by $\mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$.

Theorem

For any site $(\mathcal{C}, J)$ and Grothendieck topos $\mathcal{E}$, the above-mentioned equivalence between geometric morphisms and flat functors restricts to an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$$

natural in $\mathcal{E}$.

Sketch of proof.

Appeal to the previous theorem

- identifying the geometric morphisms $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ with the geometric morphisms $\mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ which factor through the canonical geometric inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$, and
- using the characterization of such morphisms as the geometric morphisms $f : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ such that the composite $f^* \circ y$ of the inverse image functor f^* of f with the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ sends J-covering sieves to epimorphic families in $\mathcal{E}$.

□

Morphisms of sites

Definition

A morphism of sites $(\mathcal{C}, J) \to (\mathcal{C}', J')$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ such that, denoting by $l : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$ and $l' : \mathcal{C}' \to \mathbf{Sh}(\mathcal{C}', J')$, the canonical functors, there is a geometric morphism $u : \mathbf{Sh}(\mathcal{C}', J') \to \mathbf{Sh}(\mathcal{C}, J)$ making the following square commutative:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle l'} \\ \mathbf{Sh}(\mathcal{C}, J) & \xrightarrow{u^*} & \mathbf{Sh}(\mathcal{C}', J') \end{array} .$$

Proposition

(i) *If $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ are cartesian sites (that is, $\mathcal{C}$ and $\mathcal{D}$ are cartesian categories) then a functor $\mathcal{C} \to \mathcal{D}$ is a morphism of sites if and only if it preserves finite limits and sends J-covering sieves to K-covering sieves. [In the general case, there is also an explicit, though more sophisticated, characterization of morphisms of sites.]*

(ii) *The geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by a morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ admits the following explicit description: the direct image $\mathbf{Sh}(F)_*$ is simply given by composition with F^{op}, while the inverse image $\mathbf{Sh}(F)^*$ assigns to a J-sheaf P on $\mathcal{C}$ the K-sheafification of the presheaf given by the following formula:*

$$\varinjlim_{F^{\mathrm{op}}}(P)(b) = \varinjlim_{\phi : b \to Fa} P(a),$$

for any $b \in \mathcal{D}$, where the colimit is taken over the opposite of the comma category $(b \downarrow f)$.

The basic setting for homological algebra

Abelian categories provide the basic setting for the development of homological algebra. As we shall see, they notably comprise

- categories of modules over a commutative ring;
- categories of chain complexes of abelian groups;
- categories of sheaves of modules over a commutative ring.

The concept of abelian category is technically very well-behaved; notably, unlike that of category of modules, it is self-dual, that is, the opposite of an abelian category is abelian.

The notion of abelian category can be considered an axiomatization of the key properties of the category of abelian groups. In fact, by a general metatheorem, all the basic categorical techniques which apply to abelian groups, notably including diagram chasing, extend to the setting of abelian categories.

To introduce abelian categories, it is convenient to first talk about additive categories.

Additive categories

We say that an object of a category is a zero object if it is both initial and terminal; it is usually denoted by 0.

Definition

A category $\mathcal{C}$ is additive if

(a) Hom(A, B) is an abelian group for any $A, B \in \mathcal{C}$ (the neutral element of $\mathrm{Hom}(a, b)$ will be denoted by 0_{AB});

(b) addition of morphisms distributes over composition on the left and on the right: for any morphisms $f, g : a \to b$, $\xi : x \to a$ and $\chi : b \to y$,

$$\chi \circ (f + g) = (\chi \circ f) + (\chi \circ g)$$

and

$$(f + g) \circ \xi = f \circ \xi + g \circ \xi \,.$$

(c) $\mathcal{C}$ has a zero object.

(d) $\mathcal{C}$ has finite products and finite coproducts: for any objects $A, B \in \mathcal{C}$, both $A \times B$ and $A \coprod B$ exist.

Additive categories

Remark

An additive category can be seen as a category enriched over the category of abelian groups which moreover has a zero object, binary products and binary coproducts.

Examples

- the category R-**mod** of R-modules for a commutative ring R;
- the category of functors $[\mathcal{C}, \mathbf{Ab}]$, where $\mathcal{C}$ is a small category and **Ab** is the category of abelian groups;
- the category $\mathbf{Sh}(X, \mathbf{Ab})$ of sheaves of abelian groups on a topological space X.

Finite biproducts

Lemma

(i) In an additive category $\mathcal{C}$, for any object A of $\mathcal{C}$ the following conditions are equivalent:

- *(a) A is terminal;*
- *(b) A is initial;*
- *(c) $1_A = 0 : A \longrightarrow A$.*

(ii) Given three objects A, B, C in an additive category $\mathcal{C}$, the following conditions are equivalent:

- *(a) there are arrows $p_1 : C \longrightarrow A$ and $p_2 : C \longrightarrow B$ making C a product of A and B;*
- *(b) there are arrows $i_1 : A \longrightarrow C$ and $i_2 : B \longrightarrow C$ making C a coproduct of A and B;*
- *(c) there are arrows p_1, p_2, i_1, i_2 satisfying $\pi_1 \circ i_1 = 1_A$, $\pi_2 \circ i_2 = 1_B$, $\pi_1 \circ i_2 = 0_{BA}$, $\pi_2 \circ i_1 = 0_{AB}$ and $i_1 \circ \pi_1 + i_2 \circ \pi_2 = 1_C$.*

From the lemma, it follows immediately that in any additive category, finite products coincide with finite coproducts.

Definition

An object which is simultaneously a product and coproduct of A and B is called a biproduct and denoted $A \oplus B$.

Additive functors

Definition

A functor $T : \mathcal{C} \to \mathcal{D}$ between additive categories is said to be additive if for any objects $A, B \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(T(A), T(B))$$

given by $f \mapsto T(f)$ is an abelian group homomorphism.

Remark

An additive functor $\mathcal{C} \to \mathcal{D}$ between additive categories $\mathcal{C}$ is precisely an **Ab**-*enriched functor (with respect to the canonical* **Ab**-*enriched structures on $\mathcal{C}$ and $\mathcal{D}$).*

Examples

- The hom functor $\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Ab}$ is additive (and in each of its arguments separately);
- For any R-module N, the functor $(-) \otimes N : R\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$ is additive.

Recovery of the additive structure from the categorical one

In an additive category $\mathcal{C}$, the structure of abelian group on the morphism sets $\text{Hom}_{\mathcal{C}}(A, B)$ can be recovered from the categorical one given by finite biproducts, as follows:

- The zero arrow $0_{AB} : A \to B$ is the composite

$$A \longrightarrow 1 \cong 0 \longrightarrow B\ .$$

- Given $f, g \in \text{Hom}_{\mathcal{C}}(A, B)$, $f + g$ is equal to both composites

$$A \xrightarrow{\langle 1_A, 1_A \rangle} A \times A \xrightarrow{\cong} A \coprod A \xrightarrow{[f,g]} B$$

  and

$$A \xrightarrow{\langle f, g \rangle} B \times B \xrightarrow{\cong} B \coprod B \xrightarrow{[1_B, 1_B]} B$$

Notice that an arrow $f : \coprod_{1 \leq j \leq n} A_j \to \prod_{1 \leq i \leq m} B_i$ in an additive category $\mathcal{C}$ can be represented as a $m \times n$ matrix (f_{ij}), where $f_{ij} = \pi_i \circ f \circ i_j$.

Additive functors

Lemma

Let $T : \mathcal{C} \to \mathcal{D}$ be a functor between additive categories $\mathcal{C}$ and $\mathcal{D}$. Then the following conditions are equivalent:

(i) T *is additive;*

(ii) $T(0) = 0$ *and the canonical arrow* $T(A) \oplus T(B) \to T(A \oplus B)$ *is an isomorphism for any objects* $A, B \in \mathcal{C}$*;*

(iii) $T(0) = 0$ *and the canonical arrow* $T(A \oplus B) \to T(A) \oplus T(B)$ *is an isomorphism for any objects* $A, B \in \mathcal{C}$*.*

Sketch of proof.

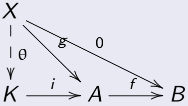
- The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow from the equational characterization of biproducts in an additive category.
- Conditions (ii) and (iii) are equivalent since the two arrows are inverse to each other.
- Condition (i) follows from the fact that the additive structure can be recovered from the categorical one in any additive category.

Kernels and cokernels

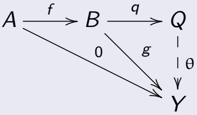
Kernels and cokernels are the additive analogue of equalizers and coequalizers:

Definition

- Given an arrow $f : A \to B$ in an additive category, the kernel $\ker(f)$ of f is a morphism $i : K \to A$ characterized by the following universal property: $u \circ f = 0$ and for every $g : X \to A$ with $f \circ g = 0$, there exists a unique $\theta : X \to K$ with $i \circ \theta = g$:



- Dually, the cokernel $\mathrm{coker}(f)$ of f is an arrow $q : B \to C$ characterized by the following universal property: $q \circ f = 0$ and for every $g : B \to Y$ such that $g \circ f = 0$ there exists a unique arrow $\theta : Q \to Y$ such that $\theta \circ q = g$:



For example, in R-mod, $\ker(f) = \{x \in A \mid f(x) = 0\}$, while $\mathrm{coker}(f) = B/\mathrm{im}(f)$.

Monomorphisms and epimorphisms in additive categories

Monomorphisms and epimorphisms in additive categories can be characterized in terms of zero arrows:

Remark

An arrow $f : A \to B$ in an additive category is a monomorphism (resp. an epimorphism) if and only if $f \circ g = 0$ (resp. $g \circ f = 0$) implies $g = 0$.

Proposition

Let $f : A \to B$ be a morphism in an additive category.

(i) If $\ker(f)$ exists, then f is monic if and only if $\ker(f) = 0$;

(ii) Dually, if $\mathrm{coker}(f)$ exists, then f is epimorphism if and only if $\mathrm{coker}(f) = 0$.

Recall that a subobject of an object A is an isomorphism class of monomorphisms with codomain A. Dually, a quotient of an object A is an isomorphism class of epimorphisms with domain A.

Abelian categories

Definition

An abelian category is an additive category such that

(i) every morphism has a kernel and a cokernel;

(ii) every monomorphism is a kernel and every epimorphism is a cokernel.

Notice that every abelian category is balanced.

Examples

- $\mathbf{Ab}$ is abelian
- For any small category $\mathcal{C}$ and abelian category $\mathcal{A}$, the category $[\mathcal{C}, \mathcal{A}]$ is abelian (all the structure is defined pointwise).
- For any ring R (not necessarily commutative), R-mod is abelian.
- If $\mathcal{C}$ is a small additive category and $\mathcal{A}$ is abelian, then the full subcategory $\mathrm{Add}(\mathcal{C}, \mathcal{A})$ of $[\mathcal{C}, \mathcal{A}]$ on the additive functors is abelian (notice that, regarding a ring R as an additive category, we have $\mathrm{Add}(R, \mathcal{A}) \simeq R$-mod).

Still, the most important class of abelian categories in the context of this course will be that of categories of modules over a ring internal to a Grothendieck topos, as they are used for defining (co)homology.

We shall say that a functor between abelian categories is exact (resp. left-exact, right-exact) if it preserves finite limits and colimits (resp. finite limits, finite colimits).

Images and coimages

Definition

Let $f : A \to B$ be a morphism in an abelian category.

- The image of f is
$$\mathrm{im}(f) := \ker(\mathrm{coker}(f));$$
- Dually, the coimage of f is
$$\mathrm{coim}(f) := \mathrm{coker}(\ker(f)) \,.$$

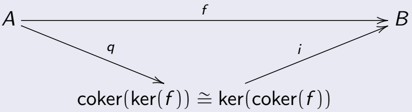
Proposition

Let $f : A \to B$ be a morphism in an abelian category $\mathcal{A}$.

(i) For any morphism in $\mathcal{A}$, $\mathrm{im}(f)$ is the least subobject through which f factors.

(ii) The canonical morphism $\ker(\mathrm{coker}(f)) \to \mathrm{coker}(\ker(f))$ is an isomorphism.

(iii) The diagram



provides the (unique up to commuting isomorphism) factorization of the arrow f as an epimorphism followed by a monomorphism.

Exact sequences

Thanks to the notion of kernel and cokernel, it is possible to define the notion of exact sequence in the general setting of abelian categories:

Definition

A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is said to be exact if we have an equality of subobjects

$$\ker(g) = \mathrm{im}(f) \; .$$

Remark

Notice that the inclusion $\mathrm{im}(f) \subseteq \ker(g)$ *is equivalent to the condition* $g \circ f = 0$*. If the above sequence merely satisfies this condition then we say that it (once completed with zeros on the left and on the right) is a complex.*

Characterization of abelian categories

Definition

- A category $\mathcal{C}$ is said to be regular if it has finite limits and its morphisms have images which are stable under pullback.
- A regular category $\mathcal{C}$ is said to be effective (or exact) if every equivalence relation in $\mathcal{C}$ has a quotient and coincides with the kernel pair of it.

The following result shows that abelian categories are the result of marrying the categorical structure of an exact category with the datum of an additive structure:

Theorem

A category is abelian if and only if it is exact and (semi-)additive.

Working with abelian categories

There are several features of the concept of abelian category which make it very convenient as a setting for developing homological algebra:

- Unlike the notion of a category enriched over abelian groups, which involves the specification of an additional additive structure on morphism sets, the property of a category to be abelian is a categorical invariant.
- The concept of abelian category is self-dual. This allows us to profitably employ the duality principle and hence to obtain "two theorems at the cost of one" provided that their statements and proofs can be lifted to the setting of abelian categories.
- By the Freyd-Mitchell embedding theorem, every abelian category can be fully embedded in a category of modules over a ring. This result, which is not fully constructive, justifies reasoning with objects and arrows of an abelian category by using "elements", that is, as they were respectively modules and homomorphisms between them. [A more intrisic, constructive justification for this comes from the realisation of the fact that any abelian category is the (effective-regular) syntactic category for its canonical regular theory, and hence supports an internal language (cf. the paper *Syntactic categories for Nori motives*).]

Modules internal to a Grothendieck topos

Let $(\mathcal{C}, J)$ be a site. Recall that

(i) A group object [resp. ring object, resp. module object over a ring object $\mathcal{O}$] of $\mathbf{Sh}(\mathcal{C}, J)$ is a sheaf of sets

$$U \to \mathcal{G}(U) \qquad [\text{resp.} \quad \mathcal{O}(U)\,,\ \text{resp. } \mathcal{M}(U)\ ]$$

endowed with a structure of group [resp. ring, module over the ring $\mathcal{O}(U)$] on each

$$\mathcal{G}(U) \qquad [\text{resp.} \quad \mathcal{O}(U)\,,\ \text{resp. } \mathcal{M}(U)\ ]$$

such that all the maps

$$\mathcal{G}(U) \to \mathcal{G}(V) \qquad [\text{resp.} \quad \mathcal{O}(U) \to \mathcal{O}(V)\,,\ \text{resp. } \mathcal{M}(U) \to \mathcal{M}(V)\ ]$$

correponding to arrows $V \to U$ in $\mathcal{C}$ are groups [resp. ring, resp. module] morphisms.

(ii) A morphism of group objects [resp. ring objects, resp. module objects over some ring object $\mathcal{O}$] is a morphism of sheaves

$$\mathcal{G}_1 \to \mathcal{G}_2 \qquad [\text{resp.} \quad \mathcal{O}_1 \to \mathcal{O}_2\,,\ \text{resp. } \mathcal{M}_1 \to \mathcal{M}_2\ ]$$

such that all maps

$$\mathcal{G}_1(U) \to \mathcal{G}_2(U) \qquad [\text{resp.} \quad \mathcal{O}_1(U) \to \mathcal{O}_2(U)\,,\ \text{resp. } \mathcal{M}_1(U) \to \mathcal{M}_2(U)\ ]$$

are group [resp. ring, resp. module] morphisms.

The abelian categories of Modules I

Definition

Let $(\mathcal{C}, J)$ be a site and $\mathcal{O}$ a ring object in the topos $\mathbf{Sh}(\mathcal{C}, J)$.

Then module objects over $\mathcal{O}$ in $\mathbf{Sh}(\mathcal{C}, J)$
are called $\mathcal{O}$-Modules, and their category is denoted

$$\mathcal{M}od_{\mathcal{O}}(\mathbf{Sh}(\mathcal{C}, J)) \text{ (or simply) } \mathcal{M}od_{\mathcal{O}} \,.$$

Proposition

For any site $(\mathcal{C}, J)$ and ring object $\mathcal{O}$ in $\mathbf{Sh}(\mathcal{C}, J)$,

$$\mathcal{M}od_{\mathcal{O}}(\mathbf{Sh}(\mathcal{C}, J))$$

is an abelian category with arbitrary limits and colimits.

The abelian categories of Modules II

In particular:

- Finite limits in $\mathcal{M}od_{\mathcal{O}(\mathbf{Sh}(\mathcal{C},J))}$ are computed as in $\mathbf{Sh}(\mathcal{C},J)$.
- The kernel $\ker(\alpha)$ of a morphism $\alpha : F \to G$ in $\mathcal{M}od_{\mathcal{O}(\mathbf{Sh}(\mathcal{C},J))}$ is given by

$$(\ker(\alpha))(c) = \{x \in F(c) \mid \alpha(c)(x) = 0\},$$

  for any $c \in \mathcal{C}$.
- Finite products and finite coproducs coincide and are given by finite cartesian products (computed pointwise).
- The image $\mathrm{im}(\alpha)$ of a morphism $\alpha : F \to G$ in $\mathrm{Ab}(\mathbf{Sh}(\mathcal{C},J))$ is given by

$$\mathrm{im}(\alpha)(c) = \{y \in G(c) \mid \{f : d \to c \mid G(f)(y) \in \mathrm{im}(\alpha(d))\} \in J(c)\}$$

  Indeed, it is the J-closure of the image of α calculated in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.
- Cokernels are obtained as the result of applying the associated sheaf functor to them as calculated (pointwise) in the presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ (endowed with the natural induced module structure).

Change of structure ring-sheaf

Proposition

Let $(\mathcal{C}, J) =$ *site,*
$(\mathcal{O}_1 \to \mathcal{O}_2) =$ *morphism of sheaves of rings in* $\mathbf{Sh}(\mathcal{C}, J)$.

Then the forgetful functor

$$\begin{array}{ccc} \mathcal{M}od_{\mathcal{O}_2} & \to & \mathcal{M}od_{\mathcal{O}_1}\,, \\ \mathcal{M} & \to & \mathcal{M}\,, \end{array}$$

has a left adjoint denoted

$$\begin{array}{ccc} \mathcal{M}od_{\mathcal{O}_1} & \to & \mathcal{M}od_{\mathcal{O}_2}\,, \\ \mathcal{M} & \to & \mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{M} \;. \end{array}$$

Remarks

(i) *For any object* $\mathcal{M}$ *of* $\mathcal{M}od_{\mathcal{O}_1}$,

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{M}$$

is constructed as the sheafification of the presheaf

$$U \to \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{M}(U) \;.$$

(ii) *The forgetful functor respects arbitrary limits and colimits while its left adjoint*

$$\mathcal{M} \to \mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{M}$$

respects arbitrary colimits.

Exponentials (or "inner $\mathcal{H}om$") and tensor products

Definition

For any object c of a site $(\mathcal{C}, J)$, the sheaf $I(c)$ (where I is the canonical functor $\mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$) induces a geometric morphism $\mathbf{Sh}(\mathcal{C}, J)/I(c) \to \mathbf{Sh}(\mathcal{C}, J)$ whose inverse image $!_{I(c)}$ is the cartesian product with $I(c)$:

$$\begin{array}{rcl} !_{I(c)} : \mathbf{Sh}(\mathcal{C}, J) & \to & \mathbf{Sh}(\mathcal{C}, J)/I(c)\,, \\ F & \to & F \times I(c)\ . \end{array}$$

Remarks

(i) *Functors of the form $!_{I(c)}$ respect arbitrary limits and colimits. In particular, they transform any ring object $\mathcal{O}$ of $\mathbf{Sh}(\mathcal{C}, J)$ into ring objects $\mathcal{O}_c$ and induce additive exact functors*

$$\mathcal{M}od_{\mathcal{O}} \to \mathcal{M}od_{\mathcal{O}_c}\ .$$

(ii) *For any J-sheaves F_1 and F_2 on $\mathcal{C}$, the presheaf*

$$c \to \mathrm{Hom}(!_{I(c)}(F_1), !_{I(c)}(F_2))$$

is a sheaf denoted $F_2^{F_1}$ or $\mathcal{H}om(F_1, F_2)$. It is characterised by the property that, for any J-sheaf G,

$$\mathrm{Hom}(G, \mathcal{H}om(F_1, F_2)) \quad \textit{identifies with} \quad \mathrm{Hom}(G \times F_1, F_2)\ .$$

(iii) *In the same way, for any $\mathcal{O}$-Modules $\mathcal{M}_1, \mathcal{M}_2$, the presheaf*

$$c \to \mathrm{Hom}_{\mathcal{O}_c}(!_{I(c)}(\mathcal{M}_1), !_{I(c)}(\mathcal{M}_2))$$

is a J-sheaf denoted $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}_1, \mathcal{M}_2)$.

Proposition

Let $\mathcal{N}$ *be an* $\mathcal{O}$*-Module, for a ring* $\mathcal{O}$ *in a topos* $\mathbf{Sh}(\mathcal{C}, J)$*.*

Then the functor

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}} & \to & \mathcal{M}od_{\mathcal{O}}\,, \\ \mathcal{L} & \to & \mathcal{H}om_{\mathcal{O}}(\mathcal{N}, \mathcal{L}) \end{array}$$

has a left adjoint denoted

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}} & \to & \mathcal{M}od_{\mathcal{O}}\,, \\ \mathcal{M} & \to & \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N} \ . \end{array}$$

Furthermore, $\otimes$ *extends as a double functor*

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}} \times \mathcal{M}od_{\mathcal{O}} & \to & \mathcal{M}od_{\mathcal{O}_X}\,, \\ (\mathcal{M}, \mathcal{N}) & \to & \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N} \end{array}$$

such that the two triple functors

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}} & \to & \textit{abelian groups}\,, \\ (\mathcal{M}, \mathcal{N}, \mathcal{L}) & \to & \mathrm{Hom}_{\mathcal{O}}(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}, \mathcal{L})\,, \\ (\mathcal{M}, \mathcal{N}, \mathcal{L}) & \to & \mathrm{Hom}_{\mathcal{O}}(\mathcal{M}, \mathcal{H}om_{\mathcal{O}}(\mathcal{N}, \mathcal{L})) \end{array}$$

are isomorphic.

Remarks

(i) *The tensor product* $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ *is constructed as the sheafification of the functor*

$$U \to \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U) \ .$$

(ii) *The two functors* $\mathcal{M}od_{\mathcal{O}} \times \mathcal{M}od_{\mathcal{O}} \to \mathcal{M}od_{\mathcal{O}}$

$$\begin{array}{rrcl} & (\mathcal{M}, \mathcal{N}) & \to & \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N} \\ \textit{and} & (\mathcal{M}, \mathcal{N}) & \to & \mathcal{N} \otimes_{\mathcal{O}} \mathcal{M} \end{array}$$

are canonically isomorphic.

(iii) *The double functor*

$$(\mathcal{M}, \mathcal{N}) \to \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$$

respects arbitrary colimits in $\mathcal{M}$ *or* $\mathcal{N}$*,*
while the double functor

$$(\mathcal{N}, \mathcal{L}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{N}, \mathcal{L})$$

respects arbitrarily limits in $\mathcal{L}$
and transforms arbitrary colimits in $\mathcal{N}$ *into limits.*

Push-forward and pull-back functors

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism between Grothendieck toposes. Then both functors f_* and f^* are left-exact.

So they transform group objects into group objects, ring objects into ring objects and, for any ring objects $\mathcal{O}$ in $\mathcal{F}$ and $\mathcal{O}'$ in $\mathcal{E}$, define additive functors

$$\begin{aligned} f_* : \mathcal{M}od_{\mathcal{O}} &\to \mathcal{M}od_{f_*\mathcal{O}} && \text{(which is right-exact)}\,, \\ f^{-1} : \mathcal{M}od_{\mathcal{O}'} &\to \mathcal{M}od_{f^{-1}\mathcal{O}'} && \text{(which is exact)}\ . \end{aligned}$$

Definition

A morphism of ringed toposes $(\mathcal{F}, \mathcal{O}') \to (\mathcal{E}, \mathcal{O})$ is a pair consisting of a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ and a ring-object homomorphism $\mathcal{O}' \to f_*\mathcal{O}$ or, equivalently, $f^{-1}\mathcal{O}' \to \mathcal{O}$.

Corollary

For any morphism of ringed toposes $(\mathcal{F}, \mathcal{O}') \to (\mathcal{E}, \mathcal{O})$*:*

(i) *The composition of the functor*

$$f_* : \mathcal{M}od_{\mathcal{O}} \to \mathcal{M}od_{f_*\mathcal{O}}$$

and of the forgetful functor defines a functor

$$f_* : \mathcal{M}od_{\mathcal{O}} \to \mathcal{M}od_{\mathcal{O}'} \ .$$

(ii) *This functor* $f_* : \mathcal{M}od_{\mathcal{O}} \to \mathcal{M}od_{\mathcal{O}'}$ *has a left adjoint functor*

$$f^* : \mathcal{M}od_{\mathcal{O}'} \to \mathcal{M}od_{\mathcal{O}}$$

constructed as the composition of the functors

$$f^{-1} : \mathcal{M}od_{\mathcal{O}'} \to \mathcal{M}od_{f^{-1}\mathcal{O}'}$$

and

$$\begin{array}{rcl} \mathcal{M}od_{f^{-1}\mathcal{O}'} & \to & \mathcal{M}od_{\mathcal{O}} \,, \\ \mathcal{M} & \to & \mathcal{O} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{M} \ . \end{array}$$

Remarks

$f_* : \mathcal{M}od_{\mathcal{O}_X} \to \mathcal{M}od_{\mathcal{O}_Y}$ *respects limits,*

$f^* : \mathcal{M}od_{\mathcal{O}_Y} \to \mathcal{M}od_{\mathcal{O}_X}$ *respects colimits.*

For further reading

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