Cohomology of toposes

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Chapter II:

Categorical preliminaries

The categorical point of view

- Category Theory, introduced by Samuel Eilenberg and Saunders Mac Lane in the years 1942-45 in the context of Algebraic Topology, is a branch of Mathematics which provides an abstract language for expressing mathematical concepts and reasoning about them. In fact, the concepts of Category Theory are unifying notions whose instances can be found in essentially every field of Mathematics.
- The underlying philosophy of Category Theory is to replace the primitive notions of set and belonging relationship between sets, which constitute the foundations of Set Theory, with abstractions of the notions of set and function, namely the concepts of object and arrow.
- Since it was introduced, this approach has entailed a deep paradigmatic shift in the way Mathematicians could look at their subject, and has paved the way to important discoveries which would have hardly been possible before. One of the great achievements of Category Theory is Topos Theory, a subject entirely written in categorical language.

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The definition of category (1/2)

Definition

A (small) category $\mathcal{C}$ consists of

(i) a set $\mathrm{Ob}(\mathcal{C})$,

(ii) for any $a, b \in Ob(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(a, b)$,

(iii) for any $a, b, c \in Ob(\mathcal{C})$, a map:

$$\circ_{\mathcal{C}} : \mathrm{Hom}_{\mathcal{C}}(a, b) \times \mathrm{Hom}_{\mathcal{C}}(b, c) \to \mathrm{Hom}_{\mathcal{C}}(a, c)$$

called the composition and denoted by $(f, g) \to g \circ f$,

these data satisfying

a) the composition $\circ$ is associative, i.e., for $f \in \mathrm{Hom}_{\mathcal{C}}(a, b), g \in \mathrm{Hom}_{\mathcal{C}}(b, c)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(c, d)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$,

b) for each $a \in Ob(\mathcal{C})$, there exists $id_a \in \mathrm{Hom}_{\mathcal{C}}(a, a)$ such that $f \circ id_a = f$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ and $id_a \circ g = g$ for all $g \in \mathrm{Hom}_{\mathcal{C}}(b, a)$.

The definition of category (2/2)

- An element of $\mathrm{Ob}(\mathcal{C})$ is called an object of $\mathcal{C}$.
- For $a, b \in \mathrm{Ob}(\mathcal{C})$, an element f of $\mathrm{Hom}_{\mathcal{C}}(a, b)$ is called an arrow (from a to b) in $\mathcal{C}$; we say that a is the domain of f, b is the codomain of f, and we write $f : a \to b$, $a = \mathrm{dom}(f)$ and $b = \mathrm{cod}(f)$.
- The arrow id_a is called the identity arrow on a.

Remark

The concept of category has a first-order axiomatization, in a language having two sorts $\mathbf{O}$ and $\mathbf{A}$ (respectively for objects and arrows), two unary function symbols (for domain and codomain) $\mathbf{A} \to \mathbf{O}$, one unary function symbol $\mathbf{O} \to \mathbf{A}$ (formalizing the concept of identity arrow) and a ternary predicate of type $\mathbf{A}$ (formalizing the notion of composition of arrows).

We will also consider *large* categories, that is categories with a proper class (rather than a *set*) of objects or arrows.

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The dual category

The concept of category is self-dual i.e. the axioms in the definition of category continue to hold if we formally reverse the direction of arrows while mantaining the same objects.

Definition

Given a category $\mathcal{C}$, the dual category $\mathcal{C}^{\mathrm{op}}$ is defined by setting

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C}), \qquad \mathrm{Hom}^{\mathrm{op}}_{\mathcal{C}}(a, b) = \mathrm{Hom}_{\mathcal{C}}(b, a),$$

and defining the composition $g \circ_{\mathcal{C}^{\mathrm{op}}} f$ of $f \in \mathrm{Hom}^{\mathrm{op}}_{\mathcal{C}}(a, b)$ and $f \in \mathrm{Hom}^{\mathrm{op}}_{\mathcal{C}}(b, c)$ by

$$g \circ_{\mathcal{C}^{\mathrm{op}}} f = f \circ_{\mathcal{C}} g.$$

Note that $\mathcal{C}^{\mathrm{op}\,\mathrm{op}} = \mathcal{C}$ for any category $\mathcal{C}$.

The duality principle

Every statement formulated in the language of Category Theory has a dual, obtained by formally reversing the arrows and the order of composition of them.

- A statement is true in a category $\mathcal{C}$ if and only if the dual statement is true in the dual category $\mathcal{C}^{\text{op}}$. Hence a statement is valid in all categories if and only if its dual is.
- Anyway, two dual statements in the language of Category Theory, when interpreted in a given 'concrete' category, may specialize to two very different-looking (and even inequivalent!) mathematical statements.
- Sometimes, it is possible to lift ordinary mathematical statements to the level of categories (or at least to classes of categories closed under duality) and obtain abstract proofs of them in the language of Category Theory; if this is the case, one can then invoke the duality principle to derive dual versions of them which can be specialized to the original context.

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Properties of arrows

We can consider various properties of arrows in a category, expressed in categorical language. An arrow $f : a \to b$ is:

- a monomorphism (or monic) if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all arrows $g_1, g_2 : x \to a$.
- an epimorphism (or epic) if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all arrows $g_1, g_2 : b \to x$.
- an isomorphism if there exists an arrow $g : b \to a$ with $f \circ g = 1_b$ and $g \circ f = 1_a$.

Notice that monomorphisms are dual to epimorphisms i.e. an arrow f of a category $\mathcal{C}$ is a monomorphism in $\mathcal{C}$ if and only if it is an epimorphism in $\mathcal{C}^{\mathrm{op}}$ (regarded as an arrow in $\mathcal{C}^{\mathrm{op}}$).

Example

In the category **Set**, an arrow is:

- a monomorphism if and only if it is an injective function.
- an epimorphism if and only if it is a surjective function.
- an isomorphism if and only if it is a bijection.

Categories of mathematical objects

Important mathematical objects can be organized into categories.

Examples

- The category $\mathbf{Set}$ of sets and functions between them.
- The category $\mathbf{Top}$ of topological spaces and continuous maps between them.
- The category of $\mathbf{Gr}$ of groups and group homomorphisms, the category $\mathbf{Rng}$ of rings and ring homomorphisms, the category $\mathbf{Vect}_K$ of vector spaces over a field K and K-linear maps between them, etc.
  In fact, given a first-order theory $\mathbb{T}$, we have a category $\mathbb{T}\text{-mod}(\mathbf{Set})$ having as objects the (set-based) models of $\mathbb{T}$ and as arrows the structure-preserving maps between them.

Mathematical objects as categories

On the other hand, important mathematical objects arise as particular kinds of categories:

- A set can be seen as a discrete category i.e. a category whose only arrows are the identity arrows.
- A preorder can be seen as a preorder category i.e. a category having at most one arrow from one object to another.
- A monoid can be seen as a category with just one object.
- A groupoid is a category whose arrows are all isomorphisms; in particular, a group is a groupoid with just one object.

Functors

Functors are the natural structure-preserving maps between categories.

Definition

Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ consists of a map $F : Ob(\mathcal{C}) \to Ob(\mathcal{C}')$ and of maps $F : \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}'}(F(a), F(b))$ for all $a, b \in \mathcal{C}$, such that

- $F(id_a) = id_{F(a)}$ for all $a \in \mathcal{C}$,
- $F(g \circ f) = F(g) \circ F(f)$ for all $f : a \to b, g : b \to c$.

Functors from the dual $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ to the category $\mathbf{Set}$ of sets are called **presheaves** on $\mathcal{C}$.

Composition of functors is defined in the obvious way and on each category $\mathcal{C}$ we have the identity functor $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.
In fact, (small) categories and functors form themselves a (large) category, denoted by $\mathbf{Cat}$.

Full and faithful functors

Definition

- A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be faithful if $F : \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}'}(F(a), F(b))$ is injective for all $a, b \in \mathcal{C}$.
- A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be full if $F : \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}'}(F(a), F(b))$ is surjective for all $a, b \in \mathcal{C}$.
- A functor $F : \mathcal{C} \to \mathcal{D}$ is (essentially) surjective if every object $d \in Ob(\mathcal{D})$ is (isomorphic to one) of the form $F(c)$ for some $c \in Ob(\mathcal{C})$.
- A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is a category $\mathcal{D}$ such that $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{D}}(a, b) \subseteq \mathrm{Hom}_{\mathcal{C}}(a, b)$ for any $a, b \in Ob(\mathcal{D})$, the composition in $\mathcal{D}$ is induced by the composition in $\mathcal{C}$ and the identity arrows in $\mathcal{D}$ are identity arrows in $\mathcal{C}$; $\mathcal{D}$ is said to be a full subcategory of $\mathcal{C}$ if the inclusion functor $i : \mathcal{D} \to \mathcal{C}$ is full.

Natural transformations

Definition

Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories and let F_1 and F_2 be two functors from $\mathcal{C}$ to $\mathcal{C}'$. A natural transformation $\alpha : F_1 \to F_2$ is a function assigning to each object $a \in Ob(\mathcal{C})$ an arrow $\alpha(a) : F_1(a) \to F_2(a)$ in $\mathcal{C}'$ in such a way that for all arrows $f : a \to b$ in $\mathcal{C}$ the diagram below commutes:

$$\begin{array}{ccc} F_1(a) & \xrightarrow{\alpha(a)} & F_2(a) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(b) & \xrightarrow[\alpha(b)]{} & F_2(b) \end{array}$$

A natural isomorphism is an invertible natural transformation.

Example

Let $\mathbf{Vect}_K$ be the category of vector spaces over a field K and $* : \mathbf{Vect}_K^{\mathrm{op}} \to \mathbf{Vect}_K$ be the duality functor which assigns to a vector space $V \in Ob(\mathbf{Vect}_K)$ the vector space $V^* = \mathrm{Hom}_{\mathbf{Vect}_K}(V, K)$. Then $id_{\mathbf{Vect}_K} \to **$ is a natural transformation of functors from $\mathbf{Vect}_K$ to itself.

Representable functors

Given a category $\mathcal{C}$ and an object $c \in Ob(\mathcal{C})$, we have a functor $\mathrm{Hom}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathbf{Set}$ defined by

- $\mathrm{Hom}_{\mathcal{C}}(c, -)(a) = \mathrm{Hom}_{\mathcal{C}}(c, a)$ for $a \in Ob(\mathcal{C})$,
- $\mathrm{Hom}_{\mathcal{C}}(c, -)(f) : \mathrm{Hom}_{\mathcal{C}}(c, a) \to \mathrm{Hom}_{\mathcal{C}}(c, b)$ given by $g \to f \circ g$, for $f : a \to b$ in $\mathcal{C}$.

Functors naturally isomorphic to those of the form $\mathrm{Hom}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathbf{Set}$ are said to be representable.
Note that, dually, we have functors $\mathrm{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

The Yoneda Lemma

Given a category $\mathcal{C}$, we define the Yoneda embedding to be the functor $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ given by:

- $y(a) = \mathrm{Hom}_{\mathcal{C}}(-, a)$, for an object $a \in Ob(\mathcal{C})$.
- $y(f) = f \circ_{\mathcal{C}} -$, for an arrow $f : a \to b$ in $\mathcal{C}$.

Theorem (Yoneda Lemma)

Let $\mathcal{C}$ be a category and $F : \mathcal{C}^{op} \to \mathbf{Set}$ be a functor. Then, for any object $c \in Ob(\mathcal{C})$, we have a bijection

$$\mathrm{Hom}_{[\mathcal{C}^{op}, \mathbf{Set}]}(y_{\mathcal{C}}(c), F) \cong F(c)$$

natural in c (and in F).

Sketch of proof.

The proof essentially amounts to checking that the any natural transformation $\alpha : \mathrm{Hom}_{\mathcal{C}}(-, c) \to F$ is uniquely determined by its value $\alpha(c)(id_c)$ at the identity on c. □

Corollary

The Yoneda embedding is full and faithful.

Representable functors: a characterization

Fact

A functor $F : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ is representable if and only if there is an object c_0 and an element $x_0 \in F(c_0)$ such that for any object c of $\mathcal{C}$ and any element $x \in F(c)$ there exists a unique arrow $f : c \to c_0$ in $\mathcal{C}$ such that $x = F(f)(x_0)$.

Indeed, by the Yoneda Lemma, specifying a natural isomorphism $\text{Hom}_{\mathcal{C}}(-, c_0) \cong F$ amounts precisely to giving an element $x_0 \in F(c_0)$ satisfying the above universal property.

Remark

All the information contained in a representable functor F is therefore condensed in the representing object c_0 and the universal element $x_0 \in F(c_0)$, which 'generates' all the other elements $x \in F(c)$ by applying functions of the form $F(f)$ to it.

Equivalence of categories

Two functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ are said to be naturally isomorphic if there exists an invertible natural transformation $\alpha : F_1 \to F_2$.

When can two categories be considered the same, from the point of view of the categorical properties they satisfy?

Definition (Equivalence of categories)

Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $F \circ G \cong id_{\mathcal{D}}$, $G \circ F \cong id_{\mathcal{C}}$.

Theorem

Under AC, a functor is part of an equivalence of categories if and only if it is full, faithful and essentially surjective.

Functor categories

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. The functor category $[\mathcal{C}, \mathcal{D}]$ is the category having as objects the functors $\mathcal{C} \to \mathcal{D}$ and as arrows the natural transformations between them.

Examples

- If $\mathcal{C}$ is the category having two distinct objects and exactly one non-identical arrow $0 \to 1$, the functor category $[\mathcal{C}, \mathcal{D}]$ becomes the category $\mathcal{D}^{\to}$ of arrows in $\mathcal{D}$ and commutative squares between them.
- If $\mathcal{C}$ is the category corresponding to a monoid M and $\mathcal{D} = \mathbf{Set}$, then $[\mathcal{C}, \mathcal{D}]$ becomes the category $M\text{-}\mathbf{Set}$ of sets equipped with a M-action and action-preserving maps between them.
- If $\mathcal{C}$ is a discrete category on a set I and $\mathcal{D} = \mathbf{Set}$ then $[\mathcal{C}, \mathcal{D}]$ becomes the category $\mathbf{Bn}(I)$ of I-indexed collections of sets and functions between them.

Other basic constructions

Definition (Slice category)

Let $\mathcal{C}$ be a category and a be an object of $\mathcal{C}$. The slice category $\mathcal{C}/a$ of $\mathcal{C}$ on a has as objects the arrows in $\mathcal{C}$ with codomain a and as arrows the commutative triangles between them (composition and identities are the obvious ones).

Notice that the slice category $\mathbf{Set}/I$ is equivalent to the functor category $\mathbf{Bn}(I)$ introduced above.

Two monomorphisms in a category $\mathcal{C}$ with common codomain a are said to be isomorphic if they are isomorphic as objects of $\mathcal{C}/a$. An isomorphism class of monomorphisms with common codomain a is called a subobject of a.

Definition (Product category)

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. The product category $\mathcal{C} \times \mathcal{D}$ has as objects the pairs $\langle a, b\rangle$ where a is an object of $\mathcal{C}$ and b is an object of $\mathcal{D}$ and as arrows $\langle a, b\rangle \to \langle c, d\rangle$ the pairs $\langle f, g\rangle$ where $f : a \to c$ is an arrow in $\mathcal{C}$ and $g : b \to d$ is an arrow in $\mathcal{D}$ (composition and identities are defined componentwise).

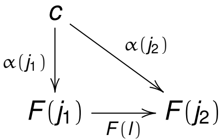
Universal properties

- It is a striking fact that one can often define mathematical objects not by means of their internal structure (that is, as in the classical spirit of set-theoretic foundations) bur rather in terms of their relations with the other objects of the mathematical environment in which one works (that is, in terms of the objects and arrows of the category in which one works), by means of so-called universal properties.
- Of course, isomorphic objects in a category are indistinguishable from the point of view of the categorical properties that they satisfy; in fact, definitions via universal property do not determine the relevant objects 'absolutely' but only up to isomorphism in the given category.

The technical embodiment of the idea of universal property is given by the notion of limit (dually, colimit) of a functor.

Limits and colimits I

Note that a functor $F : \mathcal{J} \to \mathcal{C}$ can be thought as a 'diagram in $\mathcal{C}$ of shape $\mathcal{J}$'. For every object c of $\mathcal{C}$, there is a 'constant' functor $\Delta(c) : \mathcal{J} \to \mathcal{C}$, which sends all the objects of $\mathcal{J}$ to the object c and all the arrows in $\mathcal{J}$ to the identity arrow on c. This defines a diagonal functor $\Delta : \mathcal{C} \to [\mathcal{J}, \mathcal{C}]$. A natural transformation α from $\Delta(c)$ to a functor $F : \mathcal{J} \to \mathcal{C}$ is called a cone from c to (the diagram given by) F; in fact, it is as a collection of arrows $\{\alpha(j) : c \to F(j) — j \in Ob(\mathcal{J})\}$ such that for any arrow $l : j_1 \to j_2$ in $\mathcal{J}$ the triangle



commutes.

Limits and colimits II

Definition

Let $F : \mathcal{J} \to \mathcal{C}$ be a functor. A limit for F in $\mathcal{C}$ is an object c together with a cone $\alpha : \Delta(c) \to F$ which is universal among the cones from objects of $\mathcal{C}$ to F i.e. such that for every cone $\beta : \Delta(c') \to F$ there exists a unique map $g : c' \to c$ in $\mathcal{C}$ such that $\beta(j) = \alpha(j) \circ g$ for each object j of $\mathcal{J}$.
A colimit is the dual notion to that of limit.

Of course, by the universal property, if the limit of a functor exists then it is unique up to isomorphism.

Definition

Let $F : \mathcal{J} \to \mathcal{C}$ be a functor and $\alpha : \Delta(c) \to F$ be a limit for F in $\mathcal{C}$. We say that a functor $G : \mathcal{C} \to \mathcal{D}$ preserves the limit of F if the cone in $\mathcal{D}$ from $F(c)$ to the composite functor $G \circ F$ obtained by applying G to α is universal i.e. gives a limit for the functor $G \circ F$.

Special kinds of limits

Examples

- A limit of the unique functor from the empty category to a category $\mathcal{C}$ can be identified with a terminal object, that is with an object 1 of $\mathcal{C}$ such that for any object a of $\mathcal{C}$ there exists exactly one arrow $a \to 1$ (in $\mathbf{Set}$, terminal objects are exactly the singleton sets).
- When $\mathcal{J}$ is a discrete category, a limit for a functor $\mathcal{J} \to \mathcal{C}$ is called a product in $\mathcal{C}$ (in $\mathbf{Set}$, this notion specializes to that of cartesian product).
- When $\mathcal{J}$ is the category having three objects j, k, m and two non-identity arrows $j \to m$ and $k \to m$, a limit for a functor $\mathcal{J} \to \mathcal{C}$ is called a pullback (in $\mathbf{Set}$, this notion specializes to that of fibred product).
- When $\mathcal{J}$ is the category having two objects i, j and two non-identity arrows $i \to j$, a limit for a functor $\mathcal{J} \to \mathcal{C}$ is called an equalizer.

Limits in $\mathbf{Set}$

Theorem

The limit of a diagram $H : \mathcal{I} \to \mathbf{Set}$ *is the equalizer* $e : \lim(H) \rightarrowtail \prod_{i \in I} H(i)$ *of the pair of arrows*

$$a, b : \prod_{i \in I} H(i) \to \prod_{u: i \to j \text{ in } \mathcal{I}} H(j)$$

defined by the conditions

$$\pi_u \circ a = \pi'_j$$

and

$$\pi_u \circ b = H(u) \circ \pi'_i$$

for every arrow $u : i \to j$ *in* $\mathcal{I}$*, where* $\pi_u : \prod_{u: i \to j \text{ in } \mathcal{I}} H(i) \to H(\mathrm{cod}(u))$ *and* $\pi'_i : \prod_{i \in I} H(i) \to H(i)$ *are the canonical projection arrows.*

N.B. The equalizer of a pair of arrows $f, g : A \to B$ in $\mathbf{Set}$ is the subset of A consisting of all the elements a such that $f(a) = g(a)$.

Colimits in Set

Theorem

Dually, the colimit of a diagram $H : \mathcal{I} \to \mathbf{Set}$ *is the coequalizer* $q : \coprod_{i \in I} H(i) \to \mathrm{colim}(H)$ *of the pair of arrows*

$$a, b : \coprod_{u:i \to j \text{ in } \mathcal{I}} H(i) \to \coprod_{i \in \mathcal{I}} H(i)$$

defined by the conditions

$$a \circ \lambda_u = \kappa_i$$

and

$$b \circ \lambda_u = \kappa_j \circ H(u)$$

for every arrow $u : i \to j$ *in* $\mathcal{I}$*, where* $\lambda_u : H(\mathrm{dom}(u)) \to \coprod_{u:i \to j \text{ in } \mathcal{I}} H(i)$ *and* $\kappa_i : H(i) \to \coprod_{i \in I} H(i)$ *are the canonical coproduct arrows.*

N.B. The coequalizer of a pair of arrows $f, g : A \to B$ in $\mathbf{Set}$ is the quotient of B by the smallest equivalence relation containing all the pairs of the form $(f(a), g(a))$ for $a \in A$.

Limits and colimits in functor categories

Theorem

(Co)Limits in functor categories $[\mathcal{C}, \mathcal{D}]$ are computed pointwise.

More precisely, for any diagram $H : \mathcal{I} \to [\mathcal{C}, \mathcal{D}]$, the (co)limits of the functors $H_c : \mathcal{I} \to \mathcal{D}$ (given by $H_c(i) = H(i)(c)$) for $c \in \mathcal{C}$, if they exist, yield together a functor $\mathcal{C} \to \mathcal{D}$ which is the (co)limit of H.

In fact, the evaluation functors $\mathrm{ev}_c : [\mathcal{C}, \mathcal{D}] \to \mathcal{D}$ (for $d \in \mathcal{D}$) preserve and jointly reflect (co)limits.

N.B. We shall notably apply this to categories $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on a small category $\mathcal{C}$.

Adjoint functors: definition

"Adjoint functors arise everywhere"
(S. Mac Lane, Categories for the working mathematician)

Adjunction is a very special relationship between two functors, of great importance for its ubiquity in Mathematics.

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An adjunction between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors

$$F : \mathcal{C} \to \mathcal{D} \text{ and } G : \mathcal{D} \to \mathcal{C}$$

together with a natural isomorphism between the functors $\mathrm{Hom}_{\mathcal{D}}(F(-), -), \mathrm{Hom}_{\mathcal{C}}(-, G(-)) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ i.e. a family of bijections

$$\mathrm{Hom}_{\mathcal{D}}(F(a), b) \cong \mathrm{Hom}_{\mathcal{C}}(a, G(b))$$

natural in $a \in Ob(\mathcal{C})$ and $b \in Ob(\mathcal{D})$ (notice that naturality can be checked separately in each component).

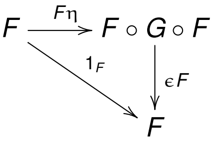
The functor F is said to be **left adjoint** to G, while G is said to be **right adjoint** to F, and we write $F \dashv G$.

Adjoint functors: unit and counit

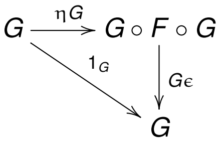
A pair of adjoint functors $F \dashv G$ induces two natural transformations $\eta : 1_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to 1_{\mathcal{D}}$, respectively called the unit and the counit, which are defined as follows:

- For any $c \in \mathcal{C}$, $\eta(c) : c \to G(F(c))$ is the arrow corresponding to the identity arrow on $F(c)$ under the adjunction;
- For any $d \in \mathcal{D}$, $\epsilon(d) : F(G(d)) \to d$ is the arrow corresponding to the identity arrow on $G(d)$ under the adjunction.

The unit and counit satisfy the triangular identities, that is the following triangles commute:



and



In fact, an adjunction $F \dashv G$ can be alternatively presented as a pair of natural transformations $\eta : 1_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to 1_{\mathcal{D}}$ satisfying these identities.

Adjoint functors: examples and properties

Examples

- Free constructions and forgetful functors
- Limits and diagonal functors
- Diagonal functors and colimits
- Hom-tensor adjunctions in Algebra
- Stone-Čech compactification in Topology
- Quantifiers as adjoints in Logic

Useful properties of adjoint functors include:

- Uniqueness: The left (resp. right) ajoint of a given functor, if it exists, is unique (up to natural isomorphism).
- Continuity: Any functor which has a left (resp. right) adjoint preserves limits (resp. colimits).

Exponentials and cartesian closed categories

For any two sets X and Y, we can always form the set Y^X of the functions $X \to Y$. This set enjoys the following (universal) property in the category $\mathbf{Set}$ of sets: the familiar bijection

$$\mathrm{Hom}_{\mathbf{Set}}(Z, Y^X) \cong \mathrm{Hom}_{\mathbf{Set}}(Z \times X, Y)$$

is natural in both Y and Z and hence it gives rise to an adjunction between the functor $- \times X : \mathbf{Set} \to \mathbf{Set}$ (left adjoint) and the functor $(-)^X : \mathbf{Set} \to \mathbf{Set}$ (right adjoint).
Expressing this property in categorical language, we arrive at the following notion of exponential for an object X of a category $\mathcal{C}$ with binary products: an exponential for X is a functor $(-)^X : \mathcal{C} \to \mathcal{C}$ which is right adjoint to the product functor $X \times - : \mathcal{C} \to \mathcal{C}$.

The counit of the adjunction yields an 'evaluation arrow' $X \times Y^X \to Y$.

Definition

A category $\mathcal{C}$ is said to be cartesian closed if it has finite products and exponentials for each object $c \in Ob(\mathcal{C})$.

For example, both the category $\mathbf{Set}$ of sets and the (large) category $\mathbf{Cat}$ of small categories are cartesian closed.

Exponentials in presheaf categories

Theorem

Every presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed. The finite products are computed pointwise, while the exponentials are defined as follows: for any $P, Q : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, we have

$$Q^P(c) = \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}(y_{\mathcal{C}}(c), Q^P) \cong \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}(y_{\mathcal{C}}(c) \times P \to Q)$$

for any $c \in \mathcal{C}$.

For any $R : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, the bijective correspondence between the natural transformations $\alpha : R \to \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}(y_{\mathcal{C}}(-) \times P \to Q)$ and the natural transformations $\beta : R \times P \to Q$ is defined by:

- For any $c \in \mathcal{C}$ and $x \in R(c)$, the natural transformation $\alpha(c)(x) : \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}(y_{\mathcal{C}}(d) \times P \to Q)$ is given by $\alpha(c)(x)(d)(f, y) = \beta(d)(Rf(x), y)$;
- For any $c \in \mathcal{C}$, we have $\beta(c)(x, z) = \alpha(c)(x)(c)(1_c, z)$.

For further reading

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