Topos Theory

Lectures 7-8: Basic properties of categories of sheaves

Olivia Caramello

The notion of Grothendieck topos I

Definition

- A site is a pair $(\mathscr{C}, J)$ where $\mathscr{C}$ is a small category and J is a Grothendieck topology on $\mathscr{C}$.
- A presheaf on a (small) category $\mathscr{C}$ is a functor $P : \mathscr{C}^{\text{op}} \to \mathbf{Set}$.
- Let $P : \mathscr{C}^{\text{op}} \to \mathbf{Set}$ be a presheaf on $\mathscr{C}$ and S be a sieve on an object c of $\mathscr{C}$. A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

$$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d\,.$$

An amalgamation for such a family is a single element $x \in P(c)$ such that

$$P(f)(x) = x_f \quad \text{for all } f \text{ in } S\,.$$

The notion of Grothendieck topos II

- Given a site $(\mathscr{C}, J)$, a presheaf on $\mathscr{C}$ is a J-sheaf if every matching family for any J-covering sieve on any object of $\mathscr{C}$ has a unique amalgamation.
- The category $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on the site $(\mathscr{C}, J)$ is the full subcategory of $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.
- A Grothendieck topos is any category of sheaves on a site.

Examples

- For any (small) category $\mathscr{C}$, $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ is the category of sheaves $\mathbf{Sh}(\mathscr{C}, T)$ where T is the trivial topology on $\mathscr{C}$.
- For any topological space X, $\mathbf{Sh}(\mathscr{O}(X), J_{\mathscr{O}(X)})$ is equivalent to the usual category $\mathbf{Sh}(X)$ of sheaves on X.

Basic properties of Grothendieck toposes

Theorem

Let $(\mathscr{C}, J)$ be a site. Then

- *the inclusion* $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$ *has a left adjoint* $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ *(called the associated sheaf functor), which preserves finite limits.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has all (small) limits, which are preserved by the inclusion functor* $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$*; in particular, limits are computed pointwise and the terminal object* $1_{\mathbf{Sh}(\mathscr{C}, J)}$ *of* $\mathbf{Sh}(\mathscr{C}, J)$ *is the functor* $T : \mathscr{C}^{op} \to \mathbf{Set}$ *sending each object* $c \in Ob(\mathscr{C})$ *to the singleton* $\{*\}$.
- *The associated sheaf functor* $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ *preserves colimits; in particular,* $\mathbf{Sh}(\mathscr{C}, J)$ *has all (small) colimits.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has exponentials, which are constructed as in the topos* $[\mathscr{C}^{op}, \mathbf{Set}]$.
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has a subobject classifier.*

Corollary

Every Grothendieck topos is an elementary topos.

Basic properties of Grothendieck toposes

Theorem

Let $(\mathscr{C}, J)$ be a site. Then

- *the inclusion* $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$ *has a left adjoint* $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ *(called the associated sheaf functor), which preserves finite limits.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has all (small) limits, which are preserved by the inclusion functor* $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$*; in particular, limits are computed pointwise and the terminal object* $1_{\mathbf{Sh}(\mathscr{C}, J)}$ *of* $\mathbf{Sh}(\mathscr{C}, J)$ *is the functor* $T : \mathscr{C}^{op} \to \mathbf{Set}$ *sending each object* $c \in Ob(\mathscr{C})$ *to the singleton* $\{*\}$.
- *The associated sheaf functor* $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ *preserves colimits; in particular,* $\mathbf{Sh}(\mathscr{C}, J)$ *has all (small) colimits.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has exponentials, which are constructed as in the topos* $[\mathscr{C}^{op}, \mathbf{Set}]$.
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has a subobject classifier.*

Corollary

Every Grothendieck topos is an elementary topos.

Basic properties of Grothendieck toposes

Theorem

Let $(\mathscr{C}, J)$ be a site. Then

- *the inclusion* $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$ *has a left adjoint* $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ *(called the associated sheaf functor), which preserves finite limits.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has all (small) limits, which are preserved by the inclusion functor* $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$*; in particular, limits are computed pointwise and the terminal object* $1_{\mathbf{Sh}(\mathscr{C}, J)}$ *of* $\mathbf{Sh}(\mathscr{C}, J)$ *is the functor* $T : \mathscr{C}^{op} \to \mathbf{Set}$ *sending each object* $c \in Ob(\mathscr{C})$ *to the singleton* $\{*\}$.
- *The associated sheaf functor* $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ *preserves colimits; in particular,* $\mathbf{Sh}(\mathscr{C}, J)$ *has all (small) colimits.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has exponentials, which are constructed as in the topos* $[\mathscr{C}^{op}, \mathbf{Set}]$.
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has a subobject classifier.*

Corollary

Every Grothendieck topos is an elementary topos.

Basic properties of Grothendieck toposes

Theorem

Let $(\mathscr{C}, J)$ be a site. Then

- *the inclusion* $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$ *has a left adjoint* $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ *(called the associated sheaf functor), which preserves finite limits.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has all (small) limits, which are preserved by the inclusion functor* $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$*; in particular, limits are computed pointwise and the terminal object* $1_{\mathbf{Sh}(\mathscr{C},J)}$ *of* $\mathbf{Sh}(\mathscr{C}, J)$ *is the functor* $T : \mathscr{C}^{op} \to \mathbf{Set}$ *sending each object* $c \in Ob(\mathscr{C})$ *to the singleton* $\{*\}$.
- *The associated sheaf functor* $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ *preserves colimits; in particular,* $\mathbf{Sh}(\mathscr{C}, J)$ *has all (small) colimits.*
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has exponentials, which are constructed as in the topos* $[\mathscr{C}^{op}, \mathbf{Set}]$.
- *The category* $\mathbf{Sh}(\mathscr{C}, J)$ *has a subobject classifier.*

Corollary

Every Grothendieck topos is an elementary topos.

Basic properties of Grothendieck toposes

Theorem

Let $(\mathscr{C}, J)$ be a site. Then

- *the inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$ has a left adjoint $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ (called the associated sheaf functor), which preserves finite limits.*
- *The category $\mathbf{Sh}(\mathscr{C}, J)$ has all (small) limits, which are preserved by the inclusion functor $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$; in particular, limits are computed pointwise and the terminal object $1_{\mathbf{Sh}(\mathscr{C},J)}$ of $\mathbf{Sh}(\mathscr{C}, J)$ is the functor $T : \mathscr{C}^{op} \to \mathbf{Set}$ sending each object $c \in Ob(\mathscr{C})$ to the singleton $\{*\}$.*
- *The associated sheaf functor $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ preserves colimits; in particular, $\mathbf{Sh}(\mathscr{C}, J)$ has all (small) colimits.*
- *The category $\mathbf{Sh}(\mathscr{C}, J)$ has exponentials, which are constructed as in the topos $[\mathscr{C}^{op}, \mathbf{Set}]$.*
- *The category $\mathbf{Sh}(\mathscr{C}, J)$ has a subobject classifier.*

Corollary

Every Grothendieck topos is an elementary topos.

Basic properties of Grothendieck toposes

Theorem

Let $(\mathscr{C}, J)$ be a site. Then

- *the inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$ has a left adjoint $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ (called the associated sheaf functor), which preserves finite limits.*
- *The category $\mathbf{Sh}(\mathscr{C}, J)$ has all (small) limits, which are preserved by the inclusion functor $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$; in particular, limits are computed pointwise and the terminal object $1_{\mathbf{Sh}(\mathscr{C}, J)}$ of $\mathbf{Sh}(\mathscr{C}, J)$ is the functor $T : \mathscr{C}^{op} \to \mathbf{Set}$ sending each object $c \in Ob(\mathscr{C})$ to the singleton $\{*\}$.*
- *The associated sheaf functor $a : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}, J)$ preserves colimits; in particular, $\mathbf{Sh}(\mathscr{C}, J)$ has all (small) colimits.*
- *The category $\mathbf{Sh}(\mathscr{C}, J)$ has exponentials, which are constructed as in the topos $[\mathscr{C}^{op}, \mathbf{Set}]$.*
- *The category $\mathbf{Sh}(\mathscr{C}, J)$ has a subobject classifier.*

Corollary

Every Grothendieck topos is an elementary topos.

The subobject classifier in $\mathbf{Sh}(\mathscr{C}, J)$

- Given a site $(\mathscr{C}, J)$ and a sieve S in $\mathscr{C}$ on an object c, we say that S is J-closed if for any arrow $f : d \to c$, $f^*(S) \in J(d)$ implies that $f \in S$.
- Let us define $\Omega_J : \mathscr{C}^{\text{op}} \to \mathbf{Set}$ by:
  $\Omega_J(c) = \{R \mid R$ is a J-closed sieve on $c\}$ (for an object $c \in \mathscr{C}$),
  $\Omega_J(f) = f^*(-)$ (for an arrow f in $\mathscr{C}$),
  where $f^*(-)$ denotes the operation of pullback of sieves in $\mathscr{C}$ along f.
  Then the arrow $true : 1_{\mathbf{Sh}(\mathscr{C},J)} \to \Omega_J$ defined by:
  $true(*)(c) = M_c$ for each $c \in Ob(\mathscr{C})$
  is a subobject classifier for $\mathbf{Sh}(\mathscr{C}, J)$.
- The classifying arrow $\chi_{A'} : A \to \Omega_J$ of a subobject $A' \subseteq A$ in $\mathbf{Sh}(\mathscr{C}, J)$ is given by:

$$\chi_{A'}(c)(x) = \{f : d \to c \mid A(f)(x) \in A'(d)\}$$

  where $c \in Ob(\mathscr{C})$ and $x \in A(c)$.

Subobjects in a Grothendieck topos

Theorem

- *For any Grothendieck topos $\mathscr{E}$ and any object a of $\mathscr{E}$, the poset $\mathrm{Sub}_{\mathscr{E}}(a)$ of all subobjects of a in $\mathscr{E}$ is a complete Heyting algebra.*
- *For any arrow $f : a \to b$ in a Grothendieck topos $\mathscr{E}$, the pullback functor $f^* : \mathrm{Sub}_{\mathscr{E}}(b) \to \mathrm{Sub}_{\mathscr{E}}(a)$ has both a left adjoint $\exists_f : \mathrm{Sub}_{\mathscr{E}}(a) \to \mathrm{Sub}_{\mathscr{E}}(b)$ and a right adjoint $\forall_f : \mathrm{Sub}_{\mathscr{E}}(a) \to \mathrm{Sub}_{\mathscr{E}}(b)$.*

For further reading

S. Mac Lane and I. Moerdijk.
Sheaves in geometry and logic: a first introduction to topos theory
Springer-Verlag, 1992.