Topos Theory

Lectures 5-6: Sheaves on a site

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Presheaves on a topological space

Definition

Let X be a topological space. A presheaf $\mathscr{F}$ on X consists of the data:

(i) for every open subset U of X, a set $\mathscr{F}(U)$ and

(ii) for every inclusion $V \subseteq U$ of open subsets of X, a function $\rho_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ subject to the conditions
- $\rho_{U,U}$ is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$ and
- if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

The maps $\rho_{U,V}$ are called restriction maps, and we sometimes write $s|_V$ instead of $\rho_{U,V}(s)$, if $s \in \mathscr{F}(U)$.

A morphism of presheaves $\mathscr{F} \to \mathscr{G}$ on a topological space X is a collection of maps $\mathscr{F}(U) \to \mathscr{G}(U)$ which is compatible with respect to restriction maps.

Remark

*Categorically, a presheaf $\mathscr{F}$ on X is a functor $\mathscr{F} : \mathscr{O}(X)^{op} \to$ **Set**, where $\mathscr{O}(X)$ is the poset category corresponding to the lattice of open sets of the topological space X (with respect to the inclusion relation).*

A morphism of presheaves is then just a natural transformation between the corresponding functors.

So we have a category $[\mathscr{O}(X)^{op}, \textbf{Set}]$ of presheaves on X.

Sheaves on a topological space

Definition

A sheaf $\mathscr{F}$ on a topological space X is a presheaf on X satisfying the additional conditions

(i) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if $s, t \in \mathscr{F}(U)$ are elements such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$;

(ii) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each $i, j \in I$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathscr{F}(U)$ (necessarily unique by (i)) such that $s|_{V_i} = s_i$ for each i.

A morphism of sheaves is defined as a morphism of the underlying presheaves.

Remark

Categorically, a sheaf is a functor $\mathscr{O}(X)^{op} \to \mathbf{Set}$ which satisfies certain conditions expressible in categorical language entirely in terms of the poset category $\mathscr{O}(X)$ and of the usual notion of covering on it. The category $\mathbf{Sh}(X)$ of sheaves on a topological space X is thus a full subcategory of the category $[\mathscr{O}(X)^{op}, \mathbf{Set}]$ of presheaves on X.

This paves the way for a significant categorical generalization of the notion of sheaf, leading to the notion of Grothendieck topos.

The associated sheaf functor

Theorem

Given a presheaf $\mathscr{F}$, there is a sheaf $a(\mathscr{F})$ and a morphism $\theta : \mathscr{F} \to a(\mathscr{F})$, with the property that for any sheaf $\mathscr{G}$, and any morphism $\phi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism $\psi : a(\mathscr{F}) \to \mathscr{G}$ such that $\psi \circ \theta = \phi$.

The sheaf $a(\mathscr{F})$ is called the sheaf associated to the presheaf $\mathscr{F}$.

Remark

Categorically, this means that the inclusion functor $i : \mathbf{Sh}(X) \to [\mathscr{O}(X)^{op}, \mathbf{Set}]$ has a left adjoint $a : [\mathscr{O}(X)^{op}, \mathbf{Set}] \to \mathbf{Sh}(X)$.

The left adjoint $a : [\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(X)$ is called the associated sheaf functor.

Examples of sheaves

Examples

- the sheaf of continuous real-valued functions on any topological space
- the sheaf of regular functions on a variety
- the sheaf of differentiable functions on a differentiable manifold
- the sheaf of holomorphic functions on a complex manifold

In each of the above examples, the restriction maps of the sheaf are the usual set-theoretic restrictions of functions to a subset.

Remark

Sheaves arising in Mathematics are often equipped with more structure than the mere set-theoretic one; for example, one may wish to consider sheaves of modules (resp. rings, abelian groups, ...) on a topological space X.
The natural categorical way of looking at these notions is to consider them as models of certain (geometric) theories in a category $\mathbf{Sh}(X)$ of sheaves of sets.

The sheaf of cross-sections of a bundle

Definition

- For any topological space X, a continuous map $p : Y \to X$ is called a bundle over X. In fact, the category of bundles is the slice category $\mathbf{Top}/X$.
- Given an open subset U of X, a cross-section over U of a bundle $p : Y \to X$ is a continuous map $s : U \to Y$ such that the composite $p \circ s$ is the inclusion $i : U \hookrightarrow X$. Let

$$\Gamma_p U = \{ s \mid s : U \to Y \text{ and } p \circ s = i : U \to X \}$$

denote the set of all such cross-sections over U.
- If $V \subseteq U$, one has a restriction operation $\Gamma_p U \to \Gamma_p V$. The functor $\Gamma_p : \mathcal{O}(X)^{\text{op}} \to \mathbf{Set}$ obtained in this way is a sheaf and is called the sheaf of cross-sections of the bundle p.

The bundle of germs of a presheaf

Definition

- Given any presheaf $\mathscr{F} : \mathscr{O}(X)^{\text{op}} \to \textbf{Set}$ on a space X, a point $x \in X$, two open neighbourhoods U and V of x, and two elements $s \in \mathscr{F}(U), t \in \mathscr{F}(V)$. We say that s and t have the same germ at x when there is some open set $W \subseteq U \cap V$ with $x \in W$ and $s|_W = t|_W$. This relation 'to have the same germ at x' is an equivalence relation, and the equivalence class of any one such s is called the germ of s at x, in symbols $germ(s)$.
- Let
$$\mathscr{F}_x = \{germ(s) \mid s \in \mathscr{F}(U),\ x \in U \text{ open in } X\}$$
be the set of all germs at x.
- Let Γ_p be the disjoint union of the $\mathscr{F}_x$
$$\Lambda_p = \{\langle x, r\rangle \mid x \in X, r \in \mathscr{F}_x\}$$
topologized by taking as a base of open sets all the image sets $\tilde{s}(U)$, where $\tilde{s} : U \to \Lambda_p$ is the map induced by an element $s \in \mathscr{F}(U)$ by taking its germs at points in U.
- With respect to this topology, the natural projection map $\Lambda_p \to X$ becomes a continuous map, called the bundle of germs of the presheaf $\mathscr{F}$.

Sheaves as étale bundles

Definition

- A bundle $p : E \to X$ is said to be étale (over X) when p is a local homeomorphism in the following sense: for each $e \in E$ there is an open set V, with $e \in V$, such that $p(V)$ is open in X and $p|_V$ is a homeomorphism $V \to p(V)$.
- The full subcategory of $\mathbf{Top}/X$ on the étale bundles is denoted by $\mathbf{Etale}(X)$.

Theorem

- *For any topological space X, there is a pair of adjoint functors*

$$\Gamma : \mathbf{Top}/X \to [\mathscr{O}(X)^{op}, \mathbf{Set}], \quad \Lambda : [\mathscr{O}(X)^{op}, \mathbf{Set}] \to \mathbf{Top}/X,$$

where Γ assigns to each bundle $p : Y \to X$ the sheaf of cross-sections of p, while its left adjoint Λ assigns to each presheaf $\mathscr{F}$ the bundle of germs of $\mathscr{F}$.
- *The adjunction restricts to an equivalence of categories*

$$\mathbf{Sh}(X) \simeq \mathbf{Etale}(X) .$$

Grothendieck topologies I

In order to 'categorify' the notion of sheaf of a topological space, the first step is to introduce an abstract notion of covering on a category.

Definition

- Given a category $\mathscr{C}$ and an object $c \in Ob(\mathscr{C})$, a sieve S in $\mathscr{C}$ on c is a collection of arrows in $\mathscr{C}$ with codomain c such that

$$f \in S \Rightarrow f \circ g \in S$$

  whenever this composition makes sense.
- We say that a sieve S is generated by a given family of arrows (with common codomain) if it is the smallest sieve which contains all the arrows of the family.

If S is a sieve on c and $h : d \to c$ is any arrow to c, then

$$h^*(S) := \{g \mid cod(g) = d,\ h \circ g \in S\}$$

is a sieve on d.

Grothendieck topologies II

Definition

A Grothendieck topology on a small category $\mathscr{C}$ is a function J which assigns to each object c of $\mathscr{C}$ a collection $J(c)$ of sieves on c in such a way that

(i) (maximality axiom) the maximal sieve $M_c = \{f \mid cod(f) = c\}$ is in $J(c)$;

(ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;

(iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

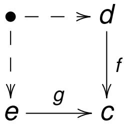
The sieves S which belong to $J(c)$ for some object c of $\mathscr{C}$ are said to be J-covering.

Examples of Grothendieck topologies I

- For any (small) category $\mathscr{C}$, the trivial topology on $\mathscr{C}$ is the Grothendieck topology in which the only sieve covering an object c is the maximal sieve M_c.
- The dense topology D on a category $\mathscr{C}$ is defined by: for a sieve S,

$$S \in D(c) \quad \text{if and only if} \quad \text{for any } f : d \to c \text{ there exists } g : e \to d \text{ such that } f \circ g \in S\,.$$

If $\mathscr{C}$ satisfies the right Ore condition i.e. the property that any two arrows $f : d \to c$ and $g : e \to c$ with a common codomain c can be completed to a commutative square



then the dense topology on $\mathscr{C}$ specializes to the atomic topology on $\mathscr{C}$ i.e. the topology J_{at} defined by: for a sieve S,

$$S \in J_{at}(c) \text{ if and only if } S \neq \emptyset\,.$$

Examples of Grothendieck topologies II

- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathscr{O}(X)}$ on the poset category $\mathscr{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathscr{O}(X))$,

$$S \in J_{\mathscr{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U \, .$$

- More generally, given a complete Heyting algebra H, i.e. a Heyting algebra with arbitrary joins $\bigvee$ (and meets), we can define a Grothendieck topology J_H by:

$$\{a_i \mid i \in I\} \in J_H(a) \text{ if and only if } \bigvee_{i \in I} a_i = a \, .$$

The notion of Grothendieck topos I

Definition

- A site is a pair $(\mathscr{C}, J)$ where $\mathscr{C}$ is a small category and J is a Grothendieck topology on $\mathscr{C}$.
- A presheaf on a (small) category $\mathscr{C}$ is a functor $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$.
- Let $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf on $\mathscr{C}$ and S be a sieve on an object c of $\mathscr{C}$. A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

$$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d \, .$$

An amalgamation for such a family is a single element $x \in P(c)$ such that

$$P(f)(x) = x_f \quad \text{for all } f \text{ in } S \, .$$

The notion of Grothendieck topos II

- Given a site $(\mathscr{C}, J)$, a presheaf on $\mathscr{C}$ is a J-sheaf if every matching family for any J-covering sieve on any object of $\mathscr{C}$ has a unique amalgamation.
- The category $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on the site $(\mathscr{C}, J)$ is the full subcategory of $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.
- A Grothendieck topos is any category of sheaves on a site.

Examples

- For any (small) category $\mathscr{C}$, $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ is the category of sheaves $\mathbf{Sh}(\mathscr{C}, T)$ where T is the trivial topology on $\mathscr{C}$.
- For any topological space X, $\mathbf{Sh}(\mathscr{O}(X), J_{\mathscr{O}(X)})$ is equivalent to the usual category $\mathbf{Sh}(X)$ of sheaves on X.

For further reading

S. Mac Lane and I. Moerdijk.
Sheaves in geometry and logic: a first introduction to topos theory
Springer-Verlag, 1992.