Topos Theory

Lectures 3-4: Categorical preliminaries II

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Functor categories

Definition

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. The functor category $[\mathscr{C}, \mathscr{D}]$ is the category having as objects the functors $\mathscr{C} \to \mathscr{D}$ and as arrows the natural transformations between them.

Examples

- If $\mathscr{C}$ is the category having two distinct objects and exactly one non-identical arrow $0 \to 1$, the functor category $[\mathscr{C}, \mathscr{D}]$ becomes the category $\mathscr{D}^{\to}$ of arrows in $\mathscr{D}$ and commutative squares between them.
- If $\mathscr{C}$ is the category corresponding to a monoid M and $\mathscr{D} = \mathbf{Set}$, then $[\mathscr{C}, \mathscr{D}]$ becomes the category M-**Set** of sets equipped with a M-action and action-preserving maps between them.
- If $\mathscr{C}$ is a discrete category on a set I and $\mathscr{D} = \mathbf{Set}$ then $[\mathscr{C}, \mathscr{D}]$ becomes the category **Bn**(I) of I-indexed collections of sets and functions between them.

Other basic constructions

Definition (Slice category)

Let $\mathscr{C}$ be a category and a be an object of $\mathscr{C}$. The slice category $\mathscr{C}/a$ of $\mathscr{C}$ on a has as objects the arrows in $\mathscr{C}$ with codomain a and as arrows the commutative triangles between them (composition and identities are the obvious ones).

Notice that the slice category $\mathbf{Set}/I$ is equivalent to the functor category $\mathbf{Bn}(I)$ introduced above.

Two monomorphisms in a category $\mathscr{C}$ with common codomain a are said to be isomorphic if they are isomorphic as objects of $\mathscr{C}/a$. An isomorphism class of monomorphisms with common codomain a is called a subobject of a.

Definition (Product category)

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. The product category $\mathscr{C} \times \mathscr{D}$ has as objects the pairs $\langle a, b\rangle$ where a is an object of $\mathscr{C}$ and b is an object of $\mathscr{D}$ and as arrows $\langle a, b\rangle \to \langle c, d\rangle$ the pairs $\langle f, g\rangle$ where $f : a \to c$ is an arrow in $\mathscr{C}$ and $g : b \to d$ is an arrow in $\mathscr{D}$ (composition and identities are defined componentwise).

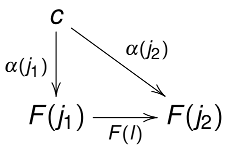
Universal properties

- It is a striking fact that one can often define mathematical objects not by means of their internal structure (that is, as in the classical spirit of set-theoretic foundations) bur rather in terms of their relations with the other objects of the mathematical environment in which one works (that is, in terms of the objects and arrows of the category in which one works), by means of so-called universal properties.
- Of course, isomorphic objects in a category are indistinguishable from the point of view of the categorical properties that they satisfy; in fact, definitions via universal property do not determine the relevant objects 'absolutely' but only up to isomorphism in the given category.

The technical embodiment of the idea of universal property is given by the notion of limit (dually, colimit) of a functor.

Limits and colimits I

Note that a functor $F : \mathscr{J} \to \mathscr{C}$ can be thought as a 'diagram in $\mathscr{C}$ of shape $\mathscr{J}$'.
For every object c of $\mathscr{C}$, there is a 'constant' functor $\Delta(c) : \mathscr{J} \to \mathscr{C}$, which sends all the objects of $\mathscr{J}$ to the object c and all the arrows in $\mathscr{J}$ to the identity arrow on c. This defines a diagonal functor $\Delta : \mathscr{C} \to [\mathscr{J}, \mathscr{C}]$. A natural transformation α from $\Delta(c)$ to a functor $F : \mathscr{J} \to \mathscr{C}$ is called a cone from c to (the diagram given by) F; in fact, it is as a collection of arrows $\{\alpha(j) : c \to F(j) \mid j \in Ob(\mathscr{J})\}$ such that for any arrow $l : j_1 \to j_2$ in $\mathscr{J}$ the triangle



commutes.

Limits and colimits II

Definition

Let $F : \mathscr{J} \to \mathscr{C}$ be a functor. A limit for F in $\mathscr{C}$ is an object c together with a cone $\alpha : \Delta(c) \to F$ which is universal among the cones from objects of $\mathscr{C}$ to F i.e. such that for every cone $\beta : \Delta(c') \to F$ there exists a unique map $g : c' \to c$ in $\mathscr{C}$ such that $\beta(j) = \alpha(j) \circ g$ for each object j of $\mathscr{J}$.
A colimit is the dual notion to that of limit.

Of course, by the universal property, if the limit of a functor exists then it is unique up to isomorphism.

Definition

Let $F : \mathscr{J} \to \mathscr{C}$ be a functor and $\alpha : \Delta(c) \to F$ be a limit for F in $\mathscr{C}$. We say that a functor $G : \mathscr{C} \to \mathscr{D}$ preserves the limit of F if the cone in $\mathscr{D}$ from $F(c)$ to the composite functor $G \circ F$ obtained by applying G to α is universal i.e. gives a limit for the functor $G \circ F$.

Special kinds of limits

Examples

- A limit of the unique functor from the empty category to a category $\mathscr{C}$ can be identified with a terminal object, that is with an object 1 of $\mathscr{C}$ such that for any object a of $\mathscr{C}$ there exists exactly one arrow $a \to 1$ (in **Set**, terminal objects are exactly the singleton sets).
- When $\mathscr{J}$ is a discrete category, a limit for a functor $\mathscr{J} \to \mathscr{C}$ is called a product in $\mathscr{C}$ (in **Set**, this notion specializes to that of cartesian product).
- When $\mathscr{J}$ is the category having three objects j, k, m and two non-identity arrows $j \to m$ and $k \to m$, a limit for a functor $\mathscr{J} \to \mathscr{C}$ is called a pullback (in **Set**, this notion specializes to that of fibred product).

Adjoint functors: definition

"Adjoint functors arise everywhere"
(S. Mac Lane, Categories for the working mathematician)

Adjunction is a very special relationship between two functors, of great importance for its ubiquity in Mathematics.

Definition

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. An adjunction between $\mathscr{C}$ and $\mathscr{D}$ is a pair of functors

$$F : \mathscr{C} \to \mathscr{D} \text{ and } G : \mathscr{D} \to \mathscr{C}$$

together with a natural isomorphism between the functors $Hom_{\mathscr{D}}(F(-), -), Hom_{\mathscr{C}}(-, G(-)) : \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Set}$ i.e. a family of bijections

$$Hom_{\mathscr{D}}(F(a), b) \cong Hom_{\mathscr{C}}(a, G(b))$$

natural in $a \in Ob(\mathscr{C})$ and $b \in Ob(\mathscr{D})$.

The functor F is said to be **left adjoint** to G, while G is said to be **right adjoint** to F, and we write $F \dashv G$.

Adjoint functors: examples and properties

Examples

- Free constructions and forgetful functors
- Limits and diagonal functors
- Diagonal functors and colimits
- Hom-tensor adjunctions in Algebra
- Stone-Čech compactification in Topology
- Quantifiers as adjoints in Logic

Useful properties of adjoint functors include:

- Uniqueness: The left (resp. right) ajoint of a given functor, if it exists, is unique (up to natural isomorphism).
- Continuity: Any functor which has a left (resp. right) adjoint preserves limits (resp. colimits).

The Yoneda Lemma

Given a category $\mathscr{C}$, we define the Yoneda embedding to be the functor $y_{\mathscr{C}} : \mathscr{C} \to [\mathscr{C}^{\text{op}}, \textbf{Set}]$ given by:

- $y(a) = Hom_{\mathscr{C}}(-, a)$, for an object $a \in Ob(\mathscr{C})$.
- $y(f) = f \circ_{\mathscr{C}} -$, for an arrow $f : a \to b$ in $\mathscr{C}$.

Theorem (Yoneda Lemma)

Let $\mathscr{C}$ be a category and $F : \mathscr{C}^{op} \to \textbf{Set}$ be a functor. Then, for any object $c \in Ob(\mathscr{C})$, we have a bijection

$$Hom_{[\mathscr{C}^{op}, \textbf{Set}]}(y_{\mathscr{C}}(c), F) \cong F(c)$$

natural in c.

Sketch of proof.

The proof essentially amounts to checking that the any natural transformation $\alpha : Hom_{\mathscr{C}}(-, c) \to F$ is uniquely determined by its value $\alpha(c)(id_c)$ at the identity on c. □

Corollary

The Yoneda embedding is full and faithful.

Exponentials and cartesian closed categories

For any two sets X and Y, we can always form the set Y^X of the functions $X \to Y$. This set enjoys the following (universal) property in the category **Set** of sets: the familiar bijection

$$Hom_{\mathbf{Set}}(Z, Y^X) \cong Hom_{\mathbf{Set}}(Z \times X, Y)$$

is natural in both Y and Z and hence it gives rise to an adjunction between the functor $- \times X : \mathbf{Set} \to \mathbf{Set}$ (left adjoint) and the functor $(-)^X : \mathbf{Set} \to \mathbf{Set}$ (right adjoint).
Expressing this property in categorical language, we arrive at the following notion of exponential for an object X of a category $\mathscr{C}$ with binary products: an exponential for X is a functor $(-)^X : \mathscr{C} \to \mathscr{C}$ which is right adjoint to the product functor $X \times - : \mathscr{C} \to \mathscr{C}$. (Note that exponentials are unique up to natural isomorphism, if they exist.)

Definition

A category $\mathscr{C}$ is said to be cartesian closed if it has finite products and exponentials for each object $c \in Ob(\mathscr{C})$.

For example, both the category **Set** of sets and the (large) category **Cat** of small categories are cartesian closed.

Heyting algebras

Definition

A Heyting algebra is a lattice H with 0 and 1 which is cartesian closed when regarded as a preorder category with products, i.e. such that for any pair of elements $x, y \in H$ there is an element $x \Rightarrow y$ satisfying the adjunction $z \leq (x \Rightarrow y)$ if and only if $z \wedge x \leq y$ (for any $z \in H$). For $x \in H$, we put $\neg x := x \Rightarrow 0$ and call it the pseudocomplement of x in H.

Remark

(i) *For any topological space X, the collection $\mathcal{O}(X)$ of open sets of X, endowed with the subset-inclusion order, is a Heyting algebra.*

(ii) *More generally, any frame (i.e. complete lattice in which the infinite distributive law holds) is a Heyting algebra.*

(iii) *Any Boolean algebra is a Heyting algebra.*

The concept of subobject classifier I

In the category **Set** of sets, subsets S of a given set X can be identified with their characteristic functions $\chi_S : X \to \{0, 1\}$; in fact, denoted by true : $\{*\} = 1_{\mathbf{Set}} \to \{0, 1\}$ the function which sends $*$ to 1, we have a pullback square

$$
\begin{CD}
S @>{!}>> \{*\} \\
@V{i}VV @VV{\text{true}}V \\
X @>>{\chi_S}> \{0,1\}
\end{CD}
$$

where $i : S \to X$ is the inclusion and $! : S \to \{*\}$ is the unique arrow in **Set** to the terminal object $1_{\mathbf{Set}} = \{*\}$.

The concept of subobject classifier II

Definition

In a category $\mathscr{C}$ with finite limits, a subobject classifier is a monomorphism $\text{true} : 1_{\mathscr{C}} \to \Omega$, such that for every monomorphism $m : a' \to a$ there is a unique arrow $\chi_m : a \to \Omega$, called the classifying arrow of m, such that we have a pullback square

$$\begin{array}{ccc} a' & \xrightarrow{\;!\;} & 1_{\mathscr{C}} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \text{true}} \\ a & \xrightarrow[\chi_m]{} & \Omega \end{array}$$

Note that, for any object A of $\mathscr{C}$, we have an arrow $\in_A : A \times \Omega^A \to \Omega$, generalizing the belonging relation $\in$ of Set Theory.

The notion of elementary topos

Definition

An elementary topos is a category with all finite limits, exponentials and a subobject classifier.

Remark

The notion of elementary topos admits a first-order axiomatization in the language of Category Theory.

We will see in the next lectures that an elementary topos can be considered as a mathematical universe in which one can perform most of the usual set-theoretic constructions, and in which one can consider models of arbitrary finitary first-order theories.

Examples of elementary toposes

Example

The following categories are all elementary toposes.

(i) **Set**.

(ii) $\mathbf{Set}^{\to}$.

(iii) Categories $\mathbf{Sh}(X)$ of sheaves on a topological space.

(iv) Categories of set-valued functors $[\mathscr{C}, \mathbf{Set}]$ (in particular, categories M-**Set** of monoid actions).

(v) Categories of sheaves on a site (this subsumes all the examples above).

For further reading

- F. Borceux.
  Handbook of categorical algebra, vol. 1.
  Cambridge University Press, 1994.
- S. Mac Lane.
  Categories for the working mathematician, Graduate Texts in Math. no 5.
  Springer-Verlag, 1971 (revised edition 1998).
- S. Mac Lane and I. Moerdijk.
  Sheaves in geometry and logic: a first introduction to topos theory
  Springer-Verlag, 1992.