Topos Theory

Lectures 21 and 22: Classifying toposes

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Toposes as mathematical universes

- Recall that every Grothendieck topos $\mathscr{E}$ is an elementary topos. Thus, given the fact that arbitrary colimits exist in $\mathscr{E}$, we can consider models of any kind of first-order (even infinitary) theory in $\mathscr{E}$. In particular, we can consider models of geometric theories in $\mathscr{E}$.
- Inverse image functors of geometric morphisms of toposes preserve finite limits (by definition) and arbitrary colimits (having a right adjoint); in particular, they are geometric functors and hence they preserve the interpretation of (arbitrary) geometric formulae. In general, they are *not* Heyting functors, which explains why the next definition only makes sense for geometric theories.

The notion of classifying topos

Definition

Let $\mathbb{T}$ be a geometric theory over a given signature. A classifying topos of $\mathbb{T}$ is a Grothendieck topos $\textbf{Set}[\mathbb{T}]$ such that for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\textbf{Geom}(\mathscr{E}, \textbf{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

natural in $\mathscr{E}$.

Naturality means that for any geometric morphism $f : \mathscr{E} \to \mathscr{F}$, we have a commutative square

$$\begin{array}{ccc} \textbf{Geom}(\mathscr{F}, \textbf{Set}[\mathbb{T}]) & \xrightarrow{\ \simeq\ } & \mathbb{T}\text{-mod}(\mathscr{F}) \\ \big\downarrow {\scriptstyle -\circ f} & & \big\downarrow {\scriptstyle \mathbb{T}\textbf{-mod}(f^*)} \\ \textbf{Geom}(\mathscr{E}, \textbf{Set}[\mathbb{T}]) & \xrightarrow{\ \simeq\ } & \mathbb{T}\text{-mod}(\mathscr{E}) \end{array}$$

Theorem

Every geometric theory (over a given signature) has a classifying topos.

Representability of the $\mathbb{T}$-model functor

Remark

- *The classifying topos of a geometric theory $\mathbb{T}$ can be seen as a representing object for the (pseudo-)functor*

$$\mathbb{T}\text{-}mod : \mathfrak{BTop}^{op} \to \textbf{Cat}$$

which assigns

  - *to a topos $\mathscr{E}$ the category $\mathbb{T}$-**mod**$(\mathscr{E})$ of models of $\mathbb{T}$ in $\mathscr{E}$ and*
  - *to a geometric morphism $f : \mathscr{E} \to \mathscr{F}$ the functor $\mathbb{T}$-**mod**$(f^*) : \mathbb{T}$-**mod**$(\mathscr{F}) \to \mathbb{T}$-**mod**$(\mathscr{E})$ sending a model $M \in \mathbb{T}$-**mod**$(\mathscr{F})$ to its image $f^*(M)$ under the functor f^*.*

- *In particular, classifying toposes are unique up to categorical equivalence.*

Universal models

Definition

Let $\mathbb{T}$ be a geometric theory. A universal model of a geometric theory $\mathbb{T}$ is a model $U_{\mathbb{T}}$ of $\mathbb{T}$ in a Grothendieck topos $\mathscr{G}$ such that for any $\mathbb{T}$-model M in a Grothendieck topos $\mathscr{F}$ there exists a unique (up to isomorphism) geometric morphism $f_M : \mathscr{F} \to \mathscr{G}$ such that $f_M^*(U_{\mathbb{T}}) \cong M$.

Remark

- *By the (2-dimensional) Yoneda Lemma, if a topos $\mathscr{G}$ contains a universal model of a geometric theory $\mathbb{T}$ then $\mathscr{G}$ satisfies the universal property of the classifying topos of $\mathbb{T}$. Conversely, if a topos $\mathscr{E}$ classifies a geometric theory $\mathbb{T}$ then $\mathscr{E}$ contains a universal model of $\mathbb{T}$.*
- *In particular classifying toposes, and hence universal models, are unique up to equivalence. In fact, if M and N are universal models of a geometric theory $\mathbb{T}$ lying respectively in toposes $\mathscr{F}$ and $\mathscr{G}$ then there exists a unique (up to isomorphism) geometric equivalence between $\mathscr{F}$ and $\mathscr{G}$ such that its inverse image functors send M and N to each other (up to isomorphism).*

The Morleyization of a first-order theory

It is a matter of fact that most of the theories important in Mathematics have a geometric axiomatization. Anyway, if a finitary first-order theory $\mathbb{T}$ is not geometric, we can canonically construct a coherent theory over a larger signature, called the Morleyization of $\mathbb{T}$ whose models in **Set** (more generally, in any Boolean coherent category) can be identified with those of $\mathbb{T}$.

Definition

A homomorphism of **Set**-models of a first-order theory $\mathbb{T}$ is an elementary embedding if it preserves the interpretation of all first-order formulae in the signature of $\mathbb{T}$. The category of $\mathbb{T}$-models in **Set** and elementary embeddings between them will be denoted by $\mathbb{T}\textbf{-mod}_e(\textbf{Set})$.

Theorem

Let $\mathbb{T}$ be a first-order theory over a signature Σ. Then there is a signature Σ' containing Σ, and a coherent theory $\mathbb{T}'$ over Σ', called the Morleyization of $\mathbb{T}$, such that we have

$$\mathbb{T}\textbf{-mod}_e(\textbf{Set}) \simeq \mathbb{T}'\textbf{-mod}(\textbf{Set})$$

Syntactic categories I

Definition

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. The syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the 'renaming'-equivalence classes of geometric formulae-in-context $\{\vec{x} \,.\, \phi\}$ over Σ and as arrows $\{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, \psi\}$ (where the contexts $\vec{x}$ and $\vec{y}$ are disjoint) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\vec{x}, \vec{y})$ which are $\mathbb{T}$-provably functional i.e. such that the sequents

$$(\phi \vdash_{\vec{x}} (\exists y)\theta),$$
$$(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi), \text{ and}$$
$$((\theta \wedge \theta[\vec{z}/\vec{y}]) \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{y} = \vec{z}))$$

are provable in $\mathbb{T}$.

Syntactic categories II

- The composite of two arrows

$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\theta]} \{\vec{y} \,.\, \psi\} \xrightarrow{[\gamma]} \{\vec{z} \,.\, \chi\}$$

is defined as the $\mathbb{T}$-provable-equivalence class of the formula $(\exists \vec{y})\theta \wedge \gamma$.

- The identity arrow on an object $\{\vec{x} \,.\, \phi\}$ is the arrow

$$\{\vec{x} \,.\, \phi\} \xrightarrow{[\phi \wedge \vec{x'} = \vec{x}]} \{\vec{x'} \,.\, \phi[\vec{x'}/\vec{x}]\}$$

- For a regular (resp. coherent, first-order) theory $\mathbb{T}$ one can define the regular (resp. coherent, first-order) syntactic category $\mathcal{C}_{\mathbb{T}}^{\text{reg}}$ (resp. $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$, $\mathcal{C}_{\mathbb{T}}^{\text{fo}}$) of $\mathbb{T}$ by replacing the word 'geometric' with 'regular' (resp. 'coherent', 'first-order') in the definition above. If $\mathbb{T}$ is a Horn theory then one can construct the cartesian syntactic category $\mathcal{C}_{\mathbb{T}}^{\text{cart}}$ by allowing as objects and arrows of the category those formulae which can be built from atomic formulae by binary conjunction, truth and 'unique-existential' quantifications (relative to $\mathbb{T}$).

Properties of syntactic categories

Theorem

(i) *For any Horn theory* $\mathbb{T}$, $\mathscr{C}_{\mathbb{T}}^{cart}$ *is a cartesian category.*
(ii) *For any regular theory* $\mathbb{T}$, $\mathscr{C}_{\mathbb{T}}^{reg}$ *is a regular category.*
(iii) *For any coherent theory* $\mathbb{T}$, $\mathscr{C}_{\mathbb{T}}^{coh}$ *is a coherent category.*
(iv) *For any first-order theory* $\mathbb{T}$, $\mathscr{C}_{\mathbb{T}}^{fo}$ *is a Heyting category.*
(v) *For any geometric theory* $\mathbb{T}$, $\mathscr{C}_{\mathbb{T}}$ *is a geometric category.*

Conversely, any regular (resp. coherent, geometric) category is, up to categorical equivalence, the regular (resp. coherent, geometric) syntactic category of some regular (resp. coherent, geometric) theory.

Lemma

Any subobject of $\{\vec{x} \,.\, \phi\}$ *in* $\mathscr{C}_{\mathbb{T}}$ *is isomorphic to one of the form*

$$\{\vec{x'} \,.\, \psi[\vec{x'}/\vec{x}]\} \xrightarrow{[\psi \wedge \vec{x'} = \vec{x}]} \{\vec{x} \,.\, \phi\}$$

where ψ *is a formula such that the sequent* $\psi \vdash_{\vec{x}} \phi$ *is provable in* $\mathbb{T}$*. We will denote this subobject simply by* $[\psi]$.
Moreover, for two such subobjects $[\psi]$ *and* $[\chi]$*, we have* $[\psi] \leq [\chi]$ *in* $\mathrm{Sub}_{\mathscr{C}_{\mathbb{T}}}(\{\vec{x} \,.\, \phi\})$ *if and only if the sequent* $\psi \vdash_{\vec{x}} \chi$ *is provable in* $\mathbb{T}$.

The universal model in $\mathscr{C}_{\mathbb{T}}$

Definition

Let $\mathbb{T}$ be a geometric theory over a signature Σ. The universal model of $\mathbb{T}$ in $\mathscr{C}_{\mathbb{T}}$ is defined as the structure $M_{\mathbb{T}}$ which assigns

- to a sort A the object $\{x^A . \top\}$ where x^A is a variable of sort A,
- to a function symbol $f : A_1 \cdots A_n \to B$ the morphism

$$\{x_1^{A_1}, \dots, x_n^{A_n} . \top\} \xrightarrow{[f(x_1^{A_1},\dots,x_n^{A_n})=y^B]} \{y^B . \top\}$$

  and
- to a relation symbol $R \rightarrowtail A_1 \cdots A_n$ the subobject

$$\{x_1^{A_1}, \dots, x_n^{A_n} . R(x_1^{A_1}, \dots, x_n^{A_n})\} \xrightarrow{[R(x_1^{A_1},\dots,x_n^{A_n})]} \{x_1^{A_1}, \dots, x_n^{A_n} . \top\}$$

Theorem

- *For any geometric formula-in-context $\{\vec{x} . \phi\}$ over Σ, the interpretation $[[\vec{x} . \phi]]_{M_{\mathbb{T}}}$ in $M_{\mathbb{T}}$ is the subobject $[\phi] : \{\vec{x} . \phi\} \rightarrowtail \{\vec{x} . \top\}$.*
- *A geometric sequent $\phi \vdash_{\vec{x}} \psi$ is satisfied in $M_{\mathbb{T}}$ if and only if it is provable in $\mathbb{T}$.*

Logical topologies I

- In a regular category, every arrow $f : a \to b$ factors uniquely through its image $Im(f) \rightarrowtail b$ as the composite $a \to Im(f) \to b$ of $Im(f) \rightarrowtail b$ with an arrow $c(f) : a \to Im(f)$; arrows of the form $c(f)$ for some f are called covers. In fact, every arrow in a regular category can be factored uniquely as a cover followed by a monomorphism, and covers are precisely the arrows g such that $Im(g) = id_{cod(g)}$.
- In a coherent (resp. geometric) category, a finite (resp. small) covering family is a family of arrows such that the union of their images is the maximal subobject.

Definition

- For a regular theory $\mathbb{T}$, the regular topology is the Grothendieck topology $J_{\mathbb{T}}^{\mathrm{reg}}$ on $\mathscr{C}_{\mathbb{T}}^{\mathrm{reg}}$ whose covering sieves are those which contain a cover.
- For a coherent theory $\mathbb{T}$, the coherent topology is the Grothendieck topology $J_{\mathbb{T}}^{\mathrm{coh}}$ on $\mathscr{C}_{\mathbb{T}}^{\mathrm{coh}}$ whose covering sieves are those which contain finite covering families.
- For a geometric theory $\mathbb{T}$, the geometric topology is the Grothendieck topology $J_{\mathbb{T}}$ on $\mathscr{C}_{\mathbb{T}}$ whose covering sieves are those which contain small covering families.

Logical topologies II

Notation: we denote by $\mathbf{Reg}(\mathscr{C}_{\mathbb{T}}^{\text{reg}}, \mathscr{D})$ (resp. $\mathbf{Coh}(\mathscr{C}_{\mathbb{T}}^{\text{coh}}, \mathscr{D})$, $\mathbf{Geom}(\mathscr{C}_{\mathbb{T}}, \mathscr{D})$) the categories of regular (resp. coherent, geometric) functors from $\mathscr{C}_{\mathbb{T}}^{\text{reg}}$ (resp. $\mathscr{C}_{\mathbb{T}}^{\text{coh}}$, $\mathscr{C}_{\mathbb{T}}$) to a regular (resp. coherent, geometric) category $\mathscr{D}$.

Fact

A cartesian functor $\mathscr{C}_{\mathbb{T}}^{reg} \to \mathscr{D}$ (resp. $\mathscr{C}_{\mathbb{T}}^{coh} \to \mathscr{D}$, $\mathscr{C}_{\mathbb{T}} \to \mathscr{D}$) is regular (resp. coherent, geometric) if and only it sends $J_{\mathbb{T}}^{reg}$-covering (resp. $J_{\mathbb{T}}^{coh}$-covering, $J_{\mathbb{T}}$-covering) sieves to covering families.

Models as functors

Theorem

(i) *For any Horn theory* $\mathbb{T}$ *and cartesian category* $\mathscr{D}$*, we have an equivalence of categories* ***Cart***$(\mathscr{C}_{\mathbb{T}}^{cart}, \mathscr{D}) \simeq \mathbb{T}$***-mod***$(\mathscr{D})$ *natural in* $\mathscr{D}$*.*

(ii) *For any regular theory* $\mathbb{T}$ *and regular category* $\mathscr{D}$*, we have an equivalence of categories* ***Reg***$(\mathscr{C}_{\mathbb{T}}^{reg}, \mathscr{D}) \simeq \mathbb{T}$***-mod***$(\mathscr{D})$ *natural in* $\mathscr{D}$*.*

(iii) *For any coherent theory* $\mathbb{T}$ *and coherent category* $\mathscr{D}$*, we have an equivalence of categories* ***Coh***$(\mathscr{C}_{\mathbb{T}}^{coh}, \mathscr{D}) \simeq \mathbb{T}$***-mod***$(\mathscr{D})$ *natural in* $\mathscr{D}$*.*

(iv) *For any geometric theory* $\mathbb{T}$ *and geometric category* $\mathscr{D}$*, we have an equivalence of categories* ***Geom***$(\mathscr{C}_{\mathbb{T}}, \mathscr{D}) \simeq \mathbb{T}$***-mod***$(\mathscr{D})$ *natural in* $\mathscr{D}$*.*

Sketch of proof.

- One half of the equivalence sends a model $M \in \mathbb{T}\text{-mod}(\mathscr{E})$ to the functor $F_M : \mathscr{C}_{\mathbb{T}} \to \mathscr{E}$ assigning to a formula $\{\vec{x} \,.\, \phi\}$ (the domain of) its interpretation $[[\phi(\vec{x})]]_M$ in M.
- The other half of the equivalence sends a functor $F : \mathscr{C}_{\mathbb{T}} \to \mathscr{D}$ to the image $F(M_{\mathbb{T}})$ of the universal model $M_{\mathbb{T}}$ under F.

Classifying toposes via syntactic sites

Corollary

- *For any Horn theory* $\mathbb{T}$*, the topos* $[(\mathscr{C}_{\mathbb{T}}^{cart})^{op}, \mathbf{Set}]$ *classifies* $\mathbb{T}$*.*
- *For any regular theory* $\mathbb{T}$*, the topos* $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}^{reg}, J_{\mathbb{T}}^{reg})$ *classifies* $\mathbb{T}$*.*
- *For any coherent theory* $\mathbb{T}$*, the topos* $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}^{coh}, J_{\mathbb{T}}^{coh})$ *classifies* $\mathbb{T}$*.*
- *For any geometric theory* $\mathbb{T}$*, the topos* $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ *classifies* $\mathbb{T}$*.*

The duality theorem I

Definition

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. A quotient of $\mathbb{T}$ is a geometric theory $\mathbb{T}'$ over Σ such that every axiom of $\mathbb{T}$ is provable in $\mathbb{T}'$.
- Let $\mathbb{T}$ and $\mathbb{T}'$ be geometric theories over a signature Σ. We say that $\mathbb{T}$ and $\mathbb{T}'$ are syntactically equivalent, and we write $\mathbb{T} \equiv_s \mathbb{T}'$, if for every geometric sequent σ over Σ, σ is provable in $\mathbb{T}$ if and only if σ is provable in $\mathbb{T}'$.

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the $\equiv_s$-equivalence classes of quotients of $\mathbb{T}$ and the subtoposes of the classifying topos **Set**$[\mathbb{T}]$ *of $\mathbb{T}$.*

The duality theorem II

If $i_J : \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J) \hookrightarrow \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the subtopos of $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ corresponding to a quotient $\mathbb{T}'$ of $\mathbb{T}$ via the duality theorem, we have a commutative (up to natural isomorphism) diagram in **Cat** (where i is the obvious inclusion)

$$\begin{array}{ccc} \mathbb{T}'\text{-mod}(\mathscr{E}) & \xrightarrow{\simeq} & \mathbf{Geom}(\mathscr{E}, \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J)) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_J \circ -} \\ \mathbb{T}\text{-mod}(\mathscr{E}) & \xrightarrow{\simeq} & \mathbf{Geom}(\mathscr{E}, \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})) \end{array}$$

naturally in $\mathscr{E} \in \mathfrak{BTop}$.

A simple example

Suppose to have a duality between two geometric theories $\mathbb{T}$ and $\mathbb{S}$.

Question: If $\mathbb{T}'$ is a quotient of $\mathbb{T}$, is there a quotient $\mathbb{S}'$ of $\mathbb{S}$ such that the given duality restricts to a duality between $\mathbb{T}'$ and $\mathbb{S}'$?

The duality theorem gives a straight positive answer to this question. In fact, both quotients of $\mathbb{T}$ and quotients of $\mathbb{S}$ correspond bijectively with subtoposes of the classifying topos $\mathbf{Set}[\mathbb{T}] = \mathbf{Set}[\mathbb{S}]$.

Note the role of the classifying topos as a 'bridge' between the two theories!

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Classifying toposes for propositional theories I

Definition

- A propositional theory is a geometric theory over a signature Σ which has no sorts.
- A localic topos is any topos of the form $\mathbf{Sh}(L)$ for a locale L.

Theorem

Localic toposes are precisely the classifying toposes of propositional theories.

Classifying toposes for propositional theories II

Specifically, given a locale L, we can consider the propositional theory $\mathbb{P}_L$ of completely prime filters in L, defined as follows. We take one atomic proposition F_a (to be thought of as the assertion that a is in the filter) for each $a \in L$; the axioms are

$$(\top \vdash F_1),$$

all the sequents of the form

$$(F_a \wedge F_b \vdash F_{a \wedge b}),$$

for any $a, b \in L$, and all the sequents of the form

$$F_a \vdash \bigvee_{i \in I} F_{a_i}$$

whenever $\bigvee_{i \in I} a_i = a$ in L.

In fact, for any locale L, the topos $\mathbf{Sh}(L)$ classifies $\mathbb{P}_L$.

Classifying toposes for Horn theories I

Definition

Let $\mathbb{T}$ be a Horn theory over a signature Σ. We say that a $\mathbb{T}$-model M in **Set** is finitely presented by a Horn formula $\phi(\vec{x})$, where $A_1 \cdots A_n$ is the string of sorts associated to $\vec{x}$, if there exists a string of elements $(\xi_1, \ldots, \xi_n) \in MA_1 \times \ldots \times MA_n$, called the generators of M, such that for any $\mathbb{T}$-model N in **Set** and string of elements $\vec{b} = (b_1, \ldots, b_n) \in MA_1 \times \ldots \times MA_n$ such that $(b_1, \ldots, b_n) \in [[\phi]]_N$, there exists a unique arrow $f^{\vec{b}} : M \to N$ in $\mathbb{T}\text{-mod}(\mathbf{Set})$ such that $(f^{\vec{b}}_{A_1} \times \ldots \times f^{\vec{b}}_{A_n})((\xi_1, \ldots, \xi_n)) = (b_1, \ldots, b_n)$. We denote by $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ the full subcategory of $\mathbb{T}\text{-mod}(\mathbf{Set})$ on the finitely presented models.

Classifying toposes for Horn theories II

Theorem

For any Horn theory $\mathbb{T}$, we have an equivalence of categories

$$f.p.\mathbb{T}\text{-}mod(\mathbf{Set}) \simeq (\mathscr{C}_{\mathbb{T}}^{cart})^{op}$$

In particular, $\mathbb{T}$ is classified by the topos $[f.p.\mathbb{T}\text{-}mod(\mathbf{Set}), \mathbf{Set}]$.

Examples

- The theory of Boolean algebras is classified by the topos $[\mathbf{Bool}_{fin}, \mathbf{Set}]$, where $\mathbf{Bool}_{fin}$ is the category of finite Boolean algebras.
- The theory of commutative rings with unit is classified by the topos $[\mathbf{Rng}_{f.g.}, \mathbf{Set}]$, where $\mathbf{Rng}_{f.g.}$ is the category of finitely generated rings.

Theories of presheaf type I

Definition

- A geometric theory $\mathbb{T}$ over a signature Σ is said to be of presheaf type if it is classified by a presheaf type.
- A model M of a theory of presheaf type $\mathbb{T}$ in the category **Set** is said to be finitely presentable if the functor $Hom_{\mathbb{T}\text{-mod}(\mathbf{Set})}(M, -) : \mathbb{T}\text{-mod}(\mathbf{Set}) \to \mathbf{Set}$ preserves filtered colimits.

Examples

- Any Horn theory
- The theory of decidable objects
- The theory of linear orders

Theories of presheaf type II

Theorem

Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ. Then

(i) *Any finitely presentable $\mathbb{T}$-model in **Set** is presented by a $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ;*

(ii) *Conversely, any $\mathbb{T}$-irreducible geometric formula $\phi(\vec{x})$ over Σ presents a finitely presentable $\mathbb{T}$-model.*
In particular, the category $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op}$ is equivalent to the full subcategory of $\mathcal{C}_{\mathbb{T}}^{geom}$ on the $\mathbb{T}$-irreducible formulae.

Fact

For any theory $\mathbb{T}$ of presheaf type, $\mathbb{T}$ is classified by the topos $[f.p.\mathbb{T}\text{-}mod(\mathbf{Set}), \mathbf{Set}]$.

Quotients of a theory of presheaf type I

- Suppose that $\mathbb{T}$ is a theory of presheaf type and $\mathbb{T}'$ is a quotient of $\mathbb{T}$ obtained from $\mathbb{T}$ by adding axioms σ of the form $\phi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{y_i})\theta_i$, where, for any $i \in I$, $[\theta_i] : \{\vec{y_i} \,.\, \psi\} \to \{\vec{x} \,.\, \phi\}$ is an arrow in $\mathscr{C}_{\mathbb{T}}$ and $\phi(\vec{x})$, $\psi(\vec{y_i})$ are formulae presenting respectively $\mathbb{T}$-models M_ϕ and M_{ψ_i}.
- For each such axiom $\phi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{y_i})\theta_i$, consider the cosieve S_σ on M_ϕ in f.p.$\mathbb{T}$-mod(**Set**) defined as follows. For each $i \in I$, $[[\theta_i]]_{M_{\psi_i}}$ is the graph of a morphism $[[\vec{y_i} \,.\, \psi_i]]_{M_{\psi_i}} \to [[\vec{x} \,.\, \phi]]_{M_{\psi_i}}$; then the image of the generators of M_{ψ_i} via this morphism is an element of $[[\vec{x} \,.\, \phi]]_{M_{\psi_i}}$ and this in turn determines, by definition of M_ϕ, a unique arrow $s_i : M_\phi \to M_{\psi_i}$ in $\mathbb{T}$-mod(**Set**). We define S_σ as the sieve in f.p.$\mathbb{T}$-mod(**Set**)$^{\text{op}}$ on M_ϕ generated by the arrows s_i as i varies in I. We define the associated $\mathbb{T}$-topology of $\mathbb{T}'$ as the Grothendieck topology generated by the sieves S_σ, where σ varies among the axioms of $\mathbb{T}'$, as above.

Quotients of a theory of presheaf type II

Theorem

Let $\mathbb{T}$ be a theory of presheaf type and $\mathbb{T}'$ be a quotient of $\mathbb{T}$ as above with associated $\mathbb{T}$-topology J on f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{op}$. Then the subtopos $\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}, J) \hookrightarrow [\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$ corresponds to the quotient $\mathbb{T}'$ via the duality theorem. In particular, $\mathbb{T}'$ is classified by the topos $\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}, J)$.

Quotients of a theory of presheaf type III

The following result provides a link between 'geometrical' properties of J and syntactic properties of $\mathbb{T}'$.

We say that a site $(\mathscr{C}, J)$ is locally connected if every J-covering sieve is connected i.e. for any $R \in J(c)$, R is connected as a full subcategory of $\mathscr{C}/c$.

Theorem

Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ, $\mathbb{T}'$ be a quotient of $\mathbb{T}$ with associated $\mathbb{T}$-topology J on f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{op}$ and $\phi(\vec{x})$ be a geometric formula over Σ which presents a $\mathbb{T}$-model M. Then

- (i) *If the site $(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}, J)$ is locally connected (for example when f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{op}$ satisfies the right Ore condition and every J-covering sieve is non-empty) then $\phi(\vec{x})$ is $\mathbb{T}'$-indecomposable.*
- (ii) *If f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{op}$ satisfies the right Ore condition and J is the atomic topology on $(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{op}$ then $\phi(\vec{x})$ is $\mathbb{T}'$-complete.*
- (iii) *If every J-covering sieve on M contains a J-covering sieve generated by a finite family of arrows then $\phi(\vec{x})$ is $\mathbb{T}'$-compact.*

The Zariski topos

Let Σ be the one-sorted signature for the theory $\mathbb{T}$ of commutative rings with unit i.e. the signature consisting of two binary function symbols $+$ and $\cdot$, one unary function symbol $-$ and two constants 0 and 1.
The coherent theory of local rings is obtained from $\mathbb{T}$ by adding the sequents

$$((0 = 1) \vdash_{[]} \bot)$$

and

$$((\exists z)((x + y) \cdot z = 1) \vdash_{x,y} ((\exists z)(x \cdot z = 1) \vee (\exists z)(y \cdot z = 1))),$$

Definition

The Zariski topos is the topos $\mathbf{Sh}(\mathbf{Rng}_{f.g.}^{\mathrm{op}}, J)$ of sheaves on the opposite of the category $\mathbf{Rng}_{f.g.}$ of finitely generated rings with respect to the topology J on $\mathbf{Rng}_{f.g.}^{\mathrm{op}}$ defined by: given a cosieve S in $\mathbf{Rng}_{f.g.}$ on an object A, $S \in J(A)$ if and only if S contains a finite family $\{\xi_i : A \to A[{s_i}^{-1}] \mid 1 \leq i \leq n\}$ of canonical inclusions $\xi_i : A \to A[{s_i}^{-1}]$ in $\mathbf{Rng}_{f.g.}$ where $\{s_1, \ldots, s_n\}$ is any set of elements of A which is not contained in any proper ideal of A.

Fact

The (coherent) theory of local rings is classified by the Zariski topos.

The classifying topos for integral domains

The theory of integral domains is the theory obtained from the theory of commutative rings with unit by adding the axioms

$$((0 = 1) \vdash_{[]} \bot)$$

$$((x \cdot y = 0) \ \vdash_{x,y} \ ((x = 0) \vee (y = 0)))\ .$$

Fact

The theory of integral domains is classified by the topos $\mathbf{Sh}(\mathbf{Rng}_{f.g.}^{op}, J)$ *of sheaves on the opposite of the category* $\mathbf{Rng}_{f.g.}$ *of finitely generated rings with respect to the topology* J *on* $\mathbf{Rng}_{f.g.}^{op}$ *defined by: given a cosieve* S *in* $\mathbf{Rng}_{f.g.}$ *on an object* A*,* $S \in J_2(A)$ *if and only if*

- *either* A *is the zero ring and* S *is the empty sieve on it or*
- S *contains a non-empty finite family* $\{\pi_{a_i} : A \to A/(a_i) \mid 1 \leq i \leq n\}$ *of canonical projections* $\pi_{a_i} : A \to A/(a_i)$ *in* $\mathbf{Rng}_{f.g.}$ *where* $\{a_1, \ldots, a_n\}$ *is any set of elements of* A *such that* $a_1 \cdot \ldots \cdot a_n = 0$.

For further reading

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