Topos Theory

Lectures 14-15: Morphisms of sites

Olivia Caramello

Morphisms of sites: the controvariant case

Definition

- A morphism of sites $(\mathscr{C}, J) \to (\mathscr{D}, K)$, where $\mathscr{C}$ and $\mathscr{D}$ are cartesian categories, is a cartesian functor $\mathscr{C} \to \mathscr{D}$ which sends J-covering sieves to K-covering sieves.
- Given a site $(\mathscr{C}, J)$, the Grothendieck topology J is said to be subcanonical if all the representable functors $\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ are J-sheaves.

Theorem

- *A morphism of sites* $f : (\mathscr{C}, J) \to (\mathscr{D}, K)$ *induces a geometric morphism* $\dot{f} : \mathbf{Sh}(\mathscr{D}, K) \to \mathbf{Sh}(\mathscr{C}, J)$.
- *If* J *and* K *are subcanonical then a geometric morphism* $g : \mathbf{Sh}(\mathscr{D}, K) \to \mathbf{Sh}(\mathscr{C}, J)$ *is of the form* $\dot{f}$ *for some* f *if and only if the inverse image functor* g^* *sends representables to representables; if this is the case then* f *is isomorphic to the restriction of* g^* *to the full subcategories of representables.*

Corollary

The assignment $L \to \mathbf{Sh}(L)$ *from locales to Grothendieck toposes is a full and faithful* 2*-functor.*

Morphisms of sites: the covariant case

Definition

A geometric morphism $f : \mathscr{E} \to \mathscr{F}$ is said to be essential if the inverse image functor $f^* : \mathscr{F} \to \mathscr{E}$ has a left adjoint.

Theorem

- *Every functor $f : \mathscr{C} \to \mathscr{D}$ induces an essential geometric morphism*

  $$E(f) : [\mathscr{C}^{op}, \mathbf{Set}] \to [\mathscr{D}^{op}, \mathbf{Set}],$$

  whose inverse image functor is given by composition with f^{op}.
- *If $\mathscr{C}$ and $\mathscr{D}$ are Cauchy-complete categories, a geometric morphism $[\mathscr{C}^{op}, \mathbf{Set}] \to [\mathscr{D}^{op}, \mathbf{Set}]$ is of the form $E(f)$ for some functor $f : \mathscr{C} \to \mathscr{D}$ if and only if it is essential; in this case, f can be recovered from $E(f)$ (up to isomorphism) as the restriction to the full subcategories of representables of the left adjoint to the inverse image of $E(f)$.*

The Comparison Lemma I

Definition

Let $\mathscr{D}$ be a full subcategory of a small category $\mathscr{C}$, and let J be a Grothendieck topology on $\mathscr{C}$. Then $\mathscr{D}$ is said to be J-dense if for every object $c \in \mathscr{C}$ there exists a sieve $S \in J(c)$ generated by a family of arrows whose domains lie in $\mathscr{D}$.

Theorem (The Comparison Lemma)

Let $(\mathscr{C}, J)$ be a site and $\mathscr{D}$ be a J-dense subcategory of $\mathscr{C}$. Then the sieves in $\mathscr{D}$ of the form $R \cap arr(\mathscr{D})$ for a J-covering sieve R in $\mathscr{C}$ form a Grothendieck topology $J|_{\mathscr{D}}$ on $\mathscr{D}$, called the induced topology, and, denoted by $i : \mathscr{D} \to \mathscr{C}$ the canonical inclusion functor, the geometric morphism

$$E(i) : [\mathscr{D}^{op}, \mathbf{Set}] \to [\mathscr{C}^{op}, \mathbf{Set}],$$

restricts to an equivalence of categories

$$E(i)| : \mathbf{Sh}(\mathscr{D}, J|_{\mathscr{D}}) \simeq \mathbf{Sh}(\mathscr{C}, J)\ .$$

The Comparison Lemma II

Corollary

- *Let B be a basis of a frame L, i.e. a subset $B \subset L$ such that every element in L can be written as a join of elements in B; then we have an equivalence of categories*

$$\mathbf{Sh}(L) \simeq \mathbf{Sh}(B, J^L|_B),$$

where J^L is the canonical topology on L.

- *Let $\mathscr{C}$ be a preorder and J be a subcanonical topology on $\mathscr{C}$. Then we have an equivalence of categories*

$$\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(Id_J(\mathscr{C})),$$

where $Id_J(\mathscr{C})$ is the frame of J-ideals on $\mathscr{C}$ (regarded as a locale).

For further reading

P. T. Johnstone.
Sketches of an Elephant: a topos theory compendium. Vols. 1-2, vols. 43-44 of *Oxford Logic Guides*
Oxford University Press, 2002.

S. Mac Lane and I. Moerdijk.
Sheaves in geometry and logic: a first introduction to topos theory
Springer-Verlag, 1992.