Topos Theory

Lectures 12 and 13: Geometric morphisms as flat functors

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Geometric morphisms as flat functors I

Theorem

Let $\mathscr{C}$ be a small category and $\mathscr{E}$ be a locally small cocomplete category. Then, for any functor $A : \mathscr{C} \to \mathscr{E}$ the functor $R_A : \mathscr{E} \to [\mathscr{C}^{op}, \mathbf{Set}]$ defined for each $e \in Ob(\mathscr{E})$ and $c \in Ob(\mathscr{C})$ by:

$$R_A(e)(c) = Hom_{\mathscr{E}}(A(c), e)$$

has a left adjoint $- \otimes_{\mathscr{C}} A : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathscr{E}$.

Definition

- A functor $A : \mathscr{C} \to \mathscr{E}$ from a small category $\mathscr{C}$ to a locally small topos $\mathscr{E}$ with small colimits is said to be flat if the functor $- \otimes_{\mathscr{C}} A : [\mathscr{C}^{\text{op}}, \mathbf{Set}] \to \mathscr{E}$ preserves finite limits.
- The full subcategory of $[\mathscr{C}, \mathscr{E}]$ on the flat functors will be denoted by $\mathbf{Flat}(\mathscr{C}, \mathscr{E})$.

Geometric morphisms as flat functors II

Theorem

Let $\mathscr{C}$ be a small category and $\mathscr{E}$ be a locally small topos with small colimits (in particular, a Grothendieck topos). Then we have an equivalence of categories

$$\textbf{\textit{Geom}}(\mathscr{E}, [\mathscr{C}^{op}, \mathbf{Set}]) \simeq \textbf{\textit{Flat}}(\mathscr{C}, \mathscr{E})$$

(natural in $\mathscr{E}$), which sends

- *a flat functor $A : \mathscr{C} \to \mathscr{E}$ to the geometric morphism $\mathscr{E} \to [\mathscr{C}^{op}, \mathbf{Set}]$ determined by the functors R_A and $- \otimes_{\mathscr{C}} A$, and*
- *a geometric morphism $f : \mathscr{E} \to [\mathscr{C}^{op}, \mathbf{Set}]$ to the flat functor given by the composite $f^* \circ y$ of $f^* : [\mathscr{C}^{op}, \mathbf{Set}] \to \mathscr{E}$ with the Yoneda embedding $y : \mathscr{C} \to [\mathscr{C}^{op}, \mathbf{Set}]$.*

Fact

Let $\mathscr{C}$ be a category with finite limits and $\mathscr{E}$ be a locally small cocomplete topos. Then a functor $\mathscr{C} \to \mathscr{E}$ is flat if and only if it preserves finite limits.

Geometric morphisms to $\mathbf{Sh}(\mathscr{C}, J)$ I

Definition

Let $\mathscr{E}$ be a topos with small colimits.

- A family $\{f_i : a_i \to a \mid i \in I\}$ of arrows in $\mathscr{E}$ with common codomain is said to be epimorphic if for any pair of arrows $g, h : a \to b$ with domain a, $g = h$ if and only if $g \circ f_i = h \circ f_i$ for all $i \in I$.
- If $(\mathscr{C}, J)$ is a site, a functor $F : \mathscr{C} \to \mathscr{E}$ is said to be J-continuous if it sends J-covering sieves to epimorphic families.

The full subcategory of $\mathbf{Flat}(\mathscr{C}, \mathscr{E})$ on the J-continuous flat functors will be denoted by $\mathbf{Flat}_J(\mathscr{C}, \mathscr{E})$.

Geometric morphisms to $\mathbf{Sh}(\mathscr{C}, J)$ II

Theorem

For any site $(\mathscr{C}, J)$ and locally small cocomplete topos $\mathscr{E}$, the above-mentioned equivalence between geometric morphisms and flat functors restricts to an equivalence of categories

$$\textbf{\textit{Geom}}(\mathscr{E}, \mathbf{Sh}(\mathscr{C}, J)) \simeq \textbf{\textit{Flat}}_J(\mathscr{C}, \mathscr{E})$$

natural in $\mathscr{E}$.

Sketch of proof.

Appeal to the previous theorem

- identifying the geometric morphisms $\mathscr{E} \to \mathbf{Sh}(\mathscr{C}, J)$ with the geometric morphisms $\mathscr{E} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ which factor through the canonical geometric inclusion $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, and
- using the characterization of such morphisms as the geometric morphisms $f : \mathscr{E} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ such that the composite $f^* \circ y$ of the inverse image functor f^* of f with the Yoneda embedding $y : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ sends J-covering sieves to colimits in $\mathscr{E}$ (equivalently, to epimorphic families in $\mathscr{E}$).

□

For further reading

S. Mac Lane and I. Moerdijk.
Sheaves in geometry and logic: a first introduction to topos theory
Springer-Verlag, 1992.