Topos Theory

Lecture 2: Categorical preliminaries I

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The categorical point of view

- Category Theory, introduced by Samuel Eilenberg and Saunders Mac Lane in the years 1942-45 in the context of Algebraic Topology, is a branch of Mathematics which provides an abstract language for expressing mathematical concepts and reasoning about them. In fact, the concepts of Category Theory are unifying notions whose instances can be found in essentially every field of Mathematics.
- The underlying philosophy of Category Theory is to replace the primitive notions of set and belonging relationship between sets, which constitute the foundations of Set Theory, with abstractions of the notions of set and function, namely the concepts of object and arrow.
- Since it was introduced, this approach has entailed a deep paradigmatic shift in the way Mathematicians could look at their subject, and has paved the way to important discoveries which would have hardly been possible before. One of the great achievements of Category Theory is Topos Theory, a subject entirely written in categorical language whose importance, both technical and conceptual, for the future of Mathematics cannot be exaggerated.

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The definition of category (1/2)

Definition

A (small) category $\mathscr{C}$ consists of

(i) a set $\mathrm{Ob}(\mathscr{C})$,

(ii) for any $a, b \in Ob(\mathscr{C})$, a set $Hom_{\mathscr{C}}(a, b)$,

(iii) for any $a, b, c \in Ob(\mathscr{C})$, a map:

$$\circ_{\mathscr{C}} : Hom_{\mathscr{C}}(a, b) \times Hom_{\mathscr{C}}(b, c) \to Hom_{\mathscr{C}}(a, c)$$

called the composition and denoted by $(f, g) \to g \circ f$,

these data satisfying

a) the composition $\circ$ is associative, i.e., for $f \in Hom_{\mathscr{C}}(a, b), g \in Hom_{\mathscr{C}}(b, c)$ and $h \in Hom_{\mathscr{C}}(c, d)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$,

b) for each $a \in Ob(\mathscr{C})$, there exists $id_a \in Hom_{\mathscr{C}}(a, a)$ such that $f \circ id_a = f$ for all $f \in Hom_{\mathscr{C}}(a, b)$ and $id_a \circ g = g$ for all $g \in Hom_{\mathscr{C}}(b, a)$.

The definition of category (2/2)

- An element of $Ob(\mathscr{C})$ is called an object of $\mathscr{C}$.
- For $a, b \in ob(\mathscr{C})$, an element f of $Hom_{\mathscr{C}}(a, b)$ is called an arrow (from a to b) in $\mathscr{C}$; we say that a is the domain of f, b is the codomain of f, and we write $f : a \to b$, $a = dom(f)$ and $b = cod(f)$.
- The arrow id_a is called the identity arrow on a.

Remark

The concept of category has a first-order axiomatization, in a language having two sorts **O** *and* **A** *(respectively for objects and arrows), two unary function symbols (for domain and codomain)* $\mathbf{A} \to \mathbf{O}$*, one unary function symbol* $\mathbf{O} \to \mathbf{A}$ *(formalizing the concept of identity arrow) and a ternary predicate of type* **A** *(formalizing the notion of composition of arrows).*

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The dual category

The concept of category is self-dual i.e. the axioms in the definition of category continue to hold if we formally reverse the direction of arrows while mantaining the same objects.

Definition

Given a category $\mathscr{C}$, the dual category $\mathscr{C}^{\mathrm{op}}$ is defined by setting

$$Ob(\mathscr{C}^{\mathrm{op}}) = Ob(\mathscr{C}), \qquad Hom_{\mathscr{C}}^{\mathrm{op}}(a,b) = Hom_{\mathscr{C}}(b,a),$$

and defining the composition $g \circ_{\mathscr{C}^{\mathrm{op}}} f$ of $f \in Hom_{\mathscr{C}}^{\mathrm{op}}(a,b)$ and $f \in Hom_{\mathscr{C}}^{\mathrm{op}}(b,c)$ by

$$g \circ_{\mathscr{C}^{\mathrm{op}}} f = f \circ_{\mathscr{C}} g.$$

Note that $\mathscr{C}^{\mathrm{op}\,\mathrm{op}} = \mathscr{C}$ for any category $\mathscr{C}$.

The duality principle

Every statement formulated in the language of Category Theory has a dual, obtained by formally reversing the arrows and the order of composition of them.

- A statement is true in a category $\mathscr{C}$ if and only if the dual statement is true in the dual category $\mathscr{C}^{\text{op}}$. Hence a statement is valid in all categories if and only if its dual is.
- Anyway, two dual statements in the language of Category Theory, when interpreted in a given 'concrete' category, may specialize to two very different-looking (and even inequivalent!) mathematical statements.
- Sometimes, it is possible to lift ordinary mathematical statements to the level of categories (or at least to classes of categories closed under duality) and obtain abstract proofs of them in the language of Category Theory; if this is the case, one can then invoke the duality principle to derive dual versions of them which can be specialized to the original context.

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Properties of arrows

We can consider various properties of arrows in a category, expressed in categorical language. An arrow $f : a \to b$ is:

- a monomorphism (or monic) if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all arrows $g_1, g_2 : x \to a$.
- an epimorphism (or epic) if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all arrows $g_1, g_2 : b \to x$.
- an isomorphism if there exists an arrow $g : b \to a$ with $f \circ g = 1_b$ and $g \circ f = 1_a$.

Notice that monomorphisms are dual to epimorphisms i.e. an arrow f of a category $\mathscr{C}$ is a monomorphism in $\mathscr{C}$ if and only if it is an epimorphism in $\mathscr{C}^{\mathrm{op}}$ (regarded as an arrow in $\mathscr{C}^{\mathrm{op}}$).

Example

In the category **Set**, an arrow is:

- a monomorphism if and only if it is an injective function.
- an epimorphism if and only if it is a surjective function.
- an isomorphism if and only if it is a bijection.

Categories of mathematical objects

Important mathematical objects can be organized into categories.

Examples

- The category **Set** of sets and functions between them.
- The category **Top** of topological spaces and continuous maps between them.
- The category of **Gr** of groups and group homomorphisms, the category **Rng** of rings and ring homomorphisms, the category $\mathbf{Vect}_K$ of vector spaces over a field K and K-linear maps between them, etc.
  In fact, given a first-order theory $\mathbb{T}$, we have a category $\mathbb{T}\text{-mod}(\mathbf{Set})$ having as objects the (set-based) models of $\mathbb{T}$ and as arrows the structure-preserving maps between them.

Mathematical objects as categories

On the other hand, important mathematical objects arise as particular kinds of categories:

- A set can be seen as a discrete category i.e. a category whose only arrows are the identity arrows.
- A preorder can be seen as a preorder category i.e. a category having at most one arrow from one object to another.
- A monoid can be seen as a category with just one object.
- A groupoid is a category whose arrows are all isomorphisms; in particular, a group is a groupoid with just one object.

Functors

Functors are the natural structure-preserving maps between categories.

Definition

Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. A functor $F : \mathscr{C} \to \mathscr{C}'$ consists of a map $F : Ob(\mathscr{C}) \to Ob(\mathscr{C}')$ and of maps $F : Hom_{\mathscr{C}}(a,b) \to Hom_{\mathscr{C}'}(F(a),F(b))$ for all $a,b \in \mathscr{C}$, such that

- $F(id_a) = id_{F(a)}$ for all $a \in \mathscr{C}$,
- $F(g \circ f) = F(g) \circ F(f)$ for all $f : a \to b, g : b \to c$.

Functors from the dual $\mathscr{C}^{\mathrm{op}}$ of a category $\mathscr{C}$ to the category **Set** of sets are called **presheaves** on $\mathscr{C}$.

Composition of functors is defined in the obvious way and on each category $\mathscr{C}$ we have the identity functor $id_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$.
In fact, (small) categories and functors form themselves a (large) category, denoted by **Cat**.

Representable functors

Given a category $\mathscr{C}$ and an object $c \in Ob(\mathscr{C})$, we have a functor $Hom_{\mathscr{C}}(c, -) : \mathscr{C} \to \mathbf{Set}$ defined by

- $Hom_{\mathscr{C}}(c, -)(a) = Hom_{\mathscr{C}}(c, a)$ for $a \in Ob(\mathscr{C})$,
- $Hom_{\mathscr{C}}(c, -)(f) : Hom_{\mathscr{C}}(c, a) \to Hom_{\mathscr{C}}(c, b)$ given by $g \to f \circ g$, for $f : a \to b$ in $\mathscr{C}$.

Functors (naturally isomorphic to those of the form) $Hom_{\mathscr{C}}(c, -) : \mathscr{C} \to \mathbf{Set}$ are said to be representable.
Note that, dually, we have functors $Hom_{\mathscr{C}}(-, c) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$.

Natural transformations

Definition

Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories and let F_1 and F_2 be two functors from $\mathscr{C}$ to $\mathscr{C}'$. A natural transformation $\alpha : F_1 \to F_2$ is a function assigning to each object $a \in Ob(\mathscr{C})$ an arrow $\alpha(a) : F_1(a) \to F_2(a)$ in $\mathscr{C}'$ in such a way that for all arrows $f : a \to b$ in $\mathscr{C}$ the diagram below commutes:

$$\begin{array}{ccc} F_1(a) & \xrightarrow{\alpha(a)} & F_2(a) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(b) & \xrightarrow[\alpha(b)]{} & F_2(b) \end{array}$$

Example

Let $\mathbf{Vect}_K$ be the category of vector spaces over a field K and $* : \mathbf{Vect}_K^{\mathrm{op}} \to \mathbf{Vect}_K$ be the duality functor which assigns to a vector space $V \in Ob(\mathbf{Vect}_K)$ the vector space $V^* = Hom_{\mathbf{Vect}_K}(V, K)$. Then $id_{\mathbf{Vect}_K} \to **$ is a natural transformation of functors from $\mathbf{Vect}_K$ to itself.

Full and faithful functors

Definition

- A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be faithful if $F : Hom_{\mathscr{C}}(a,b) \to Hom_{\mathscr{C}'}(F(a),F(b))$ is injective for all $a,b \in \mathscr{C}$.
- A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be full if $F : Hom_{\mathscr{C}}(a,b) \to Hom_{\mathscr{C}'}(F(a),F(b))$ is surjective for all $a,b \in \mathscr{C}$.
- A functor $F : \mathscr{C} \to \mathscr{D}$ is (essentially) surjective if every object $d \in Ob(\mathscr{D})$ is (isomorphic to one) of the form $F(c)$ for some $c \in Ob(\mathscr{C})$.
- A subcategory $\mathscr{D}$ of a category $\mathscr{C}$ is a category $\mathscr{D}$ such that $Ob(\mathscr{D}) \subseteq Ob(\mathscr{C})$, $Hom_{\mathscr{D}}(a,b) \subseteq Hom_{\mathscr{C}}(a,b)$ for any $a,b \in Ob(\mathscr{D})$, the composition in $\mathscr{D}$ is induced by the composition in $\mathscr{C}$ and the identity arrows in $\mathscr{D}$ are identity arrows in $\mathscr{C}$; $\mathscr{D}$ is said to be a full subcategory of $\mathscr{C}$ if the inclusion functor $i : \mathscr{D} \to \mathscr{C}$ is full.

Equivalence of categories

Two functors $F_1, F_2 : \mathscr{C} \to \mathscr{D}$ are said to be naturally isomorphic if there exists an invertible natural transformation $\alpha : F_1 \to F_2$.

When can two categories be considered the same, from the point of view of the categorical properties they satisfy?

Definition (Equivalence of categories)

Two categories $\mathscr{C}$ and $\mathscr{D}$ are said to be equivalent if there are functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ and natural isomorphisms $F \circ G \cong id_{\mathscr{D}}$, $G \circ F \cong id_{\mathscr{C}}$.

Theorem

Under AC, a functor is part of an equivalence of categories if and only if it is full, faithful and essentially surjective.

For further reading

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