Topos Theory

Lecture 16: Points of toposes

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Points of toposes

Definition

By a point of a topos $\mathscr{E}$, we mean a geometric morphism $\mathbf{Set} \to \mathscr{E}$.

Examples

- For any site $(\mathscr{C}, J)$, the points of the topos $\mathbf{Sh}(\mathscr{C}, J)$ correspond precisely to the J-continuous flat functors $\mathscr{C} \to \mathbf{Set}$;
- For any locale L, the points of the topos $\mathbf{Sh}(L)$ correspond precisely to the frame homomorphisms $L \to \{0, 1\}$;
- For any small category $\mathscr{C}$ and any object c of $\mathscr{C}$, we have a point $e_c : \mathbf{Set} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ of the topos $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, whose inverse image is the evaluation functor at c.

Fact

Any set of points P of a Grothendieck topos $\mathscr{E}$ indexed by a set X via a function $\xi : X \to P$ can be identified with a geometric morphism $\tilde{\xi} : [X, \mathbf{Set}] \to \mathscr{E}$.

Separating sets of points

Definition

- Let $\mathscr{E}$ be a topos and P be a collection of points of $\mathscr{E}$ indexed by a set X via a function $\xi : X \to P$. We say that P is separating for $\mathscr{E}$ if the points in P are jointly surjective, i.e. if the inverse image functors of the geometric morphisms in P jointly reflect isomorphisms (equivalently, if the geometric morphism $\tilde{\xi} : [X, \mathbf{Set}] \to \mathscr{E}$ is surjective).
- A topos is said to have enough points if the collection of all the points of $\mathscr{E}$ is separating for $\mathscr{E}$.

Fact

A Grothendieck topos has enough points if and only if there exists a set of points of $\mathscr{E}$ which is separating for $\mathscr{E}$.

The subterminal topology

The following notion provides a way for endowing a given set of points of a topos with a natural topology.

Definition

Let $\xi : X \to \mathrm{P}$ be an indexing of a set P of points of a Grothendieck topos $\mathscr{E}$ by a set X. We define the subterminal topology $\tau_{\xi}^{\mathscr{E}}$ as the image of the function $\phi_{\mathscr{E}} : \mathrm{Sub}_{\mathscr{E}}(1) \to \mathscr{P}(X)$ given by

$$\phi_{\mathscr{E}}(u) = \{x \in X \mid \xi(x)^*(u) \cong 1_{\mathbf{Set}}\} .$$

We denote the space X endowed with the topology $\tau_{\xi}^{\mathscr{E}}$ by $X_{\tau_{\xi}^{\mathscr{E}}}$.

The interest of this notion lies in its level of generality, as well as in its formulation as a topos-theoretic invariant admitting a 'natural behaviour' with respect to sites.

Fact

If P is a separating set of points for $\mathscr{E}$ then the frame $\mathscr{O}(X_{\tau_{\xi}^{\mathscr{E}}})$ of open sets of $X_{\tau_{\xi}^{\mathscr{E}}}$ is isomorphic to $\mathrm{Sub}_{\mathscr{E}}(1)$ (via $\phi_{\mathscr{E}}$).

Categories of toposes paired with points

The construction of the subterminal topology can be made functorial.

Definition

The category $\mathfrak{Top}_p$ toposes paired with points has as objects the pairs $(\mathscr{E}, \xi)$, where $\mathscr{E}$ is a Grothendieck topos and $\xi : X \to P$ is an indexing of a set of points P of $\mathscr{E}$, and whose arrows $(\mathscr{E}, \xi) \to (\mathscr{F}, \xi')$, where $\xi : X \to P$ and $\xi' : Y \to Q$, are the pairs (f, l) where $f : \mathscr{E} \to \mathscr{F}$ is a geometric morphism and $l : X \to Y$ is a function such that the diagram

$$\begin{array}{ccc} [X, \mathbf{Set}] & \xrightarrow{E(l)} & [Y, \mathbf{Set}] \\ \downarrow{\scriptstyle \tilde{\xi}} & & \downarrow{\scriptstyle \tilde{\xi'}} \\ \mathscr{E} & \xrightarrow{f} & \mathscr{F} \end{array}$$

commutes (up to isomorphism).

Theorem

*We have a functor $\mathfrak{Top}_p \to$ **Top** (where **Top** is the category of topological spaces) sending an object $(\mathscr{E}, \xi)$ of $\mathfrak{Top}_p$ to the space $X_{\tau_{\xi}^{\mathscr{E}}}$ and an arrow $(f, l) : (\mathscr{E}, \xi) \to (\mathscr{F}, \xi')$ in $\mathfrak{Top}_p$ to the continuous function $l : X_{\tau_{\xi}^{\mathscr{E}}} \to X_{\tau_{\xi'}^{\mathscr{F}}}$.*

Examples of subterminal topologies I

Definition

Let $(\mathscr{C}, \leq)$ be a preorder category. A J-prime filter on $\mathscr{C}$ is a subset $F \subseteq ob(\mathscr{C})$ such that F is non-empty, $a \in F$ implies $b \in F$ whenever $a \leq b$, for any $a, b \in F$ there exists $c \in F$ such that $c \leq a$ and $c \leq b$, and for any J-covering sieve $\{a_i \to a \mid i \in I\}$ in $\mathscr{C}$ if $a \in F$ then there exists $i \in I$ such that $a_i \in F$.

Theorem

Let $\mathscr{C}$ be a preorder and J be a Grothendieck topology on it. Then the space $X_{\tau_{\mathbf{Sh}(\mathscr{C},J)}}$ has as set of points the collection $\mathscr{F}^J_{\mathscr{C}}$ of the J-prime filters on $\mathscr{C}$ and as open sets the sets the form

$$\mathscr{F}_I = \{F \in \mathscr{F}^J_{\mathscr{C}} \mid F \cap I \neq \emptyset\},$$

where I ranges among the J-ideals on $\mathscr{C}$. In particular, a sub-basis for this topology is given by the sets

$$\mathscr{F}_c = \{F \in \mathscr{F}^J_{\mathscr{C}} \mid c \in F\},$$

where c varies among the objects of $\mathscr{C}$.

Examples of subterminal topologies II

- The Alexandrov topology ($\mathscr{E} = [\mathscr{P}, \mathbf{Set}]$, where $\mathscr{P}$ is a preorder and ξ is the indexing of the set of points of $\mathscr{E}$ corresponding to the elements of $\mathscr{P}$)
- The Stone topology for distributive lattices ($\mathscr{E} = \mathbf{Sh}(\mathscr{D}, J_{coh})$ and ξ is an indexing of the set of all the points of $\mathscr{E}$, where $\mathscr{D}$ is a distributive lattice and J_{coh} is the coherent topology on it)
- A topology for meet-semilattices ($\mathscr{E} = [\mathscr{M}^{\mathrm{op}}, \mathbf{Set}]$ and ξ is an indexing of the set of all the points of $\mathscr{E}$, where $\mathscr{M}$ is a meet-semilattice)
- The space of points of a locale ($\mathscr{E} = \mathbf{Sh}(L)$ for a locale L and ξ is an indexing of the set of all the points of $\mathscr{E}$)
- A logical topology ($\mathscr{E} = \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the classiying topos of a geometric theory $\mathbb{T}$ and ξ is any indexing of the set of all the points of $\mathscr{E}$ i.e. models of $\mathbb{T}$)
- The Zariski topology

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For further reading

O. Caramello.
A topos-theoretic approach to Stone-type dualities,
to shortly appear on the Mathematics ArXiv

P. T. Johnstone.
Sketches of an Elephant: a topos theory compendium. Vols. 1-2, vols. 43-44 of *Oxford Logic Guides*
Oxford University Press, 2002.

S. Mac Lane and I. Moerdijk.
Sheaves in geometry and logic: a first introduction to topos theory
Springer-Verlag, 1992.