Topos Theory

Lecture 11: Local operators

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The concept of local operator

Definition

- Let $\mathscr{E}$ be a topos, with subobject classifier $\top : 1 \to \Omega$. A local operator (or Lawvere-Tierney topology) on $\mathscr{E}$ is an arrow $j : \Omega \to \Omega$ in $\mathscr{E}$ such that the diagrams

$$\begin{array}{ccc} 1 & & \\ \downarrow{\scriptstyle \top} & \searrow^{\top} & \\ \Omega & \xrightarrow[j]{} & \Omega \end{array} \qquad \begin{array}{ccc} \Omega & & \\ \downarrow{\scriptstyle j} & \searrow^{j} & \\ \Omega & \xrightarrow[j]{} & \Omega \end{array} \qquad \begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\ \downarrow{\scriptstyle j \times j} & & \downarrow{\scriptstyle j} \\ \Omega \times \Omega & \xrightarrow[\wedge]{} & \Omega \end{array}$$

  commute (where $\wedge : \Omega \times \Omega \to \Omega$ is the meet operation of the internal Heyting algebra Ω).
- Let $\mathscr{E}$ be an elementary topos. A universal closure operation on $\mathscr{E}$ is a closure operation c on subobjects which commutes with pullback (= intersection) of subobjects.

Universal closure operations I

Definition

Let c be a universal closure operation on an elementary topos $\mathscr{E}$.

- A subobject $m : a' \to a$ in $\mathscr{E}$ is said to be *c*-dense if $c(m) = id_a$, and *c*-closed if $c(m) = m$.
- An object a of $\mathscr{E}$ is said to be a *c*-sheaf if whenever we have a diagram

$$\begin{array}{ccc} b' & \xrightarrow{f'} & a \\ {\scriptstyle m}\downarrow & & \\ b & & \end{array}$$

  where m is a c-dense subobject, there exists exactly one arrow $f : b \to a$ such that $f \circ m = f'$.
- The full subcategory of $\mathscr{E}$ on the objects which are c-sheaves will be denoted by $\mathbf{sh}_c(\mathscr{E})$.

Universal closure operations II

Theorem

For any elementary topos $\mathscr{E}$, there is a bijection between universal closure operations on $\mathscr{E}$ and local operators on $\mathscr{E}$.

Sketch of proof.

The bijection sends a universal closure operation c on $\mathscr{E}$ to the local operator $j_c : \Omega \to \Omega$ given by classifying map of the subobject $c(1 \overset{\top}{\rightarrowtail} \top)$, and a local operator j to the closure operation c_j induced by composing classifying arrows with j. □

Fact

For any local operator j on an elementary (resp. Grothendieck) topos $\mathscr{E}$, $\mathbf{sh}_{c_j}(\mathscr{E})$ is an elementary (resp. Grothendieck) topos, and the inclusion $\mathbf{sh}_{c_j}(\mathscr{E}) \hookrightarrow \mathscr{E}$ has a left adjoint $a_j : \mathscr{E} \to \mathbf{sh}_{c_j}(\mathscr{E})$ which preserves finite limits. In fact, local operators on $\mathscr{E}$ also correspond bijectively to (equivalence classes of) geometric inclusions to $\mathscr{E}$.

Local operators and Grothendieck topologies

Theorem

If $\mathscr{C}$ is a small category, the Grothendieck topologies J on $\mathscr{C}$ correspond exactly to the local operators on the presheaf topos $[\mathscr{C}^{op}, \mathbf{Set}]$.

Sketch of proof.

The correspondence sends a local operator $j : \Omega \to \Omega$ to the subobject $J \rightarrowtail \Omega$ which it classifies, that is to the Grothendieck topology J on $\mathscr{C}$ defined by:

$$S \in J(c) \text{ if and only if } j(c)(S) = M_c$$

Conversely, it sends a Grothendieck topology J, regarded as a subobject $J \rightarrowtail \Omega$, to the arrow $j : \Omega \to \Omega$ that classifies it. □

In fact, if J is the Grothendieck topology corresponding to a local operator j, an object of $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ is a J-sheaf (in the sense of Grothendieck toposes) if and only if it is a c_j-sheaf (in the sense of universal closure operations).

For further reading

S. Mac Lane and I. Moerdijk.
Sheaves in geometry and logic: a first introduction to topos theory
Springer-Verlag, 1992.