Topos Theory

Lecture 1: Overview of the course

Olivia Caramello

What is a topos?

A topos can be seen as:

- a generalized space
- a mathematical universe
- a (first-order) mathematical theory modulo 'Morita-equivalence' where 'Morita-equivalence' is the equivalence relation which identifies two theories precisely when they have equivalent categories of models in any topos $\mathscr{E}$, naturally in $\mathscr{E}$.

Recently, a new point of view has emerged: that of toposes as unifying spaces which can serve as bridges for transferring information, ideas and results between distinct mathematical theories. This approach, first introduced in my Ph.D. thesis, has already generated ramifications into distinct mathematical fields and points towards a realization of Topos Theory as a unifying theory of Mathematics.

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Toposes as bridges I

- Two mathematical theories have equivalent classifying toposes if and only if they are Morita-equivalent to each other.
- Hence, a topos can be seen as a *canonical representative* of equivalence classes of geometric theories modulo Morita-equivalence. So, we can think of a topos as embodying the 'common features' of mathematical theories which are Morita-equivalent to each other.
- The underlying intuition behind this is that a given mathematical property can manifest itself in several different forms in the context of mathematical theories which have a common 'semantical core' but a different linguistic presentation.

Toposes as bridges II

- The fact that different mathematical theories have equivalent classifying toposes translates, at the technical level, into the existence of different representations of one topos.
- The essential features of Morita-equivalences are all ‘hidden’ inside toposes, and can be revealed by using their different representations.
- For example, imagine starting with a property, say geometrical, of a certain mathematical object, and being able to find a topos and a property of it which is (logically) equivalent to the given property of our object; then one can use e.g. a logical representation for the topos to convert this property of the topos into a logical statement of a certain kind; as a result, one obtains the equivalence of our initial geometrical property with a logical one.

Toposes as bridges III

Definition

By a topos-theoretic invariant we mean a property of (or a construction involving) toposes which is stable under categorical equivalence.

- The remarkable fact is that if a property of a mathematical object is formulated as a topos-theoretic invariant on some topos then the expression of it in terms of the different theories classified by the topos is determined to a great extent by the technical relationship between the topos and the different representations of it.
- Topos-theoretic invariants can then be used to transfer properties from one theory to another.

Toposes as bridges IV

- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories.
- The fact that topos-theoretic invariants specialize to important properties or constructions of natural mathematical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications into Mathematics as a whole.

A new way of doing Mathematics I

- These methodologies define a new way of doing Mathematics which is ‘upside-down’ compared with the ‘usual’ one: instead of starting with simple ingredients and combining them to build more complicated structures, one assumes as primitive ingredients rich and sophisticated mathematical entities, namely Morita-equivalences and topos-theoretic invariants, and extracts from them a huge amount of information relevant for classical mathematics.
- The ‘working mathematician’ could very well attempt to formulate his or her properties of interest in terms of topos-theoretic invariants, and derive equivalent versions of them by using alternative representations.

A new way of doing Mathematics II

- There is an strong element of automatism in these techniques; by means of them, one can generate a great number of new mathematical results without really making any creative effort.
- The results generated in this way are in general non-trivial; in some cases they can be rather ‘weird’ according to the usual mathematical standards (although they might still be quite deep) but, with a careful choice of Morita-equivalences and invariants, one can easily get interesting and natural mathematical results.
- In fact, a lot of information that is not visible with the usual ‘glasses’ is revealed by the application of this machinery.
- On the other hand, the range of applicability of these methods is boundless within Mathematics, by the very generality of the notion of topos.

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