TOPOS THEORY EXAMPLES 4 (Lent Term 2012)

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1. Let Σ be a signature having finitely many sorts $A_1, A_2, \ldots, A_n$, and $\mathbf{T}$ a coherent theory over Σ. Show that there is a single-sorted signature $\widehat{\Sigma}$ and a coherent theory $\widehat{\mathbf{T}}$ over $\widehat{\Sigma}$ such that, for any topos $\mathcal{E}$, we have an equivalence $\mathbf{T}\text{-}\mathbf{Mod}(\mathcal{E}) \simeq \widehat{\mathbf{T}}\text{-}\mathbf{Mod}(\mathcal{E})$, which is natural in $\mathcal{E}$.

2. The class of *cartesian formulae* relative to a theory $\mathbf{T}$ is defined as follows: atomic formulae and $\top$ are cartesian, $(\phi \wedge \psi)$ is cartesian provided both ϕ and ψ are, and $(\exists x)\phi$ is cartesian provided ϕ is cartesian and the sequent $((\phi \wedge \phi[x'/x]) \vdash_{x,x',\vec{y}} (x = x'))$ is derivable in $\mathbf{T}$. (Here $x, \vec{y}$ is a suitable context for ϕ, and x' is a variable not in it.) A theory $\mathbf{T}$ is said to be *cartesian* if its axioms can be ordered in such a way that each involves formulae which are cartesian relative to the theory formed by the earlier axioms. Write down a presentation of the theory of categories (as a two-sorted theory, with function symbols for domain, codomain and identities, and a ternary relation $T(x, y, z)$ to express "z is the composite of x and y"), and verify that it is cartesian. Show also that if $\mathbf{T}$ is a cartesian theory then $\mathbf{T}\text{-}\mathbf{Mod}(\mathcal{E})$ is closed under finite limits in $\Sigma\text{-}\mathbf{Str}(\mathcal{E})$.

3. Consider the following (single-sorted) geometric theory $\mathbf{K}$: it has one n-ary relation symbol R_n for each $n \geq 0$, with axioms

$$(R_n(x_1, \ldots, x_n) \vdash_{x_1,\ldots,x_n,y} \bigvee_{i=1}^{n} (y = x_i))$$

(for $n = 0$, we interpret this as the axiom $(R_0 \vdash_y \bot)$),

$$(R_n(x_1, \ldots, x_n) \dashv\vdash_{x_1,\ldots,x_n} R_m(x_{f(1)}, \ldots, x_{f(m)}))$$

for any surjection $f\colon \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$, and

$$(\top \vdash \bigvee_{n=0}^{\infty} (\exists x_1) \cdots (\exists x_n) R_n(x_1, \ldots, x_n)) \quad .$$

Show that the category of $\mathbf{K}$-models in $\mathbf{Set}$ is equivalent to the category of finite sets and surjections between them.

4. Let $\mathcal{E}$ be a Grothendieck topos with internal language $\Sigma_{\mathcal{E}}$. We write $\mathcal{E} \models \sigma$, where σ is a sequent over $\Sigma_{\mathcal{E}}$, to mean that σ is satisfied in the canonical $\Sigma_{\mathcal{E}}$-structure in $\mathcal{E}$. Show that

(a) $1_A : A \to A$ is the identity arrow on A if and only if $\mathcal{E} \models (\top \vdash_x (\ulcorner 1_A \urcorner(x) = x))$;

(b) $f : A \to C$ is the composite of $g : A \to B$ and $h : B \to C$ if and only if $\mathcal{E} \models (\top \vdash_x (\ulcorner f \urcorner(x) = \ulcorner h \urcorner(\ulcorner g \urcorner(x))))$;

(c) $f : A \to B$ is monic if and only if $\mathcal{E} \models ((\ulcorner f \urcorner(x) = \ulcorner f \urcorner(x')) \vdash_{x,x'} (x = x'))$;

(d) $f : A \to B$ is an epimorphism if and only if $\mathcal{E} \models (\top \vdash_y (\exists x)\ulcorner f \urcorner(x) = y)$;

(e) A is a terminal object of $\mathcal{E}$ if and only if $\mathcal{E} \models (\top \vdash_{[]} (\exists x)\top)$ and $\mathcal{E} \models (\top \vdash_{x,x'} x = x')$ (here x and x' are of sort $\ulcorner A \urcorner$).

5. Let $\xi : X \to \mathrm{P}$ be an indexing of a set P of points of a Grothendieck topos $\mathcal{E}$ by a set X. Show that the image of the function $\phi_{\mathcal{E}} : \mathrm{Sub}_{\mathcal{E}}(1) \to \mathscr{P}(X)$ given by

$$\phi_{\mathcal{E}}(u) = \{x \in X \mid \xi(x)^*(u) \cong 1_{\mathbf{Set}}\} \ .$$

defines a topology on the set X, which we call the *subterminal topology.* We denote the set X endowed with this topology by $X_{\tau_\xi^{\mathcal{E}}}$. Show that

(a) If $\mathcal{P}$ is a preorder and ξ is the indexing $\{ev_p : \mathbf{Set} \to [\mathcal{P}, \mathbf{Set}] \mid p \in \mathcal{P}\}$ of set of points of the topos $[\mathcal{P}, \mathbf{Set}]$ by (the underlying set of) $\mathcal{P}$ (cf. problem **6**) then $\mathcal{P}_{\tau_\xi^{[\mathcal{P},\mathbf{Set}]}}$ is the Alexandrov space associated to $\mathcal{P}$.

(b) If X is a topological space and ξ is the indexing $\{F_x : \mathbf{Set} \to \mathbf{Sh}(X)\}$ of set of points of $\mathbf{Sh}(X)$ by the set X, where for each $x \in X$, F_x is the point of $\mathbf{Sh}(X)$ whose inverse image is the stalk functor at x (cf. problem **5** on sheet 1) then the space $\mathcal{X}_{\tau_\xi^{\mathbf{Sh}(X)}}$ is homeomorphic to X.

(c) If $\mathcal{B}$ is a Boolean algebra, J is the coherent topology on $\mathcal{B}$ and ξ is the identical indexing of the set X of points of the topos $\mathbf{Sh}(\mathcal{B}, J)$ by itself then the space $X_{\tau_\xi^{\mathbf{Sh}(\mathcal{B},J)}}$ is homeomorphic to the Stone space associated to the Boolean algebra $\mathcal{B}$.

6. Given an indexing $\xi : X \to \mathrm{P}$ of a set of points of a Grothendieck topos $\mathcal{E}$, let $\tilde{\xi} : [X, \mathbf{Set}] \to \mathcal{E}$ denote the associated geometric morphism, as in problem **6** on sheet 3. Show that if $\xi : X \to \mathrm{P}$ (resp. $\xi' : Y \to \mathrm{Q}$) is an indexing of a set of points of a Grothendieck topos $\mathcal{E}$ (resp. $\mathcal{F}$) and $f : \mathcal{E} \to \mathcal{F}$ is a geometric morphism such that the diagram

$$\begin{array}{ccc} [X, \mathbf{Set}] & \xrightarrow{E(l)} & [Y, \mathbf{Set}] \\ \downarrow{\scriptstyle \tilde{\xi}} & & \downarrow{\scriptstyle \tilde{\xi'}} \\ \mathcal{E} & \xrightarrow{f} & \mathcal{F} \end{array}$$

commutes (up to isomorphism), where $E(l) : [X, \mathbf{Set}] \to [Y, \mathbf{Set}]$ is the geometric morphism induced by the functor $l : X \to Y$, then $l : X_{\tau_\xi^{\mathcal{E}}} \to Y_{\tau_{\xi'}^{\mathcal{F}}}$ is a continuous map of topological spaces.