TOPOS THEORY EXAMPLES 3 (Lent Term 2012)

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1. Let $\mathcal{E}$ be a topos. Show that the following conditions are equivalent:

(a) For any subobject $A' \to A$, we have $A' \cup \neg A' \cong A$.

(b) $(\top, \bot)\colon 1 + 1 \to \Omega$ is an isomorphism.

(c) The sequent $(\top \vdash_x x \vee \neg x = 1)$ written in the theory of Heyting algebras is valid in the internal Heyting algebra $\Omega_{\mathcal{E}}$ of $\mathcal{E}$ given by its subobject classifier.

A topos $\mathcal{E}$ satisfying any of these conditions is said to be a *Boolean topos*. Show that a topos of sheaves $\mathbf{Sh}(\mathcal{C}, J)$ on a site $(\mathcal{C}, J)$ is Boolean if and only if for every object $c \in \mathcal{C}$ and J-closed sieve R on c,

$$\{f : d \to c \mid (f^*(R) = R_d) \text{ or } (f \in R)\} \in J(c),$$

where R_e denotes the J-closure of the empty sieve on e, i.e. the sieve $R_e := \{f : d \to e \mid \emptyset \in J(d)\}$ (for each $e \in \mathcal{C}$).
Specialize this characterization to obtain necessary and sufficient conditions on a small category $\mathcal{C}$ for the presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ to be Boolean, and on a topological space X for the topos $\mathbf{Sh}(X)$ to be Boolean.

2. If $A_1 \to A$ and $A_2 \to A$ are subobjects in an elementary topos $\mathcal{E}$, verify that we always have $\neg(A_1 \cup A_2) \cong \neg A_1 \cap \neg A_2$, but that we do not necessarily have $\neg(A_1 \cap A_2) \cong \neg A_1 \cup \neg A_2$. [Consider subobjects of 1 in $\mathbf{Sh}(X)$, for a suitable space X.] Show further that the second condition (for all pairs of subobjects in $\mathcal{E}$) is equivalent to any of the following conditions:

(a) For any subobject $A' \to A$, we have $\neg A' \cup \neg\neg A' \cong A$.

(b) Every $\neg\neg$-closed subobject is complemented.

(c) $(\top, \bot) : 2 \to \Omega_{\neg\neg}$ is an isomorphism. [Here 2 denotes the coproduct of two copies of 1, and $\Omega_{\neg\neg}$ denotes the equalizer in $\mathcal{E}$ of the pair of arrows $1_\Omega, \neg \circ \neg : \Omega \to \Omega$.]

(d) The sequent $(\top \vdash_x \neg x \vee \neg\neg x = 1)$ written in the theory of Heyting algebras is valid in the internal Heyting algebra $\Omega_{\mathcal{E}}$ of $\mathcal{E}$ given by its subobject classifier.

A topos $\mathcal{E}$ satisfying any of these conditions is said to be a *De Morgan topos*. Show that a topos of sheaves $\mathbf{Sh}(\mathcal{C}, J)$ on a site $(\mathcal{C}, J)$ is De Morgan if and only if for every object $c \in \mathcal{C}$ and J-closed sieve R on c,

$$\{f : d \to c \mid (f^*(R) = R_d) \text{ or (for any } g : e \to d,\ g^*(f^*(R)) = R_e \text{ implies } g \in R_d)\}$$

belongs to $J(c)$, where R_e denotes the J-closure of the empty sieve on e, i.e. the sieve $R_e := \{f : d \to e \mid \emptyset \in J(d)\}$ (for each $e \in \mathcal{C}$).
Specialize this characterization to obtain necessary and sufficient conditions on a small category $\mathcal{C}$ for the presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ to be De Morgan, and on a topological space X for the topos $\mathbf{Sh}(X)$ to be De Morgan.

3. Let $(\mathcal{C}, J)$ be a site. Show that the subobject classifier $\Omega_{\mathbf{Sh}(\mathcal{C},J)}$ of the topos $\mathbf{Sh}(\mathcal{C}, J)$ has the structure of an internal Heyting algebra in $\mathbf{Sh}(\mathcal{C}, J)$ (i.e., it is a model of the theory of Heyting algebras in $\mathbf{Sh}(\mathcal{C}, J)$) with respect to the following operations:

$$0 : 1 \to \Omega_{\mathbf{Sh}(\mathcal{C},J)}$$

defined by setting $0(c)(*)$ equal to the J-closure of the empty sieve on c,

$$1 : 1 \to \Omega_{\mathbf{Sh}(\mathcal{C},J)}$$

defined by setting $1(c)(*)$ equal to the maximal sieve on c,

$$\wedge : \Omega_{\mathbf{Sh}(\mathcal{C},J)} \times \Omega_{\mathbf{Sh}(\mathcal{C},J)} \to \Omega_{\mathbf{Sh}(\mathcal{C},J)}$$

defined by setting $\wedge(c)(S, T) = S \cap T$ (for any $c \in \mathcal{C}$ and any J-closed sieves S and T on c),

$$\vee : \Omega_{\mathbf{Sh}(\mathcal{C},J)} \times \Omega_{\mathbf{Sh}(\mathcal{C},J)} \to \Omega_{\mathbf{Sh}(\mathcal{C},J)}$$

defined by setting $\vee(c)(S, T) = \{f : d \to c \text{ in } \mathcal{C} \mid f^*(S \cup T) \in J(d)\}$ (for any $c \in \mathcal{C}$ and any J-closed sieves S and T on c), and

$$\Rightarrow : \Omega_{\mathbf{Sh}(\mathcal{C},J)} \times \Omega_{\mathbf{Sh}(\mathcal{C},J)} \to \Omega_{\mathbf{Sh}(\mathcal{C},J)}$$

defined by setting $\Rightarrow(c)(S, T) = \{f : d \to c \text{ in } \mathcal{C} \mid f^*(S) \subseteq f^*(T)\}$ (for any $c \in \mathcal{C}$ and any J-closed sieves S and T on c).

4. Let $\mathcal{P}$ be a preorder. The *Alexandrov space* $\mathcal{A}_\mathcal{P}$ associated to $\mathcal{P}$ is the topological space whose underlying set is $\mathcal{P}$ and whose open sets are the upper sets in $\mathcal{P}$ (i.e. the subsets $S \subseteq \mathcal{P}$ such that for any $a, b \in \mathcal{P}$ with $a \leq b$, $a \in \mathcal{P}$ implies $b \in \mathcal{P}$). Using the technique 'toposes as bridges' (applied to the Morita-equivalence $\mathbf{Sh}(Id(\mathcal{P}^{\text{op}})) \simeq [\mathcal{P}, \mathbf{Set}]$ and to the topos-theoretic invariants 'to be a Boolean topos' and 'to be a De Morgan topos', in light of the site characterizations obtained in problems **1** and **2**), show that

(a) $\mathcal{A}_\mathcal{P}$ is almost discrete if and only if for any $p, q \in \mathcal{P}$, $p \leq q$ implies $q \leq p$.

(b) $\mathcal{A}_\mathcal{P}$ is extremally disconnected if and only if $\mathcal{P}$ satisfies the amalgamation property (i.e., for any elements $a, b, c \in \mathcal{P}$ such that $c \leq a, b$ there exists $d \in \mathcal{P}$ such that $a, b \leq d$).

5. Let L and L' be frames and let $f : L \to L'$ be a surjective frame homomorphism. Show that the induced geometric morphism $\mathbf{Sh}(L') \to \mathbf{Sh}(L)$ is a geometric inclusion. Deduce, by using the technique 'toposes as bridges' (applied to the Morita-equivalence $\mathbf{Sh}(Id(\mathcal{C})) \simeq [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ holding for any preorder $\mathcal{C}$, and to the invariant notion of subtopos (in the sense of equivalence class of geometric inclusions to a given topos)) that for any preorder $\mathcal{C}$ and any surjective frame homomorphism $f : Id(\mathcal{C}) \to F$ onto a frame F there exists a Grothendieck topology J on $\mathcal{C}$ such that $F \cong Id_J(\mathcal{C})$ and f corresponds, under this isomorphism, to the frame homomorphism $cl_J : Id(\mathcal{C}) \to Id_J(\mathcal{C})$ sending an ideal I on $\mathcal{C}$ to its J-closure. Is the Grothendieck topology J necessarily unique?

6. A *point* of a topos $\mathcal{E}$ is a geometric morphism $\mathbf{Set} \to \mathcal{E}$. Show that

(a) For any preorder $\mathcal{C}$ and Grothendieck topology J on $\mathcal{C}$, the points of the topos $\mathbf{Sh}(\mathcal{C}, J)$ correspond precisely to the *J-prime filters* on $\mathcal{C}$ (by a J-prime filter on $\mathcal{C}$ we mean a subset $F \subseteq \mathcal{C}$ such that F is non-empty, $a \in F$ implies $b \in F$ whenever $a \leq b$, for any $a, b \in F$ there exists $c \in F$ such that $c \leq a$ and $c \leq b$, and for any J-covering sieve $\{a_i \to a \mid i \in I\}$ in $\mathcal{C}$ if $a \in F$ then there exists $i \in I$ such that $a_i \in F$).

(b) For any frame L, the points of the topos $\mathbf{Sh}(L)$ correspond precisely to the frame homomorphisms $L \to \{0, 1\}$, equivalently to the *completely prime filters* on L (i.e., the subsets $S \subseteq L$ such that $1 \in S$, $a \wedge b \in S$ if and only if $a \in S$ and $b \in S$, and for any family of elements $\{a_i \mid i \in I\}$ whose join is a, $a \in S$ implies $a_i \in S$ for some i).

(c) For any small category $\mathcal{C}$ and any object c of $\mathcal{C}$, there is a point $ev_c : \mathbf{Set} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ of the topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ whose inverse image is the evaluation functor at the object c.

Show further that for any Grothendieck topos $\mathcal{E}$, any set of points P of a Grothendieck topos $\mathcal{E}$ indexed by a set X via a function $\xi : X \to P$ can be naturally identified with a geometric morphism $\tilde{\xi} : [X, \mathbf{Set}] \to \mathcal{E}$.

7. Let $\mathcal{C}$ be a meet-semilattice (regarded as a preorder category) and let J be a (subcanonical) topology on $\mathcal{C}$. Show, by using the technique 'toposes as bridges' (applied to the Morita-equivalence $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(Id_J(\mathcal{C}))$ and to the invariant notion of geometric morphism from a localic topos $\mathbf{Sh}(L)$ to a given topos), that for any monotone map $f : \mathcal{C} \to L$ to a frame L, f is a meet-semilattice homomorphism which sends every J-covering sieve to a jointly covering family in L if and only if there is a (unique) frame homomorphism $\tilde{f} : Id_J(\mathcal{C}) \to L$ such that $\tilde{f} \circ \eta = f$ (given by the formula

$\tilde{f}(I) = \bigvee_{c \in I} f(c)$ for any $I \in Id_J(\mathcal{C})$). Deduce that the assignment sending a filter F on $Id_J(\mathcal{C})$ to the J-prime filter $\{c \in \mathcal{C} \mid (c)\downarrow \in F\}$ on $\mathcal{C}$ defines a bijection between the completely prime filters on the frame $Id_J(\mathcal{C})$ and the J-prime filters on $\mathcal{C}$ (cf. problem **6**).