TOPOS THEORY EXAMPLES 2 (Lent Term 2012)

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1. Show that the *degenerate topos* **1** with one object and one (identity) morphism is initial (in a suitable weak sense) in the 2-category **Top** of elementary toposes and geometric morphisms, and that the cartesian product $\mathcal{E} \times \mathcal{F}$ of any two toposes is their coproduct in **Top**. Show also that coprojections in **Top** are inclusions, and that the square

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{E} \times \mathcal{F} \end{array}$$

is a pullback in an appropriate sense (i.e., coproducts in **Top** are disjoint).

2. Show that the topos **Set** is *subterminal* in **Top**, in the sense that, for any $\mathcal{E}$, there is (up to isomorphism) at most one geometric morphism f from $\mathcal{E}$ to **Set**. Show also that there exists a geometric morphism $\mathcal{E} \to \mathbf{Set}$ iff $\mathcal{E}$ has small hom-sets and arbitrary set-indexed copowers (that is, coproducts of constant set-indexed families) of 1.

3. Let X be a topological space. Show that there is a geometric surjection $\mathbf{Set}/X \to \mathbf{Sh}(X)$ whose inverse image sends a sheaf F to the disjoint union of its stalks x^*F, $x \in X$ (cf. question 5 on sheet 1), with the obvious projection to X, and whose direct image sends $p\colon E \to X$ to the sheaf F such that $F(U)$ is the set of all sections of p over U (that is, functions $s\colon U \to E$ such that $p \circ s$ is the inclusion $U \to X$).

4.

(a) Let $\mathcal{C}$ be a small category. Show that the collection of all non-empty sieves on objects of $\mathcal{C}$ is a Grothendieck topology (the *atomic topology*) iff $\mathcal{C}$ satisfies the condition that any diagram of the form

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ B & \longrightarrow & C \end{array}$$

can be completed to a commutative square.

(b) Show that the category of pullback-preserving functors $\mathbf{Ord}_{fm} \to \mathbf{Set}$ (where $\mathbf{Ord}_{fm}$ denotes the category of finite ordinals and order-preserving injections between them) and natural transformations between them is equivalent to the topos of sheaves for the atomic topology on $\mathbf{Ord}_{fm}^{\mathrm{op}}$.

5. Let $f : A \to B$ be an epimorphism in an elementary topos $\mathcal{E}$. Show that the pullback functor $k^* : \mathcal{E}/B \to \mathcal{E}/A$ is faithful [Use the fact that pullbacks of epimorphisms in an elementary topos are epimorphisms].

6.

(a) Given a Grothendieck topology J on a small category $\mathcal{C}$, an object A of $\mathcal{C}$ is said to be J-*irreducible* if the only J-covering sieve on A is the maximal one. If A is J-irreducible, show that any morphism $A \to B$ must belong to every J-covering sieve on B.

(b) J is said to be *rigid* if every object A of $\mathcal{C}$ is covered by the sieve generated by all morphisms $B \to A$ with B irreducible. Show that in this event $\mathbf{Sh}(\mathcal{C}, J)$ is equivalent to the functor category $[\mathcal{D}^{\text{op}}, \mathbf{Set}]$, where $\mathcal{D}$ is the full subcategory of irreducible objects of $\mathcal{C}$.

(c) If J is a coverage on a finite category $\mathcal{C}$, show that every object of $\mathcal{C}$ has a smallest J-covering sieve.

(d) We say a morphism $f : B \to A$ is *essential* in a sieve R on A if, whenever we have a factorization $f = g \circ h$ with $g \in R$, then h is a split monomorphism. If every object A of $\mathcal{C}$ has a smallest J-covering sieve R_A, and f is essential in R_A, show that its domain is J-irreducible. [Consider the sieve of all composites $g \circ h$, where $g \in R_A$ and $h \in R_{\text{dom } g}$.]

(e) Now suppose again that $\mathcal{C}$ is finite, and additionally that idempotents split in $\mathcal{C}$. Show that every sieve on an object of $\mathcal{C}$ is generated by its essential members, and deduce that every coverage on $\mathcal{C}$ is rigid. [Given $f_0 \in R$, define a sequence of morphisms (f_n) as follows: if f_n can be factored as $g \circ h$ with h split epic but not iso, choose one such factorization and set $f_{n+1} = g$. If this is impossible, but f_n can be factored as $g \circ h$ with $g \in R$ and h not split monic, choose one such factorization and set $f_{n+1} = g$. If neither is possible, then stop. What happens if the sequence continues for ever?]