TOPOS THEORY EXAMPLES 1 (Lent Term 2012)

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1. Let $\mathcal{C}$ be a category such that, for each object c, the slice category $\mathcal{C}/c$ is equivalent to a small category, even though $\mathcal{C}$ may not be small. Show that the functor category $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is an elementary topos [hint: show that the usual constructions of exponentials and Ω for small $\mathcal{C}$ yield set-valued rather than class-valued functors]. By considering the case when $\mathcal{C}$ is the ordered class $\mathbf{On}$ of ordinals, or otherwise, show that such toposes need not be locally small (i.e. their 'hom-sets' may be proper classes.).

2. Let $\mathcal{E}$ be a category with finite limits and a subobject classifier Ω, and let $f\colon \Omega \to \Omega$ be a monomorphism in $\mathcal{E}$. By considering the pullback squares

$$\begin{array}{ccccc} V & \rightarrowtail & U & \rightarrowtail & 1 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \top} \\ 1 & \xrightarrow{\ \top\ } & \Omega & \xrightarrow{\ f\ } & \Omega \end{array}$$

show that $f \circ f$ is the identity morphism 1_Ω. By considering the topos $[\mathbf{N}, \mathbf{Set}]$ where $\mathbf{N}$ is the ordered set of natural numbers, show that Ω may have epic endomorphisms which are not isomorphisms.

3. Let G be a topological group. The category $\mathbf{B}(G)$ of continuous G-sets has as objects sets X equipped with a right action $\xi : X \times G \to X$ which is continuous when X is equipped with the discrete topology and as arrows $(X, \xi) \to (Y, \xi')$ the functions $X \to Y$ which respect the action. Show that $\mathbf{B}(G)$ is an elementary topos.

4. Let X be a topological space. Given an open covering $\mathcal{U} = \{U_i \mid i \in I\}$ of an open set $U \subseteq X$, we write $S_\mathcal{U}$ for the subfunctor of $\mathbf{y}U = \mathcal{O}(X)(-, U)$ which is the union of the $\mathbf{y}U_i$, i.e.

$$S_\mathcal{U}(V) = \{*\} \text{ if } V \subseteq U_i \text{ for some } i$$

and $S_\mathcal{U}(V) = \emptyset$ otherwise. Show that a functor $F\colon \mathcal{O}(X)^{\text{op}} \to \mathbf{Set}$ is a sheaf iff, for each such $\mathcal{U}$, each morphism $S_\mathcal{U} \to F$ in $[\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$ has a unique extension to a morphism $\mathbf{y}U \to F$.

5. Let x be a point of a topological space X, and F a sheaf on X. If $s \in F(U)$ for some open neighbourhood U of x, the *germ* of $s$at x (denoted s_x) is its equivalence class under the relation which identifies $s \in F(U)$ with $t \in F(V)$ iff they agree when restricted to some open W with $x \in W \subseteq U \cap V$. The *stalk* x^*F of F at x is the set of all germs at x of elements of F defined on neighbourhoods of x. Show that x^* is a functor $\mathbf{Sh}(X) \to \mathbf{Set}$, that it preserves finite limits, and that it has a right adjoint.

6. Prove that for any arrow $f : Z \to Y$ in the category $\tilde{\mathcal{C}} := [\mathcal{C}^{\text{op}}, \mathbf{Set}]$, the pullback functor

$$f^* : \text{Sub}_{\tilde{\mathcal{C}}}(Y) \to \text{Sub}_{\tilde{\mathcal{C}}}(Z)$$

has both a left adjoint $\exists_f$ and a right adjoint $\forall_f$.