Toposes as 'bridges' for mathematics and artificial intelligence

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Workshop on Semantic Information and Communication: Towards a semantic 6G

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Plan of the talk

- The theme of unification in mathematics
- The role of logic
- Toposes as unifying 'bridges'
- Applications (actual and potential) to artificial intelligence:
  - Towards a theory of semantic information
  - Modelling of learning processes via proofs
  - The logic and geometry of images
  - Structural approximation theory
  - Transformers from a topos-theoretic perspective
  - Automated theorem proving

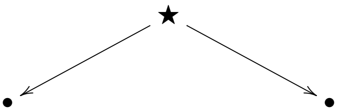
Unification and 'bridges' in mathematics

- Mathematics consists of several distinct areas (e.g., Algebra, Geometry, Analysis, Topology, Number Theory), each characterized by its own language and techniques.
- With time, various connections between the areas have been discovered, leading in some cases to the creation of actual 'bridges' between different mathematical branches (think for example of analytic geometry).
- The importance of 'bridges' between different areas lies in the fact that they make it possible to transfer knowledge and methods between the areas, so that problems formulated in the language of one field can be tacked (and possibly solved) using techniques from a different field.
- Mathematical logic and topos theory turn out to be fundamental tools for investigating the relations between different mathematical theories in a systematic and rigorous way.

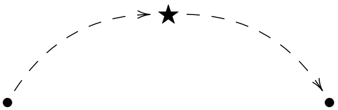
The concept of unification

We can distinguish between two different kinds of unification.

- 'Static' unification (through a generalization): two concepts are seen to be special instances of a more general one:



- 'Dynamic' unification (through a construction): two objects are related to each other through a third one (usually constructed from each of them), which acts like a 'bridge' enabling transfers of information between them.



Transfers of information arise from the process of 'translating' properties of (resp. constructions on) the 'bridge object' into properties of (resp. constructions on) the two objects.

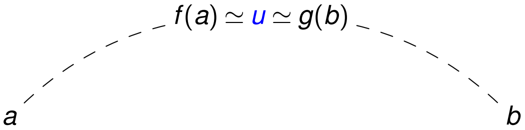
Unifying theories for mathematics

- Set theory and category theory realize a static unification of mathematics, essentially of linguistic nature. Indeed, each of these theories provide an abstract global framework in whose language most of mathematics can be formulated.

  Note that, even though each of them provides a way of expressing and organizing mathematics in one single language, these theories do not offer by themselves effective methods for an actual transfer of knowledge between distinct fields.
- Instead, as we shall see, toposes, as spaces on which the fundamental mathematical invariants are naturally defined, allow one to effectively connect different mathematical theories with each other, and also to study a given theory from a multiplicity of different points of view, thus defining a much more substantial, dynamical approach to the problem of 'unifying mathematics'.

Bridge objects

- Supposing we wish to compare two objects a and b with each other, imagine to be able to associate with a an object $f(a)$ through a certain ‘construction’ f and, analogously, to be able to associate with b an object $g(b)$ through a ‘costruction’ g, and that the objects $f(a)$ and $g(b)$ be related by a certain equivalence relation $\simeq$. Then these objects can be seen as two distinct represestations of a unique object u, equivalent on the one hand to $f(a)$ and on the other hand to $g(b)$, which can be used as a bridge objects between a and b as follows:

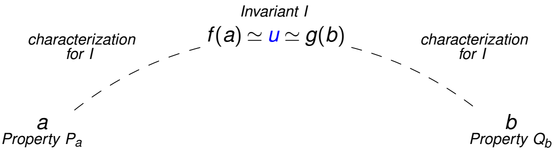


- Transfers of information arise from the process of ‘unraveling’ properties of (resp. constructions on) the ‘bridge object’ u into properties of (resp. constructions on) the two objects a and b by using the two different representations $f(a)$ and $g(b)$ of u.

Bridge objects

- More precisely, for any property I of the object u which is invariant with respect to the equivalence relation $\simeq$, we can try to characterize it in terms of a by using the representation $f(a)$ and in terms of b by using the representation $g(b)$, that is, to look for a property P_a of a equivalent to $f(a)$ satisfying I and, similarly, for a property Q_b of b equivalent to $g(b)$ satisfying I.

  The properties P_a and Q_b will thus be equivalent to each other, since each of them is equivalent to the invariant property I of the bridge object u:



- Note that, whilst the properties P_a and Q_b are *different manifestations* of a *unique* property, namely I, of the bridge object, they can be concretely completely different!

- In the topos-theoretic implementation of the 'bridge' technique, the objects a and b are mathematical contexts or theories while $f(a)$ and $g(b)$ are toposes associated with them. Note that a and b may also be seen as different 'points of view' on the bridge object u.

Logic, or the power of formalization

Mathematics can be considered, in a broad sense, as the discipline which studies everything that is formalizable, that is, definable in abstract and completely rigorous terms.

The branch of mathematics that specifically studies the various ways for formalizing concepts and reasonings is Logic. Logic allows one to study mathematical concepts and their relations in a fully rigorous way, to define the concept of mathematical theory and also, in a sense, to give a precise meaning to the intuitive notion of 'point of view'.

Indeed, different ways of thinking about or of constructing a given objects translate into different formalizations, which can be studied in themselves as well as in relation with each other through mathematical methods.

Syntax and semantics

One of the cornerstones of logic is the fundamental distinction between syntax and semantics.

The syntax is the set of ways (in the sense of 'grammatical forms') of linguistically describing a certain content.

The semantics is the set of possible ways of attributing meanings to given syntactic expressions (which in themselves have no meaning).

For example, natural languages (Italian, English, Japanese, Chinese etc.) have each its own syntax which provides a way for denoting real objects or concepts (through its vocabulary) and specifies rules for manipulating such expressions for constructing more complex ones (through its grammar).

In light of the distinction between syntax and semantics, we can interpret the existence of different points of view on a given theme as the existence of different syntax having a common semantics.

Mathematical logic thus provides a framework for formulating and investigating the relationships between different mathematical theories in a fully rigorous way, and, when combined with topos theory, it provides powerful tools for establishing deep connections across different mathematical areas.

The "unifying notion" of topos

In this talk the term 'topos' will always mean 'Grothendieck topos'.

"C'est le thème du topos qui est ce "lit", ou cette "rivière profonde" où viennent s'épouser la géométrie et l'algèbre, la topologie et l'arithmétique, la logique mathématique et la théorie des catégories, le monde du continu et celui des structures "discontinues" ou "discrètes". Il est ce que j'ai conçu de plus vaste, pour saisir avec finesse, par un même langage riche en résonances géométriques, une "essence" commune à des situations des plus éloignées les unes des autres provenant de telle région ou de telle autre du vaste univers des choses mathématiques".

A. Grothendieck

Since the times of my Ph.D. studies, I have developed a theory and a number of techniques allowing one to exploit the unifying potential of the notion of topos for establishing 'bridges' across different mathematical theories, by building in particular on the notion of classifying topos educed by categorical logicians.

Toposes as unifying 'bridges'

This theory, introduced in the programmatic paper "*The unification of Mathematics via Topos Theory*" of 2010, allows one to exploit the technical flexibility inherent to the concept of topos - most notably, the possibility of presenting a topos in a multitude of different ways - for building unifying 'bridges' useful for transferring notions, ideas and results across different mathematical contexts.

In the last years, besides leading to the solution of a number of long-standing problems in categorical logic, these techniques have generated several substantial applications in different mathematical fields. Still, much remains to be done so that toposes become a key tool universally used for investigating mathematical theories and their relations.

In fact, these 'bridges' have proved useful not only for connecting different mathematical theories with each other, but also for investigating a given theory from multiple points of view.

A few selected applications

Since the theory of topos-theoretic ‘bridges’ was introduced, several applications of it have been obtained in different fields of Mathematics, such as:

- Model theory (topos-theoretic Fraïssé theorem)
- Proof theory (various results for first-order theories)
- Algebra (topos-theoretic generalization of topological Galois theory)
- Topology (topos-theoretic interpretation/generation of Stone-type and Priestley-type dualities)
- Functional analysis (various results on Gelfand spectra and Wallman compactifications)
- Many-valued logics and lattice-ordered groups (two joint papers with A. C. Russo)
- Cyclic homology, as reinterpreted by A. Connes (work on *“cyclic theories”*, jointly with N. Wentzlaff)
- Algebraic geometry (logical analysis of (co)homological motives, cf. the paper *“Syntactic categories for Nori motives”* joint with L. Barbieri-Viale and L. Lafforgue)

Classifying toposes

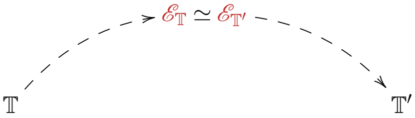
Grothendieck toposes are objects which are capable of capturing the essence of a great variety of different mathematical contexts. In particular, they can emboby the semantic content of a very wide class of theories:

Indeed, in the seventies, thanks to the work of a number of categorical logicians, notably including M. Makkai and G. Reyes, it was discovered that:

- With any mathematical theory $\mathbb{T}$ (of a very general form) one can canonically associate a topos $\mathscr{E}_{\mathbb{T}}$, called its classifying topos, which represents its 'semantical core'.
- Two given mathematical theories have the same classifying topos (up to equivalence) if and only if they have the same 'semantical core', that is, if and only if they are indistinguishable from a semantic viewpoint.
- Conversely, any topos is the classifying topos of some theory (in fact, of infinitely many theories).

Toposes as 'bridges'

- The notion of two theories having the same classifying topos formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods', which explains its ubiquity in Mathematics.
- The existence of different theories with the same classifying topos translates, at the technical level, into the existence of different representations for the same topos.
- Topos-theoretic invariants, that is properties of (or constructions on) toposes which are invariant with respect to their different representations, can thus be used to transfer information from one theory to another:



- Transfers of information take place by expressing a given invariant in terms of the different representations of the topos.

Towards a theory of semantic information

- As we explained above, toposes are objects which embody the semantic content of a wide class of theories, or the essence of a great number of mathematical contexts (pertaining to different areas of mathematics).

- It is therefore natural to develop a theory of semantic information based on toposes and on the invariants that one can define on them.

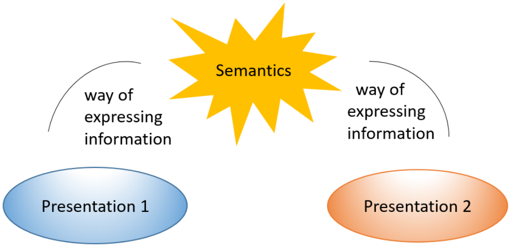
- As a matter of fact, the fundamental invariants of mathematical structures are actually invariants of toposes associated with these structures.

- Indeed, it is at the topos-theoretic level that invariants naturally live. This is due to the fact that toposes, unlike ordinary mathematical structures, have a very rich internal structure, actually being completions of concrete theories or structures with respect to all the natural operations that one might want to perform on them.

Communication through 'bridges'

Topos-theoretic 'bridges' have proved to be very effective in acting as 'universal translators', that is, as tools for unifying different presentations of a given semantic content and for transferring knoweldge between them.

We thus expect communication between different intelligent agents to be profitably understandable in terms of 'bridges' induced by equivalences (or more general relations) between toposes which describe their functioning. More generally, toposes embodying a given semantic content can act as 'bridges' across different knowledge representations:



Unification and computation

In artificial intelligence and beyond, unification and computation are two facets of the same phenomenon: the more there is understanding, the less 'brute force' is needed and calculations can be simplified.

One should thus aim for computations that are as conceptually inspired and enlightening as possible, and which therefore do not appear 'blind', but rather meaningful, to the human eye. In other words, one should aim for the smallest possible number of general methods (*unification*) and for the greatest possible number of concrete 'ingredients' to which such methods can be applied (*computational part*).

This goes hand in hand with the development of conceptual architectures which should embody a small number of fundamental principles and be as technically flexible as possible in relation to the applications that they guide and orient. This relies, in turn, on a continuous effort to try to isolate, in any situation, the conceptual part from the purely computational, 'routinary' or 'mechanical' one.

Modularity and continuity

The topos-theoretic methods which we have been elaborating in the past years are a compelling illustration of this philosophy.

They enjoy in particular strong forms of modularity and continuity, by virtue of their capacity of generating a sort of 'rain of results' around a given theme. Indeed, by enlightening the general architecture of proofs and where and how the hypotheses come into play, they put the mathematician in an ideal position to understand how different hypotheses can lead to different results, and hence to adapt or transport notions, techniques and results from one context to another.

Moreover, the study of invariants of toposes allows one to identify, in concrete mathematical contexts, the "good notions", that is those which correspond to topos-theoretic invariants (through their characterizations in terms of sites of other presentations) and which therefore admits an infinity of different equivalent reformulations in other contexts.

Modelling of learning processes via proofs

We plan to explore the possibility of modelling the functioning of (artificial) learning processes, in particular of deep neural networks, by building on the notion of mathematical proof:

- Note that any intelligent agent must, in order to get an effective understanding of (aspects of) the world, derive knowledge starting from certain 'sensory' inputs, which play a similar role to that of axioms for a mathematical theory, by following certain dynamical rules, which correspond to the inference rules of the logical system inside which the mathematical theory is formulated.
- As every mathematical theory can be enriched by the addition of new axioms, so the functioning of an agent can be updated by the integration of new information which becomes available to it.
- The functioning of a learning system can thus be modelled by a sequence of a mathematical theories, each of which more refined (that is, with more axioms, or fewer models) than the previous ones.

Modelling of learning processes via proofs

- In principle, any learning sequence, as any process 'approximating truth', is infinite, but for practical purposes one is normally satisfied by the result of a learning process when the last theory in the sequence leaves a degree of ambiguity which is sufficiently low for the desired applications. (For example, for a car with an object detection system, it is important to be able to identify the kind of animal that might cross the road, but not necessarily the color of its hair!).
- All the theories in the sequence should extend (that is, be defined over) the basic theory of the agent formalizing its essential features. More generally, every theory in the sequence can be seen as theory defined *over* each of the preceding ones.
- Note that any constraints embedded in the logical formalism (or integrated at some step of the sequence of theories) will allow to significantly reduce the space of parameters that the agent has to explore and hence correspondingly decrease the computational complexity of the learning process.

A topos-theoretic analysis of images

We also plan to investigate images from a topos-theoretic perspective:

- Logical axiomatisations of the objects (in the three-dimensional space) that images are meant to represent, together with a study of their two-dimensional projections, would greatly improve the systems for image recognition.

- On the other hand, images can also be profitably understood from a geometric, sheaf-theoretic, perspective, as arising from the glueing of local regions admitting simpler descriptions.

- The integration between logic and geometry provided by topos theory would allow one to switch from the logical point of view to the geometric one, thus taking advantage of both. In particular, a topos-theoretic treatment would allow a swift passage between different scales.

Structural approximation theory

The classical theory of neural networks is based on the approximation of real-valued functions and hence, ultimately, on a very particular metric space: $\mathbb{R}$ (with the usual Euclidean metric).

Numbers are not semantically meaningful by themselves; this makes any theory of deep learning which is based merely on them necessarily *fragile*, i.e. exposed to the risk of overfitting. In order to make a learning process *robust* and capable of *generalisation*, we need to make it structural.

As observed by L. Lafforgue, numbers should be thought as 'traces' of geometric structures. Accordingly, we should develop an approximation theory for structures rather than for mere numerical functions. These structures should result from logical constraints imposed at the outset (analogously to the *a priori* structures of our brain, which make us organize the data that we infer from our sensory inputs in a certain way) as well as from symmetries of the object of the learning process.

Ongoing experiments by the Semantics team of the Huawei Paris Research Center on training neural networks by implementing equivariance principles show how successful such an approach can be in significantly reducing the space of parameters to be explored.

Transformers from a topos-theoretic perspective

As already observed by Michael Robinson, trasformers should be formalized and investigated through a sheaf-theoretic perspective.

Indeed, the process of building a global view from a family of compatible local views is akin to the construction of elements of a sheaf as amalgamations of matching families of local data.

In order to formalize semantic information collected from different sets of sensory inputs, each of which using its own knowledge representation, it is more sensible, in order to have a common language in which all the data coming from the different inputs are formulated, to formalize the local view of each of the sensors by a topos, and to think of a transformer as a stack of such toposes.

Note that the use of toposes of local views as opposed to sets (as in Robinson's proposal) allows one to formalize non-trivial symmetries in the views of sensory inputs.

Automatic generation of theorems

Topos theory notably provides a very broad and effective setting for studying any aspects of mathematical theories, both in themselves and in relation to each other.

In fact, the theory of topos-theoretic 'bridges' could be implemented on a computer, so as to obtain a proof assistant capable to generate new results in any field of mathematics in an automatic way:

- Once an equivalence between two different presentations of the same topos is established, the calculation of how invariants express in terms of the two presentations is essentially canonical and can be automatized in many cases.

- This means that a computer could well be programmed in order to generate a huge amount of new (non-trivial) results in different mathematical fields by implementing these techniques.

  In fact, even results obtained by choosing as invariants easily computable ones are in general non-trivial.

For further reading

- O. Caramello
Grothendieck toposes as unifying 'bridges' : a mathematical morphogenesis,
in the Springer book *Philosophy of Mathematics. Objects, Structures, and Logics*.

- O. Caramello
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Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges',
Oxford University Press, 2017.