# Morita-equivalences for MV-algebras

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# Toposes

A Grothendieck topos $\mathscr{E}$ is a category that can be considered as a sort of mathematical universe enjoying the main categorical properties of the classical universe of sets:

- $\mathscr{E}$ is complete
- $\mathscr{E}$ is cocomplete
- $\mathscr{E}$ has exponentiation
- $\mathscr{E}$ has a subobject classifier

Grothendieck toposes are usually built as categories $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves of sets on a site $(\mathcal{C}, J)$. A site is by definition a pair consisting of a small category $\mathcal{C}$ and a Grothendieck topology $J$ on $\mathcal{C}$ (which defines a notion of covering of objects $c$ of $\mathcal{C}$ by families of arrows with codomain $c$).

# Model theory in toposes

We can consider models of arbitrary first-order theories in any Grothendieck topos $\mathscr{E}$.

Let $\Sigma$ (possibly multi-sorted) be a first-order signature. A *structure $M$* over $\Sigma$ in $\mathscr{E}$ is specified by the following data:

- any sort $A$ of $\Sigma$ is interpreted by an *object* $MA$ of $\mathscr{E}$
- any function symbol $f : A_1, \ldots, A_n \to B$ of $\Sigma$ is interpreted as an *arrow* $Mf : MA_1 \times \cdots \times MA_n \to MB$ in $\mathscr{E}$
- any relation symbol $R \rightarrowtail A_1, \ldots, A_n$ of $\Sigma$ is interpreted as a *subobject* $MR \rightarrowtail MA_1 \times \cdots \times MA_n$ in $\mathscr{E}$

Any formula $\vec{x}.\phi$ over $\Sigma$ is interpreted as a subobject $[[\vec{x}.\phi]]_M \rightarrowtail MA_1 \times \cdots \times MA_n$ defined recursively on the structure of the formula.

A *model* of a theory $\mathbb{T}$ over a first-order signature $\Sigma$ is a structure over $\Sigma$ in which all the axioms of $\mathbb{T}$ are satisfied.

# Geometric theories

Geometric logic represents the logic underlying Grothendieck toposes.

**Definition**

A geometric theory $\mathbb{T}$ is a theory over a first-order signature $\Sigma$ whose axioms can be presented in the form $(\phi \vdash_{\vec{x}} \psi)$, where $\phi$ and $\psi$ are *geometric formulae*, that is formulae with a finite number of free variables in the context $\vec{x}$ built up from atomic formulae over $\Sigma$ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.

Most of the first-order theories naturally arising in Mathematics are geometric; anyway, if a finitary first-order theory is not geometric, one can always canonically associate with it a geometric theory, called its *Morleyization*, having the same set-based models.

# Classifying toposes and Morita-equivalence

Every geometric theory $\mathbb{T}$ has a unique (up to categorical equivalence) classifying topos $\mathscr{E}_{\mathbb{T}}$ which satisfies the following universal property:

- for every Grothendieck topos $\mathscr{E}$ there is a categorical equivalence

$$(\text{models of } \mathbb{T} \text{ in } \mathscr{E}) \equiv (\text{morphisms } \mathscr{E} \to \mathscr{E}_{\mathbb{T}})$$

  naturally in $\mathscr{E}$.

## Definition

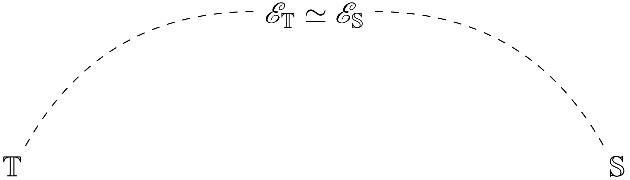
Two geometric theories $\mathbb{T}$ and $\mathbb{S}$ are said to be *Morita-equivalent* if they have equivalent classifying toposes.

## Trivial example

The theory of Boolean algebras and the theory of Boolean rings.

- More generally, any two bi-interpretable theories are (trivially) Morita-equivalent... but the really interesting Morita-equivalences are *not* of this form!

# Toposes as 'bridges'




If $\mathbb{T}$ and $\mathbb{S}$ are Morita-equivalent theories, we can transfer properties and results between them by using topos-theoretic invariants defined on their common classifying topos $\mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{S}}$.

# Main results

- Lift to Morita-equivalences of two well-known categorical equivalences between classes of MV-algebras and classes of lattice-ordered abelian groups, namely
  - Mundici's equivalence:

    category of MV-algebras $\simeq$ category of $\ell$-groups with strong unit
  - Di Nola-Lettieri's equivalence:

    category of perfect MV-algebras $\simeq$ category of $\ell$-groups
- Application of the method 'toposes as bridges' to these Morita-equivalences
- Construction (thanks to the study of certain classifying toposes) of a new class of (Morita-)equivalences that contains the one lifting Di Nola-Lettieri's equivalence

# Results in connection with Mundici's equivalence

- The theory of $\ell$-groups with strong unit is of presheaf type (i.e. classified by a presheaf topos) and in fact Morita-equivalent to an algebraic theory (namely that of MV-algebras)
- Bijective correspondence between the quotients (i.e. geometric theory extensions over the same signature) of the theory of MV-algebras and those of the theory of $\ell$-u groups (in spite of the fact that these theories are not bi-interpretable)
- Logical characterization of the finitely presentable $\ell$-u groups
- Form of compactness and completeness for the geometric theory of $\ell$-u groups (in spite of the infinitary nature of this theory)
- Sheaf-theoretic version of Mundici's equivalence

# Results in connection with Di Nola-Lettieri's equivalence

- The theory of perfect MV-algebras is of presheaf type and in fact Morita-equivalent to an algebraic theory (namely that of $\ell$-groups)
- Three levels of partial bi-interpretability for
  - irreducible formulas
  - sentences
  - imaginaries
- the finitely presentable models of the theory of perfect MV-algebras are finitely presentable also with respect to the variety generated by Chang's algebra
- Representation result: every finitely presentable MV-algebra in the variety generated by Chang's MV-algebra is a finite direct product of finitely presentable perfect MV-algebras
- a Morita-equivalence (actually, bi-interpretability) between the theory of lattice-ordered abelian groups and that of cancellative lattice-ordered abelian monoids with bottom element.

# Results for local MV-algebras in varieties

All the theories considered above are of presheaf type. This class of theories contains all the finitary algebraic theories and has many remarkable properties. We have shown that:

- The theory of local MV-algebras is NOT of presheaf type
- The theory of local MV-algebras in an arbitrary proper subvariety of MV-algebras IS of presheaf type
- The theory of local MV-algebras in an arbitrary proper subvariety of MV-algebras is Morita-equivalent to a theory extending that of lattice-ordered abelian groups
- the finitely presentable models of the theory of local MV-algebras in an arbitrary proper subvariety are finitely presentable also with respect to the variety
- The theory of simple MV-algebras is NOT of presheaf type, but the geometric theory of finite MV-chains IS
- Representation result: every finitely presentable MV-algebra in an abitrary proper subvariety of MV-algebras is a finite direct product of finitely presentable local MV-algebras in the variety

# MV-algebras

## Definition

An *MV-algebra* is an algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ with a binary operation $\oplus$, an unary operation $\neg$ and a constant 0, satisfying the following equations: for any $x, y, z \in A$

1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
2. $x \oplus y = y \oplus x$
3. $x \oplus 0 = x$
4. $\neg\neg x = x$
5. $x \oplus \neg 0 = \neg 0$
6. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

The standard MV-algebra is the real interval $([0,1], \oplus, \neg, 0)$, where

$$x \oplus y = min(1, x + y)$$
$$\neg x = 1 - x$$

We denote by $\mathbb{MV}$ the theory of MV-algebras

# Perfect MV-algebras

Let $\mathcal{A}$ be an MV-algebra

$Rad(\mathcal{A}) :=$ intersection of maximal ideals

$\neg Rad(\mathcal{A}) := \{x \in A \mid \neg x \in Rad(\mathcal{A})\}$

**Definition**

An MV-algebra $\mathcal{A}$ is called *perfect* if $\mathcal{A}$ is non-trivial and $\mathcal{A} = Rad(\mathcal{A}) \cup \neg Rad(\mathcal{A})$.

We denote by $\mathbb{P}$ the theory of perfect MV-algebras.

If $\mathcal{A}$ is an algebra in the variety generated by perfect MV-algebras, then $Rad(\mathcal{A}) = \{x \in A \mid (2x)^2 = 0\}$.

# Lattice-ordered abelian groups with strong unit

## Definition

A *lattice-ordered abelian group* ($\ell$-group for short) is a structure $\mathcal{G} = (G, +, -, \leq, \inf, \sup, 0)$, where $(G, +, -, 0)$ is an abelian group, $\leq$ is a partial order relation that induces a lattice structure and is compatible with addition, i.e. has the translation invariance property

$$\forall x, y, t \in G \quad x \leq y \Rightarrow t + x \leq t + y$$

## Definition

A distinguished element $u$ of $G$ is a *strong unit* if it satisfies the conditions:

- $u \geq 0$;
- for any $x \geq 0$ in $G$, there is a natural number $n$ such that $x \leq nu$.

We denote by $\mathbb{L}$ the theory of $\ell$-groups and by $\mathbb{L}_u$ the theory of $\ell$-groups with strong unit.

### Categorical equivalences

- Mundici's equivalence: $\Gamma : \mathbb{L}_u\text{-mod}(\mathbf{Set}) \simeq \mathbb{MV}\text{-mod}(\mathbf{Set})$
- Di Nola-Lettieri's equivalence: $\Delta : \mathbb{P}\text{-mod}(\mathbf{Set}) \simeq \mathbb{L}\text{-mod}(\mathbf{Set})$

### Lifts

- $\Gamma_{\mathcal{E}} : \mathbb{L}_u\text{-mod}(\mathcal{E}) \simeq \mathbb{MV}\text{-mod}(\mathcal{E})$
- $\Delta_{\mathcal{E}} : \mathbb{P}\text{-mod}(\mathcal{E}) \simeq \mathbb{L}\text{-mod}(\mathcal{E})$

for every Grothendieck topos $\mathcal{E}$, naturally in $\mathcal{E}$

### Morita-equivalences

- $\mathbb{MV}$ is Morita-equivalent to $\mathbb{L}_u$
- $\mathbb{P}$ is Morita-equivalent to $\mathbb{L}$

# Bijective correspondence between quotients

subtoposes

$\mathscr{E}_{\mathbb{MV}} \simeq \mathscr{E}_{\mathbb{L}_u}$

quotients of $\mathbb{MV}$

quotients of $\mathbb{L}_u$

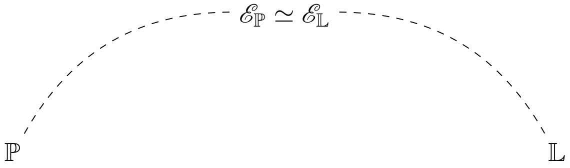
**Theorem**

*Every quotient of the theory $\mathbb{MV}$ is Morita-equivalent to a quotient of the theory $\mathbb{L}_u$, and conversely. These Morita-equivalences are the restrictions of the one between $\mathbb{MV}$ and $\mathbb{L}_u$.*

This result is non-trivial since the two theories are not bi-interpretable.

# Partial bi-interpretations

irreducible objects
subterminal objects
coherent objects




$\mathbb{P}$-irreducible formulas
geometric sentences over $\Sigma_{\mathbb{P}}$
imaginaries for $\mathbb{P}$

$\mathbb{L}$-irreducible formulas
geometric sentences over $\Sigma_{\mathbb{L}}$
imaginaries for $\mathbb{L}$

These bi-interpretations are interesting since we do not have bi-interpretability at the level of the coherent syntactic categories of the two theories.

# Representation result

**Theorem**

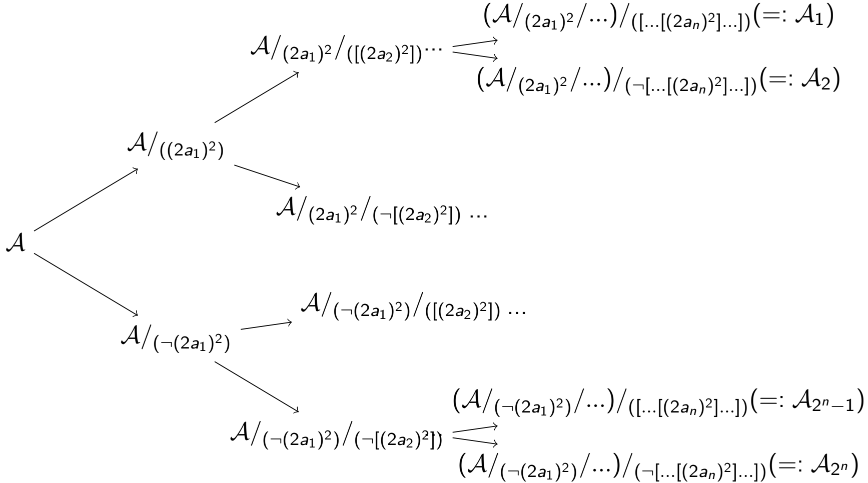
*Every finitely presentable MV-algebra in Chang's variety is a direct product of a finite family of finitely presentable perfect MV-algebras.*

Recall that the theory $\mathbb{P}$ is a quotient of the theory of Chang's variety obtained by adding the non-triviality axiom and the sequent

$$\top \vdash_x (2x)^2 = 0 \vee (2x)^2 = 1$$

If $\mathcal{A}$ is a finitely generated MV-algebra in Chang's variety and $\{a_1, \ldots, a_n\}$ is generating system for $\mathcal{A}$, then the final algebras in the following covering cosieve are perfect (or trivial) MV-algebras.

# Representation result



# Generalizing from perfect to local MV-algebras

$$\mathit{Perfect} = \mathit{Local} \cap V(S_1^{\omega})$$

We proved that $\mathbb{P}$ is a theory of presheaf type which is Morita-equivalent to the theory $\mathbb{L}$. It is natural to wonder if there is a geometric theory axiomatizing

$$\mathit{Local} \cap V$$

which is also of presheaf type and Morita-equivalent to a theory extending the theory $\mathbb{L}$.

# Local MV-algebras

Let $\mathcal{A}$ be an MV-algebra and $x \in A$
$ord(x) = min\{m \in \mathbb{N} \mid mx = 1\}$

**Definition**

An MV-algebra $\mathcal{A}$ is *local* if it is non-trivial and for every $x \in A$,

$$ord(x) < \infty \text{ or } ord(\neg x) < \infty$$

**Theorem**

*The geometric theory of local MV-algebras is not of presheaf type.*

# Local MV-algebras in a proper variety $V$

### Komori's theorem

An arbitrary proper subvariety of MV-algebras is of the form

$$V = V(\{S_i\}_{i \in I}, \{S_j^\omega\}_{j \in J})$$

where $S_i = \Gamma(\mathbb{Z}, i)$ are simple MV-algebras, $S_j^\omega = \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (j, 0))$ are called Komori chains and $I$ and $J$ are finite subsets of $\mathbb{N}$.

The least common multiple $n$ of the ranks of the generators is an invariant of the variety.

# First axiomatization

Let $\mathbb{T}_V$ be the theory of $V$ and let $\mathbb{L}oc_V$ be the theory of local MV-algebras in $V$. The axioms of $\mathbb{L}oc_V$ are the axioms of $\mathbb{T}_V$, the non-triviality axiom plus the sequent

$$\sigma_n : \top \vdash_x ((n+1)x)^2 = 0 \vee ((n+1)x)^2 = 1$$

where $n$ is the least common multiple of the ranks of the generators of $V$.

Since the theory $\mathbb{L}oc_V$ is a quotient of the theory $\mathbb{T}_V$, its classifying topos can be represented as a subtopos $\mathbf{Sh}((\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set}))^{op}, J_1)$ of the classifying topos $[\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set}), \mathbf{Set}]$ of $\mathbb{T}_V$, where $J_1$ is a (uniquely determined) Grothendieck topology on $(\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set}))^{\text{op}}$.

# Subcanonicity of $J_1$

The Grothendieck topology $J_1$ is generated by finite multicompositions of diagrams of the following form:

$$\mathcal{A} \to \mathcal{A}/((n+1)x)^2) \qquad \mathcal{A} \to \mathcal{A}/(\neg((n+1)x)^2)$$

The algebra $\mathcal{A}$ is the product of the algebras $\mathcal{A}/((n+1)x)^2)$ and $\mathcal{A}/(\neg((n+1)x)^2)$. This implies that $J_1$ is subcanonical.

# Cartesianisation of $\mathbb{L}oc_V$

**Theorem**

*Every cartesian sequent that is provable in the theory $\mathbb{L}oc_V$ is also provable in the theory $\mathbb{T}_V$.*

*Proof* The classifying topos of the theories $\mathbb{T}_V$ and $\mathbb{L}oc_V$ are the following:

$$\mathcal{E}_{\mathbb{T}_V} = [\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set}), \mathbf{Set}]$$
$$\mathcal{E}_{\mathbb{L}oc_V} = \mathbf{Sh}((\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set}))^{op}, J_1)$$

The universal model of $\mathbb{T}_V$ is the presheaf $\text{Hom}_{\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set})}(F, -)$, where $F$ is the free algebra in $V$ on one generator. By the subcanonicity of $J_1$, this also yields 'the' universal model of $\mathbb{L}oc_V$ in $\mathcal{E}_{\mathbb{L}oc_V}$. Since interpretations of cartesian formulas are the same in both classifying toposes, we have the thesis.

**Corollary**

*The radical of every algebra $\mathcal{A}$ in the variety $V$ is given by:*

$$Rad(\mathcal{A}) = \{x \in A \mid ((n+1)x)^2 = 0\}$$

# Topos-theoretic result

**Definition**

A Grothendick topology $J$ on a category $\mathcal{C}$ is rigid if every object $c \in \mathcal{C}$ has a $J$-covering generated by $J$-irreducible objects.

**Theorem (O.C.)**

*Let $\mathbb{T}'$ be a quotient of a theory of presheaf type $\mathbb{T}$ corresponding to a Grothendieck topology $J$ on the category $\text{f.p.}\mathbb{T}\text{-}mod(\mathbf{Set})^{op}$ under the duality theorem between quotients and subtoposes. Then $J$ is rigid if and only if $\mathbb{T}'$ is of presheaf type and every finitely presentable $\mathbb{T}'$-model is finitely presentable also as a $\mathbb{T}$-model.*

To prove the rigidity of the topology associated with $\mathbb{L}oc_V$ as a quotient of $\mathbb{T}_V$, we introduce a more refined axiomatization.

# Second axiomatization

A local MV-algebra $\mathcal{A}$ is of finite rank if $\mathcal{A}/Rad(\mathcal{A})$ is isomorphic to a finite simple MV-algebra $S_m$.

**Theorem (Di Nola-Esposito-Gerla)**

*Every local MV-algebra in $V$ has finite rank which divides that of one of the generators of $V$ (and hence $n$).*

So for any local MV-algebra $\mathcal{A}$ in $V$ we have a canonical MV-algebra homomorphism $\phi_{\mathcal{A}} : \mathcal{A} \to S_n$.
We exhibit Horn formulae $x \in Fin_d$ over the language of MV-algebras which define the radical classes $\phi_{\mathcal{A}}^{-1}(d)$ (for $d \in S_n$) of $\mathcal{A}$.
The axioms of $\mathbb{L}oc_V$ are the axioms of $\mathbb{T}_V$ plus the sequent

$$\rho_n : \top \vdash_x \bigvee_{d=0}^{n} x \in Fin_d$$

**Proposition**

The following sequent are provable in $\mathbb{T}_V$:

$$x \in Fin_d \wedge y \in Fin_b \vdash_{x,y} x \oplus y \in Fin_{d \oplus b}$$

$$x \in Fin_d \vdash_x \neg x \in Fin_{n-d}$$

# Main result

We call $J_2$ the Grothendieck topology associated with the second axiomatization. Its covering sieves contain finite multicompositions of diagrams of the form

$$\begin{array}{ccc} & & \mathcal{A}/(x \in Fin_0(\mathcal{A})) \\ & \nearrow & \vdots \\ \mathcal{A} & \longrightarrow & \mathcal{A}/(x \in Fin_d(\mathcal{A})) \\ & \searrow & \vdots \\ & & \mathcal{A}/(x \in Fin_n(\mathcal{A})) \end{array}$$

where $\mathcal{A}$ is a finitely presentable algebra in $V$. If we choose at each step the one of the generators of the algebra $\mathcal{A}$, the algebras at the ends of the resulting diagram are local MV-algebras. Thus $J_2$ is rigid.

**Theorem**

*The theory $\mathbb{L}oc_V$ is of presheaf type and the finitely presentable models of $\mathbb{L}oc_V$ are finitely presentable also as algebras in $V$.*

# Some corollaries

Since $J_1 = J_2$, the rigidity of $J_2$ implies that of $J_1$ and hence the following result:

**Theorem**

*Every finitely presentable algebra in $V$ is a finite product of (finitely presentable) local MV-algebras.*

The following proposition represents a constructive version of the theorem (by Di Nola-Esposito-Gerla) asserting that every MV-algebras has a biggest local subalgebra, holding for MV-algebras in a Komori variety $V$.

**Theorem**

*Let $\mathcal{A}$ be an MV-algebra in a Komori variety $V$ with invariant $n$. The biggest local subalgebra $A_{\text{loc}}$ of $\mathcal{A}$ is given by:*

$$\mathcal{A}_{\text{loc}} = \{x \in \mathcal{A} \mid x \in Fin_d(\mathcal{A}) \text{ for some } d \in \{0, \ldots, n\}\}.$$

# Representation theorem for local MV-algebras in a Komori variety

Theorem (Di Nola-Esposito-Gerla)

*Every local MV-algebra in $V$ is of finite rank and its rank divides one of the ranks of the generators of $V$. Further, any local MV-algebra of finite rank is of the form*

$$\Gamma(\mathbb{Z} \times_{lex} G, (k, g))$$

*where $G$ is an $\ell$-group, $g \in G$ and $k$ is the rank of the algebra.*

*Moreover, if $V = V(\{S_i\}_{i \in I}, \{S_j^{\omega}\}_{j \in J})$ then the class of local MV-algebras in $V$ coincides with the class consisting of the simple MV-algebras embeddable into a member of $\{S_i \mid i \in I\}$ and of the local MV-algebras $\mathcal{A}$ of finite rank such that $\mathcal{A}/Rad(\mathcal{A})$ is embeddable into a member of $\{S_j \mid j \in J\}$.*

# Extension of the theory $\mathbb{L}$

Let $\mathbb{G}_{(I,J)}$ be the theory whose signature is the one of $\ell$-groups to which we add an arbitrary constant and a 0-ary predicate $R_k$ for each divisor $k$ of the least common multiple of the numbers in $I$ and $J$. The axioms of this theory are

- axioms of $\mathbb{L}$
- $(\top \vdash R_1)$;
- $(R_k \vdash R_{k'})$, for each $k'$ which divides $k$;
- $(R_k \wedge R_{k'} \vdash R_{l.c.m.(k,k')})$, for any $k, k'$;
- $(R_k \vdash_g g = 0)$, for every $k \in \delta(I) \setminus \delta(J)$;
- $(R_k \vdash \bot)$, for any $k \notin \delta(I) \cup \delta(J)$.

where we indicate with $\delta(I)$ and $\delta(J)$ respectively the set of divisors of the numbers in $I$ and $J$.

The theory $\mathbb{G}_{(I,J)}$ is of presheaf type and models of $\mathbb{G}_{(I,J)}$ in **Set** can be identified with the triples $(G, g, k)$, where $G$ is an $\ell$-group, $g \in G$ and $k \in \delta(I) \cup \delta(J)$, such that if $k \in \delta(I) \setminus \delta(J)$ then $G$ is the trivial group.

# New (Morita-)equivalences

Let $V = V(\{S_i\}_{i \in I}, \{S_j^{\omega}\}_{j \in J})$ be an arbitrary proper subvariety of MV-algebras.

**Theorem**

*The category of local MV-algebras in $V$ and the category of (set-based) models of $\mathbb{G}_{(I,J)}$ are equivalent; since both $\mathbb{L}oc_V$ and $\mathbb{G}_{(I,J)}$ are of presheaf type, it follows that they are Morita-equivalent.*

- $\Lambda_{(I,J)} : \mathbb{L}oc_V\text{-mod}(\mathbf{Set}) \to \mathbb{G}_{(I,J)}\text{-mod}(\mathbf{Set})$

$$\Lambda_{(I,J)}(\mathcal{A}) := (G, g, k)$$

  for every $\mathcal{A} \simeq \Gamma(\mathbb{Z} \times_{lex} G, (k, g))$ local MV-algebra in $V$;
- $M_{(I,J)} : \mathbb{G}_{(I,J)}\text{-mod}(\mathbf{Set}) \to \mathbb{L}oc_V\text{-mod}(\mathbf{Set})$

$$M_{(I,J)}(G, g, k) := \Gamma(\mathbb{Z} \times_{lex} G, (k, g))$$

  for every $\mathbb{G}_{(I,J)}$-model $(G, g, k)$.

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