Stone-type dualities through topos-theoretic 'bridges'

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A machinery for generating dualities

- In this course we shall describe a general topos-theoretic machinery for building 'Stone-type' dualities, i.e. dualities or equivalences between categories of preorders and categories of posets, locales or topological spaces.
- This machinery allows one to unify the classical Stone-type dualities as instances of just one topos-theoretic phenomenon, and to generate many new such dualities.
- It results from an implementation of the view of Grothendieck toposes as unifying 'bridges' for transferring information between distinct mathematical theories.

Classical Stone-type dualities

In particular, we recover the following well-known dualities:

- Stone duality for distributive lattices (and Boolean algebras)
- Lindenbaum-Tarski duality for atomic complete Boolean algebras
- The duality between spatial frames and sober spaces
- M. A. Moshier and P. Jipsen's topological duality for meet-semilattices
- Alexandrov equivalence between preorders and Alexandrov spaces
- Birkhoff duality for finite distributive lattices
- The duality between algebraic lattices and sup-semilattices
- The duality between completely distributive algebraic lattices and posets

Grothendieck topologies

Definition

A Grothendieck topology on a (small) category $\mathscr{C}$ is a function J which assigns to each object c of $\mathscr{C}$ a collection $J(c)$ of sieves on c in such a way that

(i) (maximality axiom) the maximal sieve $M_c = \{f \mid \mathrm{cod}(f) = c\}$ is in $J(c)$;

(ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;

(iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

The sieves S which belong to $J(c)$ for some object c of $\mathscr{C}$ are said to be J-covering.

A site is a pair $(\mathscr{C}, J)$ where $\mathscr{C}$ is a small category and J is a Grothendieck topology on $\mathscr{C}$.

Categories of sheaves on a site

- A presheaf on a (small) category $\mathscr{C}$ is a functor $P : \mathscr{C}^{\text{op}} \to \mathbf{Set}$.
- Given a site $(\mathscr{C}, J)$, a presheaf on $\mathscr{C}$ is a J-sheaf if every matching family for a J-covering sieve S on any object of $\mathscr{C}$ (i.e. family of elements $\{x_f \in P(\mathrm{dom}(f)) \mid f \in S\}$ such that $x_{f \circ g} = P(g)(x_f)$ for any g composable with f) has a unique amalgamation (i.e. element x such that $P(f)(x) = x_f$ for all $f \in S$).
- The category $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on the site $(\mathscr{C}, J)$ is the full subcategory of $[\mathscr{C}^{\text{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.
- A Grothendieck topos is a category (equivalent to) the category of sheaves on a site.
- A Grothendieck topology J on a category $\mathscr{C}$ is said to be subcanonical if every representable functor on $\mathscr{C}$ is a J-sheaf, equivalently if the canonical functor $\mathscr{C} \to \mathbf{Sh}(\mathscr{C}, J)$ is a full embedding.

Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism.

Definition

A geometric morphism $f : \mathscr{E} \to \mathscr{F}$ consists of a pair of adjoint functors $f_* : \mathscr{E} \to \mathscr{F}$ (the direct image of f - the right adjoint) and $f^* : \mathscr{F} \to \mathscr{E}$ (the inverse image of f - the left adjoint) such that f^* preserves finite limits.

For example:

- For any site $(\mathscr{C}, J)$, there is a geometric morphism $\mathbf{Sh}(\mathscr{C}, J) \hookrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ whose direct image is the inclusion functor and whose inverse image is the associated sheaf functor.
- Any continuous map $f : X \to Y$ of topological spaces induces a geometric morphism $\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. More generally, any map of locales $f : L \to L'$ induces a geometric morphism $\mathbf{Sh}(L) \to \mathbf{Sh}(L')$, and any geometric morphism $\mathbf{Sh}(L) \to \mathbf{Sh}(L')$ is, up to equivalence, of this form.

One can induce geometric morphisms between Grothendieck toposes $\mathbf{Sh}(\mathscr{C}, J)$ and $\mathbf{Sh}(\mathscr{D}, K)$ starting from suitable functors between the sites $(\mathscr{C}, J)$ and $(\mathscr{D}, K)$ either contravariantly (through the so-called *morphisms of sites*) or covariantly (through the so-called *comorphisms of sites*).

Geometric morphisms

In particular:

(a) Any functor $F : \mathscr{C} \to \mathscr{C}'$ between categories $\mathscr{C}$ and $\mathscr{C}'$ with finite limits which preserves finite limits and is cover-preserving (i.e., sends J-covering sieves to families which generate a J'-covering sieve) induces a geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathscr{C}', J') \to \mathbf{Sh}(\mathscr{C}, J)$. If the topologies J and J' are subcanonical then F can be identified with the restriction of the inverse image $\mathbf{Sh}(F)^{*} : \mathbf{Sh}(\mathscr{C}, J) \to \mathbf{Sh}(\mathscr{C}', J')$ of $\mathbf{Sh}(F)$ to the representables.

(b) Any functor $f : \mathscr{C} \to \mathscr{C}'$ induces a geometric morphism $E(f) : [\mathscr{C}, \mathbf{Set}] \to [\mathscr{C}', \mathbf{Set}]$. If $\mathscr{C}$ and $\mathscr{C}'$ are Cauchy-complete then f can be identified with the restriction to the representables of the left adjoint $E(f)_{!} : [\mathscr{C}, \mathbf{Set}] \to [\mathscr{C}', \mathbf{Set}]$ to the inverse image of $E(f)$. If $\mathscr{C}$ and $\mathscr{C}'$ are Cauchy-complete, the geometric morphisms $[\mathscr{C}, \mathbf{Set}] \to [\mathscr{C}', \mathbf{Set}]$ of the form $E(f)$ for some functor $f : \mathscr{C} \to \mathscr{C}'$ can be intrinsically characterized as the *essential* ones (i.e., those whose inverse image admits a left adjoint).

The general methodology

The main idea consists in interpreting the fact that two structures $\mathscr{C}$ and $\mathscr{D}$ correspond to each other under a Stone-type duality in terms of the existence of a common topos $\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(\mathscr{D}, K)$ naturally attached to each of the structures independently from one another.

A natural source of equivalences of toposes

$$\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(\mathscr{D}, K),$$

is provided by Grothendieck's comparison lemma: $\mathscr{C}$ is a K-dense full subcategory of $\mathscr{D}$ (i.e. a full subcategory $\mathscr{C}$ of $\mathscr{D}$ such that for any object d of $\mathscr{D}$ the sieve generated by the arrows from objects of $\mathscr{C}$ to d is K-covering) and J is the induced Grothendieck topology $K|_{\mathscr{C}}$ on $\mathscr{C}$.

The general methodology

Given a bunch of such equivalences

$$\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}}) \simeq \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}}),$$

where the Grothendieck topologies $J_{\mathscr{C}}$ and $K_{\mathscr{D}}$ are intrinsically defined in terms of the categories $\mathscr{C}$ and $\mathscr{D}$, we will obtain, under some natural hypotheses which are satisfied in a large number of cases, dualities or equivalences between a category of structures $\mathscr{C}$ (whose morphisms are maps which induce geometric morphisms between the associated toposes $\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}})$, either covariantly or contravariantly) and a category of structures $\mathscr{D}$ (whose morphisms are maps which induce geometric morphisms between the associated toposes $\mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}})$, either covariantly or contravariantly).

The general methodology

The key point is the possibility, under those hypotheses, of recovering the structures $\mathscr{C}$ (resp. $\mathscr{D}$) from the corresponding toposes $\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}})$ (resp. $\mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}})$) by means of topos-theoretic invariants:

$$\begin{tikzcd}
 & \mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}}) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}}) \arrow[d] & \\
 & \mathbf{Sh}(\mathscr{C}', J_{\mathscr{C}'}) \arrow[r, phantom, "\simeq"] & \mathbf{Sh}(\mathscr{D}', K_{\mathscr{D}'}) & \\
\mathscr{C} & & & \mathscr{D} \arrow[d] \\
\mathscr{C}' \arrow[u] & & & \mathscr{D}'
\end{tikzcd}$$

(in this bridge the first arch is contravariant and the second is covariant, but all the variance possibilities are equally feasible).

Grothendieck topologies on preorders

Definition

Let $\mathscr{C}$ be a preorder.

(i) A (basis for a) Grothendieck topology on $\mathscr{C}$ is a function J which assigns to every element $c \in \mathscr{C}$ a family $J(c)$ of lower subsets of $(c)\downarrow$, called the J-covers on c, such that for any $S \in J(c)$ and any $c' \leq c$ the subset $S_{c'} = \{d \leq c' \mid d \in S\}$ belongs to $J(c')$.

(ii) A preorder site is a pair $(\mathscr{C}, J)$, where $\mathscr{C}$ is a preorder and J is a Grothendieck topology on $\mathscr{C}$.

(iii) A Grothendiek topology J on $\mathscr{C}$ is subcanonical if and only if for every $c \in \mathscr{C}$ and any subset $S \in J(c)$, c is the supremum in $\mathscr{C}$ of the elements $d \in S$ (i.e., for any element c' in $\mathscr{C}$ such that for every $d \in S$ $d \leq c'$, we have $c \leq c'$).

Examples of Grothendieck topologies

- If P is a preorder, the trivial topology on P is the one in which the only covers are the maximal ones.
- If D is a distributive lattice, the coherent topology on D is the one in which the covers are exactly those which contain finite families whose join is the given element.
- If F is a frame, the canonical topology on F is the one in which the covers are exactly the families whose join is the given element.
- If D is a disjunctively distributive lattice, the disjunctive topology on D is the one in which the covers are exactly those which contain finite families of pairwise disjoint elements whose join is the given element.
- If U is a k-frame, the k-covering topology on U is the one in which the covers are the those which contain families of less than k elements whose join is the given element.
- If V is a preframe, the directed topology on V is the one in which the covering sieves are precisely those which contain directed families of elements whose join is the given element.

J-ideals

Definition

Given a preorder site $(\mathscr{C}, J)$, a J-ideal on $\mathscr{C}$ is a subset $I \subseteq \mathscr{C}$ such that

- for any $a, b \in \mathscr{C}$ such that $b \leq a$ in $\mathscr{C}$, $a \in I$ implies $b \in I$, and
- for any J-cover R on an element c of $\mathscr{C}$, if $a \in I$ for every $a \in R$ then $c \in I$.

We denote by $Id_J(\mathscr{C})$ the set of all the J-ideals on $\mathscr{C}$.

Proposition

Let $\mathscr{C}$ be a preorder and J be a Grothendieck topology on $\mathscr{C}$. Then $(Id_J(\mathscr{C}), \subseteq)$ is a frame. In fact, we have an equivalence of toposes

$$\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(Id_J(\mathscr{C}))$$

and the J-ideals on $\mathscr{C}$ correspond precisely to the subterminal objects of this topos.

Remark

If J is subcanonical (i.e. all the principal ideals on $\mathscr{C}$ are J-ideals) and $\mathscr{C}$ is a poset then we have an embedding $\mathscr{C} \hookrightarrow Id_J(\mathscr{C})$, which identifies $\mathscr{C}$ with the set of principal ideals on $\mathscr{C}$.

Functorialization I

We can generate covariant or controvariant equivalences with categories of posets by appropriately functorializing the assignments above.

Definition

A morphism of sites $(\mathscr{C}, J) \to (\mathscr{D}, K)$, where $\mathscr{C}$ and $\mathscr{D}$ are meet-semilattices, is a meet-semilattice homomorphism $\mathscr{C} \to \mathscr{D}$ which sends J-covers to K-covers.

Theorem

1. *A morphism of sites $f : (\mathscr{C}, J) \to (\mathscr{D}, K)$ induces, naturally in f, a frame homomorphism $\dot{f} : Id_J(\mathscr{C}) \to Id_K(\mathscr{D})$. This homomorphism sends a J-ideal I on $\mathscr{C}$ to the smallest K-ideal on $\mathscr{D}$ containing the image of I under f.*
2. *If J and K are subcanonical then a frame homomorphism $Id_J(\mathscr{C}) \to Id_K(\mathscr{D})$ is of the form $\dot{f}$ for some f if and only if it sends principal ideals to principal ideals; if this is the case then f is isomorphic to the restriction of $\dot{f}$ to the principal ideals.*

Functorialization II

Theorem

Let $\mathscr{C}$ and $\mathscr{D}$ be two preorders. Then

1. *For any monotone map $f : \mathscr{C} \to \mathscr{D}$, the map $B_f : Id(\mathscr{D}) \to Id(\mathscr{C})$ sending an ideal I on $\mathscr{D}$ to the inverse image $f^{-1}(I)$ of I under f is a frame homomorphism.*
2. *A frame homomorphism $F : Id(\mathscr{D}) \to Id(\mathscr{C})$ is of the form B_f for some monotone map $f : \mathscr{C} \to \mathscr{D}$ if and only if F preserves arbitrary infima, equivalently if and only if it has a left adjoint $F_! : Id(\mathscr{C}) \to Id(\mathscr{D})$, given by the formula $F_!(I) = \bigcap_{I \subseteq F(I')} I'$ (for any $I \in Id(\mathscr{C})$).*
3. *If $\mathscr{C}$ and $\mathscr{D}$ are posets then any monotone map $f : \mathscr{C} \to \mathscr{D}$ can be recovered from B_f as the restriction of its left adjoint $(B_f)_!$ to the subsets of principal ideals.*

The general framework

We only discuss for simplicity the case of covariant equivalences with categories of frames, the other cases being conceptually similar to it.

Let $\mathscr{K}$ be a category of preordered structures, and suppose to have equipped each structure $\mathscr{C}$ in $\mathscr{K}$ with a Grothendieck topology $J_{\mathscr{C}}$ on $\mathscr{C}$ in such a way that every arrow $f : \mathscr{C} \to \mathscr{D}$ in $\mathscr{K}$ gives rise to a morphism of sites $f : (\mathscr{C}, J_{\mathscr{C}}) \to (\mathscr{D}, J_{\mathscr{D}})$.

These choices automatically induce a functor

$$A : \mathscr{K} \to \mathbf{Frm}$$

to the category **Frm** of frames sending any $\mathscr{C}$ in $\mathscr{K}$ to $Id_{J_{\mathscr{C}}}(\mathscr{C})$ and any $f : \mathscr{C} \to \mathscr{D}$ in $\mathscr{K}$ to the frame homomorphism $\dot{f} : Id_{J_{\mathscr{C}}}(\mathscr{C}) \to Id_{J_{\mathscr{D}}}(\mathscr{D})$.

With the above notation, if all the Grothendieck topologies $J_{\mathscr{C}}$ are subcanonical and the preorders in $\mathscr{K}$ are posets then the functor $A : \mathscr{K} \to \mathbf{Frm}$ yields an embedding of $\mathscr{K}$ into **Frm**.

Recovering the structures through invariants

- It would thus be desirable to have an equivalence of $\mathscr{K}$ with a subcategory of **Frm** which is closed under isomorphisms in **Frm** (namely, the closure $ExtIm(A)$ of the image of A under isomorphisms in **Frm**) and whose objects and arrows admit an intrinsic description in frame-theoretic terms.
- To achieve this, we investigate the problem of recovering a preorder $\mathscr{C}$ in $\mathscr{K}$ from the topos $\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}})$ (equivalently, from the frame $Id_{J_{\mathscr{C}}}(\mathscr{C})$) through an invariant, functorially in $\mathscr{C}$.
- It turns out that if the topologies $J_{\mathscr{C}}$ can be 'uniformly described through an invariant' C (namely C-induced in the sense of the following definition) then the principal ideals on $\mathscr{C}$ can be characterized among the elements of the frame $Id_{J_{\mathscr{C}}}(\mathscr{C})$ precisely as the ones which are C-compact.
- This enables us to define a functor on the category $ExtIm(A)$ which yields, together with A, the desired equivalence.

Topologies defined through invariants

Definition

Let C be a frame-theoretic invariant property of families of elements of a frame (for example: to be finite, to be a singleton, to be of cardinality at most k for some cardinal k, to be formed by elements which are pairwise disjoint, to be directed etc.)

- Given a structure $\mathscr{C}$ in $\mathscr{K}$, the Grothendieck topology $J_{\mathscr{C}}$ is said to be C-induced if for any J_{can}^{F}-dense monotone embedding $i : \mathscr{C} \hookrightarrow F$ into a frame F (where J_{can}^{F} is the canonical topology on F) possibly satisfying some invariant property P which is known to hold for the canonical embedding $\mathscr{C} \hookrightarrow Id_{J_{\mathscr{C}}}(\mathscr{C})$ and such that the $J_{\mathscr{C}}$-covers on $\mathscr{C}$ are sent by i to covers in F, for any family $\mathscr{A}$ of elements in $\mathscr{C}$ there exists a $J_{\mathscr{C}}$-cover S on an element $c \in \mathscr{C}$ such that the elements $a \in \mathscr{A}$ such that $a \leq c$ generate S if and only if the image $i(\mathscr{A})$ of the family $\mathscr{A}$ in F has a refinement satisfying C made of elements of the form $i(c')$ (for $c' \in \mathscr{C}$).

Proposition

The trivial (resp. coherent, canonical, k-covering, disjunctive, directed) topology is C-induced where C is the invariant 'to be a singleton' (resp. 'to be finite', 'to be any family', 'to be of cardinality at most k','to be formed by elements which are pairwise disjoint', 'to be directed').

A key result

Definition

An element u of a frame F is said to be C-compact if every covering of u in F has a refinement satisfying C.

Theorem

If all the Grothendieck topologies $J_{\mathscr{C}}$ associated to the structures $\mathscr{C}$ in $\mathscr{K}$ are C-induced and the invariant C satisfies the property that for any structure $\mathscr{C}$ in $\mathscr{K}$ and for any family $\mathscr{F}$ of principal $J_{\mathscr{C}}$-ideals on $\mathscr{C}$, $\mathscr{F}$ has a refinement satisfying C (if and) only if it has a refinement satisfying C made of principal $J_{\mathscr{C}}$-ideals on $\mathscr{C}$ then the functor $\mathrm{ExtIm}(A) \to \mathscr{K}$ sending a frame F in $\mathrm{ExtIm}(A)$ to the poset of C-compact elements of F and acting on the arrows accordingly is a categorical quasi-inverse to A.

The target categories of frames

Theorem

- *The frames in $\mathrm{ExtIm}(A)$ are precisely the frames F with a basis B_F of C-compact elements which, regarded as a poset with the induced order, belongs to $\mathscr{K}$, and such that the embedding $B_F \hookrightarrow F$ satisfies property P, the property that every covering in F of an element of B_F is refined by a covering made of elements of B_F which satisfies the invariant C, and the property that the J_{B_F}-covering sieves are sent by the embedding $B_F \hookrightarrow F$ into covering families in F (where J_{B_F} is the Grothendieck topology with which B_F comes equipped as a structure in $\mathscr{K}$).*
- *The arrows $F \to F'$ in $\mathrm{ExtIm}(A)$ are the frame homomorphisms which send C-compact elements to C-compact elements in such a way that their restriction to the subsets of C-compact elements can be identified with an arrow in $\mathscr{K}$.*

The subterminal topology

For obtaining dualities with categories of topological spaces rather than locales/frames, one can use the following construction, which provides a canonical way for endowing a given set of points of a topos with a natural topology.

Definition

Let $\xi : X \to \mathsf{P}$ be an indexing of a set P of points of a Grothendieck topos $\mathscr{E}$ by a set X. We define the subterminal topology $\tau_{\xi}^{\mathscr{E}}$ as the image of the frame homomorphism $\phi_{\mathscr{E}} : \mathrm{Sub}_{\mathscr{E}}(1) \to \mathscr{P}(X)$ given by

$$\phi_{\mathscr{E}}(u) = \{x \in X \mid \xi(x)^*(u) \cong 1_{\mathbf{Set}}\} .$$

We denote the topological space obtained by endowing the set X with the topology $\tau_{\xi}^{\mathscr{E}}$ by $X_{\tau_{\xi}^{\mathscr{E}}}$.

The interest of this notion lies in its level of generality, as well as in its formulation as a topos-theoretic invariant admitting a 'natural behaviour' with respect to sites. Moreover, the following fact will be crucial for us.

Remark

If P is a separating set of points for $\mathscr{E}$ (for example, the set of all the points of a localic topos having enough points) then the frame $\mathscr{O}(X_{\tau_{\xi}^{\mathscr{E}}})$ of open sets of the space $X_{\tau_{\xi}^{\mathscr{E}}}$ is isomorphic (via $\phi_{\mathscr{E}}$) to the frame $\mathrm{Sub}_{\mathscr{E}}(1)$ of subterminals of the topos $\mathscr{E}$.

Examples of subterminal topologies I

Definition

Let $(\mathscr{C}, \leq)$ be a preorder. A J-prime filter on $\mathscr{C}$ is a subset $F \subseteq \mathscr{C}$ such that F is non-empty, $a \in F$ implies $b \in F$ whenever $a \leq b$, for any $a, b \in F$ there exists $c \in F$ such that $c \leq a$ and $c \leq b$, and for any J-covering sieve $\{a_i \to a \mid i \in I\}$ in $\mathscr{C}$ if $a \in F$ then there exists $i \in I$ such that $a_i \in F$.

Theorem

Let $\mathscr{C}$ be a preorder and J be a Grothendieck topology on it. Then the space $X_{\tau_{\mathbf{Sh}(\mathscr{C},J)}}$ of points of the topos $\mathbf{Sh}(\mathscr{C}, J)$ has as set of points the collection $\mathscr{F}^J_{\mathscr{C}}$ of J-prime filters on $\mathscr{C}$ and as open sets the sets the form

$$\mathscr{F}_I = \{F \in \mathscr{F}^J_{\mathscr{C}} \mid F \cap I \neq \emptyset\},$$

where I ranges among the J-ideals on $\mathscr{C}$. In particular, a sub-basis for this topology is given by the sets

$$\mathscr{F}_c = \{F \in \mathscr{F}^J_{\mathscr{C}} \mid c \in F\},$$

where c varies among the elements of $\mathscr{C}$.

Examples of subterminal topologies II

- The Alexandrov topology ($\mathscr{E} = [P, \mathbf{Set}]$, where P is a preorder and ξ is the indexing of the set of points of $\mathscr{E}$ corresponding to the elements of P)
- The Stone topology for distributive lattices ($\mathscr{E} = \mathbf{Sh}(D, J_D^{coh})$ and ξ is an indexing of the set of all the points of $\mathscr{E}$, where D is a distributive lattice and J_D^{coh} is the coherent topology on it)
- A topology for meet-semilattices ($\mathscr{E} = [M^{op}, \mathbf{Set}]$ and ξ is an indexing of the set of all the points of $\mathscr{E}$, where M is a meet-semilattice)
- The space of points of a locale ($\mathscr{E} = \mathbf{Sh}(L)$ for a locale L and ξ is an indexing of the set of all the points of $\mathscr{E}$)
- A logical topology ($\mathscr{E} = \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the classiying topos of a geometric theory $\mathbb{T}$ and ξ is any indexing of the set of all the points of $\mathscr{E}$ i.e. set-based models of $\mathbb{T}$)
- The Zariski topology

...

Dualities with categories of topological spaces

- By using the subterminal topology, we can 'lift' the equivalences with frames established above to dualities with topological spaces, provided that the toposes involved have enough points.
- Indeed, the construction of the subterminal topology can be naturally made functorial.
- Thus, by assigning sets of points of the toposes corresponding to the structures in a natural way, we obtain a functor $\tilde{A} : \mathscr{K} \to \mathbf{Top}^{\mathrm{op}}$ such that $\mathscr{O} \circ \tilde{A} \cong A$, where $\mathscr{O} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Frm}$ the usual functor taking the frame of open sets of a topological space:

$$\begin{tikzcd} & \mathbf{Top}^{\mathrm{op}} \arrow[d, "\mathscr{O}"] \\ \mathscr{K} \arrow[ur, dashed, "\tilde{A}"] \arrow[r, "A"'] & \mathbf{Frm} \end{tikzcd}$$

The case of Stone duality

- Stone duality between the category of distributive lattices and that of coherent spaces is obtained by functorializing the equivalences of the form

$$\mathbf{Sh}(D, J_D^{\mathrm{coh}}) \simeq \mathbf{Sh}(X_D),$$

  where D is any distributive lattice and X_D is the Stone space associated with D.
- Indeed, the morphisms $D \to D'$ of distributive lattices are precisely the morphisms of sites $(D, J_D^{\mathrm{coh}}) \to (D', J_{D'}^{\mathrm{coh}})$, and any distributive lattice D can be recovered from $\mathbf{Sh}(D, J_D^{\mathrm{coh}})$ as the lattice of its compact subterminals; accordingly, the arrows in the target category are the continuous maps between coherent spaces whose inverse image send compact open sets to compact open sets.
- The space X_D is the space of points of the locale $Id_{J_D^{\mathrm{coh}}}(D)$ of ideals of D. As predicted by our theorem, the coherent spaces are precisely the sober topological spaces with a basis of compact open sets which forms a distributive lattice (equivalently, with a basis of compact open sets which is closed under finite intersections).

The case of Lindenbaum-Tarski duality

- Lindenbaum-Tarski duality between the category of sets and the category of complete atomic Boolean algebras and frame homomorphisms between them which preserve arbitrary infima can be obtained by functorializing the equivalences of the form
$$[A, \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{P}(A)),$$
where A is any set and $\mathscr{P}(A)$ is the powerset of A, or of the form
$$\mathbf{Sh}(B) \simeq \mathbf{Sh}(\mathrm{At}(B)),$$
where B is any complete atomic Boolean algebra and $\mathrm{At}(B)$ is the set of its atoms. Here B is viewed as a frame and equipped with the canonical topology, with respect to which the full subcategory $\mathrm{At}(B)$ of B is dense (by definition of atomic frame).
- A geometric morphism $[A, \mathbf{Set}] \to [B, \mathbf{Set}]$ (resp. a frame homomorphism $\mathscr{P}(B) \to \mathscr{P}(A)$) is of the form $E(f)$ for some map $f : A \to B$ (resp. is of the form $\mathscr{P}(f)$ for some map $f : A \to B$) if and only if it is essential (resp. it admits a left adjoint or, equivalently, it preserves arbitrary infima).

The general case

- Functorializing general equivalences

$$\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(\mathscr{D}, K)$$

(where $\mathscr{C}$ is a K-dense subcategory of $\mathscr{D}$ and J is induced by K on $\mathscr{C}$), we are able to recover all the dualities mentioned at the beginning of the talk as special cases generated through our machinery.

- At the same time, our framework allows enough flexibility to construct many new dualities with particular properties.
- In fact, our machinery has essentially four degrees of freedom:
  - (i) The choice of the structures $\mathscr{C}$;
  - (ii) The choice of the structures $\mathscr{D}$;
  - (iii) The choice of the topologies K;
  - (iv) The choice of points of the toposes $\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(\mathscr{D}, K)$.

Examples of new dualities

Among the new dualities that we obtain though our machinery, we have:

- A duality between the category of meet-semilattices and meet-semilattices homomorphisms betweeen them and the category of locales whose objects are the locales with a basis of supercompact elements which is closed under finite meets and whose arrows are the locale maps whose associated frame homomorphisms send supercompact elements to supercompact elements.
- A duality between the category of disjunctively distributive lattices and the category whose objects are the sober topological spaces which have a basis of disjunctively compact open sets which is closed under finite intersection and satisfies the property that any covering of a basic open set has a disjunctively compact refinement by basic open sets and whose arrows are the continuous maps between such spaces such that the inverse image of any disjunctively compact open set is a disjunctively compact open set.

Examples of new dualities

- For any regular cardinal k, a duality between the category of k-frames and the category whose objects are the frames which have a basis of k-compact elements which is closed under finite meets and whose arrows are the frame homomorphisms between them which send k-compact elements to k-compact elements.
- A duality between the category of disjunctive frames and the category $\mathbf{Pos}_{dis}$ which has as objects the posets $\mathscr{P}$ such that for any $a, b \in \mathscr{P}$ there exists a family $\{c_i \mid i \in I\}$ of elements of $\mathscr{P}$ such that for any $p \in \mathscr{P}$, $p \leq a$ and $p \leq b$ if and only if $p \leq c_i$ for a unique $i \in I$ and as arrows $\mathscr{P} \to \mathscr{P}'$ the monotone maps $g : \mathscr{P} \to \mathscr{P}'$ such that for any $b \in \mathscr{P}'$ there exists a family $\{c_i \mid i \in I\}$ of elements of $\mathscr{P}$ such that for any $p \in \mathscr{P}$, $g(p) \leq b$ if and only if $p \leq c_i$ for a unique $i \in I$.

Examples of new dualities

- A duality between the category **DirIrrPFrm** of directedly generated preframes whose objects are the directedly generated preframes and whose arrows $\mathscr{D} \to \mathscr{D}'$ are the preframe homomorphisms $f : \mathscr{D} \to \mathscr{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathscr{D}}}(\mathscr{D}) \to Id_{J'_{\mathscr{D}}}(\mathscr{D}')$ which sends an ideal I of $\mathscr{D}$ to the ideal of $\mathscr{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category $\mathbf{Pos}_{dir}$ having as objects the posets $\mathscr{P}$ such that for any $a, b \in \mathscr{P}$ there is $c \in \mathscr{P}$ such that $c \leq a$ and $c \leq b$ and for any elements $d, e \in \mathscr{P}$ such that $d, e \leq a$ and $d, e \leq b$ there exists $z \in \mathscr{P}$ such that $z \leq a$, $z \leq b$, $d, e \leq z$, and as arrows $\mathscr{P} \to \mathscr{P}'$ the monotone maps $g : \mathscr{P} \to \mathscr{P}'$ with the property that for any $b \in \mathscr{P}'$ there exists $a \in \mathscr{P}$ such that $g(a) \leq b$ and for any two $u, v \in \mathscr{P}$ such that $g(u) \leq b$ and $g(v) \leq b$ there exists $z \in \mathscr{P}$ such that $u, v \leq z$ and $g(z) \leq b$.
  This duality restricts to the duality between algebraic lattices and sup-semilattices.
- An equivalence between the category of meet-semilattices and the category whose objects are the the meet-semilattices F with a bottom element 0_F which have the property that for any $a, b \in F$ with $a, b \neq 0$, $a \wedge b \neq 0$ and whose arrows are the meet-semilattice homomorphisms $F \to F'$ which send 0_F to $0_{F'}$ and any non-zero element of F to a non-zero element of F'.

Examples of new dualities

- A duality between the category **IrrDLat** whose objects are the irreducibly generated distributive lattices and whose arrows $\mathscr{D} \to \mathscr{D}'$ are the distributive lattices homomorphisms $f : \mathscr{D} \to \mathscr{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathscr{D}}}(\mathscr{D}) \to Id_{J'_{\mathscr{D}}}(\mathscr{D}')$ which sends an ideal I of $\mathscr{D}$ to the ideal of $\mathscr{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category $\mathbf{Pos}_{comp}$ whose objects are the posets $\mathscr{P}$ such that for any $a, b \in \mathscr{P}$ there exists a finite set of elements $\{c_k \mid k \in K\}$ such that for any $p \in \mathscr{P}$, $p \leq a$ and $p \leq b$ if and only if $p \leq c_k$ for some $k \in K$, and whose arrows $\mathscr{P} \to \mathscr{P}'$ are the monotone maps $g : \mathscr{P} \to \mathscr{P}'$ such that for any $q \in \mathscr{P}'$, there exists a finite family $\{a_k \mid k \in K\}$ of elements of $\mathscr{P}$ such that for any $p \in \mathscr{P}$, $g(p) \leq q$ if and only if $p \leq a_k$ for some $k \in K$.
  This duality restricts to Birkhoff duality.
- A duality between the category **AtDLat** whose objects are the atomic distributive lattices and whose arrows $\mathscr{D} \to \mathscr{D}'$ are the distributive lattices homomorphisms $f : \mathscr{D} \to \mathscr{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathscr{D}}}(\mathscr{D}) \to Id_{J'_{\mathscr{D}}}(\mathscr{D}')$ which sends an ideal I of $\mathscr{D}$ to the ideal of $\mathscr{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category $\mathbf{Set}_f$ whose objects are the sets and whose arrows $A \to B$ are the functions $f : A \to B$ such that the inverse image under f of any finite subset of B is a finite subset of A.
- ...

Other applications

The construction and study of new dualities generated through our machinery is *a priori* interesting since they have essentially the same level of 'mathematical depth' as the classical Stone duality.

On the other hand, a great amount of applications can be established by applying the technique of 'toposes as bridges' in the context of toposes associated with preordered structures. Examples include:

- Representation theorems for preordered structures (arising whenever one can recover a structure from a topos intrinsically built from another structure).
- Adjunctions between categories of preorders and categories of posets, frames or topological spaces resulting from geometric morphisms between toposes associated with these structures.
- Links between Stone-type dualities and free structures

Other applications

- Translations of properties of preordered structures into properties of the corresponding posets or topological spaces by means of suitable topos-theoretic invariants. This can be particularly useful for investigating the relationships between different 'bases' for the same structure (or more generally between different representations of a given structure or different languages for describing it).
- Construction and spatial realization of structures presented by generators and relations (by using the theory of classifying toposes and syntactic categories). In fact, the toposes of sheaves on preorder sites are precisely the classifying toposes of propositional theories.
- Completeness theorems for propositional logics
- Generation of dualities for other, possibly more complex, algebraic or topological structures (e.g. Priestley-type dualities).

Duality and Morita equivalence

- The methodology that we have used to produce our machinery can be adapted in a great variety of other situations to build dualities or equivalences; in fact, in every situation in which one disposes of different representations for certain toposes by means of some objects, one can try to 'functorialize' these representations and 'reconstruct' the given objects from the associated toposes to obtain a duality or equivalence for categories of such objects.
- Even when it is not possible to recover the objects from the corresponding topos by means of invariants, one can still effectively investigate how *properties* of the given objects (or constructions on them) reformulate in terms of properties of (or constructions on) the toposes associated with them and then how these rephrase in terms of other possible representations for the same toposes.
- This indicates that the notion of Morita equivalence (i.e. toposes associated with different structures being equivalent) is in a sense more fundamental than duality/categorical equivalence since it goes well beyond the traditional notion of 'dictionary'.

For further reading

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