Extensions of flat functors and theories of presheaf type

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Geometric theories

Definition

- A geometric formula over a signature Σ is any formula (with a finite number of free variables) built from atomic formulae over Σ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.
- A geometric theory over a signature Σ is any theory whose axioms are of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are geometric formulae over Σ and $\vec{x}$ is a context suitable for both of them.

Fact

Most of the first-order theories naturally arising in Mathematics are geometric; and if a finitary first-order theory is not geometric, we can always associate to it a finitary geometric theory over a larger signature (the so-called Morleyization of the theory) with essentially the same models in the category **Set** *of sets.*

Classifying toposes

Definition

Let $\mathbb{T}$ be a geometric theory over a given signature. A classifying topos of $\mathbb{T}$ is a Grothendieck topos $\mathbf{Set}[\mathbb{T}]$ such that for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathbf{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

natural in $\mathscr{E}$.

Theorem (Joyal-Makkai-Reyes, '70s)

Every geometric theory (over a given signature) has a classifying topos. Conversely, every Grothendieck topos arises as the classifying topos of some geometric theory.

The classifying topos of a geometric theory $\mathbb{T}$ can always be constructed canonically from the theory by means of a syntactic construction, namely as the topos of sheaves $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ on the geometric syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$ with respect to the syntactic topology $J_{\mathbb{T}}$ on it (i.e. the canonical Grothendieck topology on $\mathscr{C}_{\mathbb{T}}$).

The duality theorem

Definition

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. A quotient of $\mathbb{T}$ is a geometric theory $\mathbb{T}'$ over Σ such that every axiom of $\mathbb{T}$ is provable in $\mathbb{T}'$.
- Let $\mathbb{T}$ and $\mathbb{T}'$ be geometric theories over a signature Σ. We say that $\mathbb{T}$ and $\mathbb{T}'$ are syntactically equivalent, and we write $\mathbb{T} \equiv_s \mathbb{T}'$, if for every geometric sequent σ over Σ, σ is provable in $\mathbb{T}$ if and only if σ is provable in $\mathbb{T}'$.

Theorem (O.C., 2008)

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the $\equiv_s$-equivalence classes of quotients of $\mathbb{T}$ and the subtoposes of the classifying topos **Set**$[\mathbb{T}]$ *of $\mathbb{T}$.*

Theories of presheaf type

Definition

Following T. Beke, we say that a geometric theory is of presheaf type if it is classified by a presheaf topos.

Theories of presheaf type occupy a central role in Logic and Mathematics, as they are the basic ‘building blocks’ from which every geometric theory can be built.

Indeed, as every Grothendieck topos is a subtopos of a presheaf topos, so every geometric theory is a quotient of a theory of presheaf type (cf. the above-mentioned duality theorem).

In this talk, we shall present a characterization theorem providing explicit necessary and sufficient conditions for a theory to be of presheaf type.

Some examples

The class of theories of presheaf type contains a great variety of theories pertaining to different areas of Mathematics. For instance:

- All finitary algebraic (or, more generally, all cartesian) theories (Hakim, Gabriel-Ulmer)
- The theory of abstract intervals (classified by the simplicial topos) (Joyal)
- The theory of abstract circles (classified by Connes' topos) (Moerdijk)
- The theory of decidable objects (Johnstone and Wraith)
- The theory of Diers' fields (Johnstone)
- The geometric theory of finite sets (Johnstone and Wraith)
- The theory of flat modules over a commutative ring with unit (Beke)
- ... and many more!

Finitely presentable models

Definition

A model M of a theory of presheaf type $\mathbb{T}$ in the category **Set** is said to be finitely presentable if the functor $Hom_{\mathbb{T}\text{-mod}(\mathbf{Set})}(M, -) : \mathbb{T}\text{-mod}(\mathbf{Set}) \to \mathbf{Set}$ preserves filtered colimits.

We denote by $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ the category of finitely presentable $\mathbb{T}$-models and $\mathbb{T}$-model homomorphisms between them.

The centrality of the notion of theory of presheaf type is also explained by the fact that *every small category is, up to Cauchy-completion, of the form* $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ *for some theory of presheaf type* $\mathbb{T}$.

Fact

For any theory of presheaf type $\mathscr{C}$, *we have two different representations of its classifying topos:*

$$[f.p.\mathbb{T}\text{-}mod(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

Applying the 'bridge' technique

The existence of this double representation for the classifying topos allows the 'bridge' technique to be fruitfully applied, leading to a variety of results on theories of presheaf type (cf. my papers). For instance:

Theorem

Let M be a set-based model of a theory of presheaf type $\mathbb{T}$. Then M is finitely presented by a geometric formula over the signature of $\mathbb{T}$ if and only if it is finitely presentable.

Theorem

Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ, $A_1, \ldots, A_n$ a string of sorts of Σ and suppose we are given, for every finitely presentable **Set***-model M of $\mathbb{T}$ a subset R_M of $MA_1 \times \cdots \times MA_n$ in such a way that each $\mathbb{T}$-model homomorphism $h : M \to N$ maps R_M into R_N. Then there exists a geometric formula-in-context $\phi(x^{A_1}, \ldots, x^{A_n})$ such that $R_M = [[\phi]]_M$ for each M.*

Characterizing theories of presheaf type I

By Diaconescu's theorem, a geometric theory $\mathbb{T}$ is of presheaf type if and only if there exists an equivalence

$$\mathbb{T}\text{-mod}(\mathscr{E}) \simeq \mathbf{Flat}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, \mathscr{E}),$$

natural in $\mathscr{E}$.

In fact, without loss of generality, we can suppose this equivalence to be of the following form:

$$M \rightsquigarrow Hom^{\mathscr{E}}_{\underline{\mathbb{T}\text{-mod}(\mathscr{E})}}(\gamma^*_{\mathscr{E}}(-), M)$$

$$\tilde{F}(M_{\mathbb{T}}) \leftsquigarrow F,$$

where the functor $\tilde{F} : \mathscr{C}_{\mathbb{T}} \to \mathscr{E}$ denotes the extension of the flat functor F along the canonical geometric morphism

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \to \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

and $M_{\mathbb{T}}$ denotes the universal model of $\mathbb{T}$ inside $\mathscr{C}_{\mathbb{T}}$.

Characterizing theories of presheaf type II

- As these functors are always defined for *any* geometric theory, the requirement that they should be categorical inverses to each other naturally in $\mathscr{E}$ is logically equivalent to the property of $\mathbb{T}$ to be of presheaf type.
- But these requirements look very abstract and hardly useful in practice!
- Can we express them as a family of 'concrete' conditions that can be effectively used in practice to test whether a given theory is of presheaf type?
- The following theorem provides a positive answer to this question.
- We shall first give an abstract version of the theorem, and then proceed to obtain concrete reformulations of the various conditions.

The characterization theorem I

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then $\mathbb{T}$ is of presheaf type if and only if all of the following conditions are satisfied:

(i) For any $\mathbb{T}$-model M in a Grothendieck topos $\mathscr{E}$, the functor

$$H_M := Hom^{\mathscr{E}}_{\underline{\mathbb{T}\text{-mod}(\mathscr{E})}}(\gamma^*_{\mathscr{E}}(-), M) : \text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \to \mathscr{E}$$

is flat;

(ii) The canonical morphism $\tilde{H_M}(M_{\mathbb{T}}) \to M$ is an isomorphism;

(iii) Any of the following conditions (equivalent, under the assumptions (i) and (ii)) is satisfied:

(a) The correspondence $M \to H_M$ is natural in $\mathscr{E}$; that is, for any finitely presentable $\mathbb{T}$-model c and any $\mathbb{T}$-model M in a Grothendieck topos $\mathscr{E}$, for any geometric morphism $f : \mathscr{F} \to \mathscr{E}$, the canonical morphism

$$f^*(Hom^{\mathscr{E}}_{\underline{\mathbb{T}\text{-mod}(\mathscr{E})}}(\gamma^*_{\mathscr{E}}(c), M)) \to Hom^{\mathscr{F}}_{\underline{\mathbb{T}\text{-mod}(\mathscr{F})}}(\gamma^*_{\mathscr{F}}(c), f^*(M))$$

is an isomorphism;

The characterization theorem II

(b) For any flat functor $F : \text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \to \mathscr{E}$, the canonical natural transformation

$$F \to Hom^{\mathscr{E}}_{\underline{\mathbb{T}\text{-mod}(\mathscr{E})}}(\gamma^*_{\mathscr{E}}(-), \tilde{F}(M_{\mathbb{T}})) \cong Hom^{\mathscr{E}}_{\underline{\mathbf{Flat}_{J_{\mathbb{T}}}(\mathscr{C}_{\mathbb{T}},\mathscr{E})}}(\gamma^*_{\mathscr{E}} \circ y(-), \tilde{F})$$

is an isomorphism;

(c) The canonical functor

$$\mathbf{Flat}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, \mathscr{E}) \to \mathbf{Flat}_{J_{\mathbb{T}}}(\mathscr{C}_{\mathbb{T}}, \mathscr{E}) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

is full and faithful;

(d) Any finitely presentable $\mathbb{T}$-model is presented by a geometric formula over Σ and for any finitely presentable models M and N of $\mathbb{T}$ presented respectively by formulae $\{\vec{x} \,.\, \phi\}$ and $\{\vec{y} \,.\, \psi\}$ and any $\mathbb{T}$-model homomorphism $h : M \to N$ there exists a $\mathbb{T}$-provably functional geometric formula $\theta(\vec{x}, \vec{y}) : \{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, \psi\}$ which induces h.

Concrete reformulations - condition (i)

Theorem

Let $\mathbb{T}$ be a geometric theory, and let M be a $\mathbb{T}$-model in a Grothenieck topos $\mathscr{E}$ with a separating set S. Then condition (i) of the characterization theorem holds for M if and only if

(a) *There exists an epimorphic family $\{E_i \to 1_{\mathscr{E}} \mid i \in I, E_i \in S\}$ and for each $i \in I$ a finitely presentable $\mathbb{T}$-model c_i and a Σ-structure homomorphism $c_i \to Hom_{\mathscr{E}}(E_i, M)$;*

(b) *For any finitely presentable $\mathbb{T}$-models c and d and Σ-structure homomorphisms $x : c \to Hom_{\mathscr{E}}(E, M)$ (where $E \in S$) and $y : d \to Hom_{\mathscr{E}}(E, M)$ there exists an epimorphic family $\{e_i : E_i \to E \mid i \in I, E_i \in S\}$ and for each $i \in I$ a finitely presentable $\mathbb{T}$-model b_i, $\mathbb{T}$-model homomorphisms $u_i : c \to b_i$, $v_i : d \to b_i$ and a Σ-structure homomorphism $z_i : b_i \to Hom_{\mathscr{E}}(E_i, M)$ such that $Hom_{\mathscr{E}}(e_i, M) \circ x = z_i \circ u_i$ and $Hom_{\mathscr{E}}(e_i, M) \circ y = z_i \circ v_i$;*

(c) *For any two parallel arrows $u, v : d \to c$ between finitely presentable $\mathbb{T}$-models and any Σ-structure homomorphism $x : c \to Hom_{\mathscr{E}}(E, M)$ in $\mathscr{E}$ (where $E \in S$) for which $x \circ u = x \circ v$, there is an epimorphic family $\{e_i : E_i \to E \mid i \in I, E_i \in S\}$ in $\mathscr{E}$ and for each index i a homomorphism of finitely presentable $\mathbb{T}$-models $w_i : c \to b_i$ and a Σ-structure homomorphism $y_i : b_i \to Hom_{\mathscr{E}}(E_i, M)$ such that $w_i \circ u = w_i \circ v$ and $y_i \circ w_i = Hom_{\mathscr{E}}(e_i, M) \circ x$.*

Concrete reformulations - condition (ii)

Theorem

Let $\mathbb{T}$ be a geometric theory, and let M be a $\mathbb{T}$-model in a Grothenieck topos $\mathscr{E}$ with a separating set S. Then condition (ii) of the characterization theorem holds for M if and only if for any sort A over Σ, both of the following conditions are satisfied (where $\mathscr{A}_{\{x^A.\top\}}$ denotes the collection of pairs of the form (c, z), where c is a finitely presentable $\mathbb{T}$-model and $z \in cA$):

(a) *For any generalized element $x : E \to MA$ there exists an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ and for each index $i \in I$ an element (c_i, z_i) of $\mathscr{A}_{\{x^A.\top\}}$ and a Σ-homomorphism $f_i : c_i \to Hom_{\mathscr{E}}(E_i, M)$ such that $(f_i A)(z_i) = x \circ e_i$;*

(b) *For any two elements (c, z) and (d, w) of $\mathscr{A}_{\{x^A.\top\}}$ and any Σ-structure homomorphisms $f : c \to Hom_{\mathscr{E}}(E, M)$ and $f' : d \to Hom_{\mathscr{E}}(E, M)$, we have that $f(z) = f'(w)$ if and only if there exists an epimorphic family $\{e_j : E_j \to E \mid j \in J\}$ and for each index $j \in J$ a finitely presentable $\mathbb{T}$-model b_j, a Σ-structure homomorphism $h_j : b_j \to Hom_{\mathscr{E}}(E_j, M)$ and two $\mathbb{T}$-model homomorphisms $f_j : c \to b_j$ and $f'_j : d \to b_j$ such that $f_j(z) = f'_j(w)$, $h_j \circ f_j = Hom_{\mathscr{E}}(e_j, M) \circ f$ and $h_j \circ f'_j = Hom_{\mathscr{E}}(e_j, M) \circ f'$.*

Concrete reformulations - condition (iii)

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ and let $F : f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op} \to \mathscr{E}$ be a flat functor. Then F satisfies condition (iii) of the characterization theorem if and only if the following conditions are satisfied (where for any pair (c, x) consisting of a finitely presentable $\mathbb{T}$-model c and a generalized element $x : E \to F(c)$ the Σ-structure homomorphism $\xi_{(c,x)}$ is defined by setting for each sort A over Σ $\xi_{(c,x)}A : cA \to Hom_{\mathscr{E}}(E, \tilde{F}(M_{\mathbb{T}})A)$ equal to the function $y \to \chi_{(c,y)} \circ x$, where $\chi_{(c,y)} : F(c) \to \tilde{F}(M_{\mathbb{T}})A$ is the canonical colimit arrow).

(a) *for any finitely presentable $\mathbb{T}$-model c and any generalized elements $x, x' : E \to F(c)$, the Σ-structure homomorphisms $\xi_{(c,x)}$ and $\xi_{(c,x')}$ are equal if and only if $x = x'$.*

(b) *for any finitely presentable $\mathbb{T}$-model c, any object E of $\mathscr{E}$ and any Σ-structure homomorphism $z : c \to Hom_{\mathscr{E}}(E, \tilde{F}(M_{\mathbb{T}}))$ there exists an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ and for each index $i \in I$ a generalized element $x_i : E_i \to F(c)$ such that $Hom(e_i, M) \circ z = \xi_{(c,x_i)}$ for all $i \in I$.*

Some corollaries

Corollary

Let $\mathbb{T}$ be a one-sorted geometric theory over a finite signature Σ with a finite number of axioms each of which is of the form $(\top \vdash_{\vec{x}} \bigvee_{i \in I} \phi_i)$, where the ϕ_i are atomic formulae. Suppose that for every $\mathbb{T}$-model M in a Grothendieck topos $\mathscr{E}$ any object E of $\mathscr{E}$, any finitely generated Σ-substructure of $Hom_{\mathscr{E}}(E, M)$ has only a finite number of elements besides the constants (for instance, when the signature Σ does not contain function symbols except for a finite number of constants). Then $\mathbb{T}$ is of presheaf type, classified by the category of covariant set-valued functors from the category of finite models of $\mathbb{T}$.

Corollary

Let $\mathbb{S}$ be a quotient of a theory of presheaf type $\mathbb{T}$ over a signature Σ such that all the finitely presentable $\mathbb{S}$-models are finitely presentable as $\mathbb{T}$-models. Suppose moreover that for any object E of $\mathscr{E}$, $\mathbb{S}$-model M in $\mathscr{E}$, Σ-structure homomorphism $x : c \to Hom_{\mathscr{E}}(E, M)$ and finitely presentable $\mathbb{T}$-model c, there exists an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathscr{E}$ and for each $i \in I$ a $\mathbb{T}$-model homomorphism $f_i : c \to c_i$, where c_i is a finitely presentable $\mathbb{S}$-model, and a Σ-structure homomorphism $x_i : c_i \to Hom_{\mathscr{E}}(E_i, M)$ such that $x_i \circ f_i = Hom_{\mathscr{E}}(e_i, M) \circ x$ for all $i \in I$. Then $\mathbb{S}$ is of presheaf type.

Other relevant results I

Theorem

Let $\mathbb{T}$ a geometric theory. Then $\mathbb{T}$ is of presheaf type if and only if $\mathbb{T}$ has enough finitely presentable models and

(i) *for any finitely presentable model of $\mathbb{T}$ there exists a geometric formula over the signature of $\mathbb{T}$ which presents it;*

(ii) *for any finitely presentable models M and N of $\mathbb{T}$ presented respectively by formulae $\{\vec{x} . \phi\}$ and $\{\vec{y} . \psi\}$ and any $\mathbb{T}$-model homomorphism $h : M \to N$ there exists a $\mathbb{T}$-provably functional geometric formula $\theta(\vec{x}, \vec{y}) : \{\vec{x} . \phi\} \to \{\vec{y} . \psi\}$ which induces h.*

Theorem

Let $\mathbb{T}$ be a theory of presheaf type and $\mathbb{T}'$ be a quotient of $\mathbb{T}$. Suppose that there exists a set $\mathscr{A}$ of finitely presentable $\mathbb{T}'$-models which are finitely presentable as $\mathbb{T}$-models. Then the theory $\mathbb{T}''$ consisting of the set of all geometric sequents which are valid in all models in $\mathscr{A}$ is of presheaf type, and every finitely presentable $\mathbb{T}''$-model is a retract of a model in $\mathscr{A}$. In particular, if the models in $\mathscr{A}$ are jointly conservative for $\mathbb{T}'$ then $\mathbb{T}'$ is of presheaf type, and every finitely presentable $\mathbb{T}'$-model is a retract of a model in $\mathscr{A}$.

Other relevant results II

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then $\mathbb{T}$ is of presheaf type if and only if the following conditions are satisfied:

(i) *Every finitely presentable model is presented by a geometric formula over Σ;*

(ii) *Every property of finite strings of elements of a (finitely presentable) $\mathbb{T}$-model which is preserved by $\mathbb{T}$-model homomorphisms is definable by a geometric formula over Σ;*

(iii) *The finitely presentable $\mathbb{T}$-models are jointly conservative for $\mathbb{T}$.*

Theorem

Let $\mathbb{T}$ be a geometric theory. Then there exists an expansion of $\mathbb{T}$ (by no means unique) to a theory of presheaf type classified by the topos $[f.p.\mathbb{T}\text{-}mod(\mathbf{Set}), \mathbf{Set}]$.
Any such theory will be said to be a presheaf completion of $\mathbb{T}$.

New examples

Our characterization theorem subsumes all the previous results obtained on the subject, it is fully constructive, and can be concretely applied in practice to test whether a given theory is of presheaf type. New examples of theories of presheaf type obtained through this method include:

- The theory of algebraic (resp. separable) extensions of a base field
- The theory of vector spaces with linear independence predicates;
- The theory of locally finite groups and its injectification
- The theory of *l*-groups with strong unit
- A presheaf completion of the theory of decidable groups
- The theory of Diers' fields and of abstract circles (without assuming any form of the axiom of choice)
- ...

For further reading

O. Caramello.
Extensions of flat functors and theories of presheaf type,
arxiv:math.CT/1404.4610 (2014)