Introduction to relative topos theory

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Plan of the course

- Motivation
- Topos-theoretic background
- Toposes as 'bridges'
- Functors inducing morphisms of toposes
- Relative toposes (joint work with Riccardo Zanfa)
  - Relative presheaf toposes
  - The fundamental adjunction
  - Relative sheaf toposes
- A problem of Grothendieck

Relativity techniques

- Broadly speaking, in Mathematics the relativization method consists in trying to state notions and results in terms of morphisms, rather than objects, of a given category, so that they can be 'relativized' to an arbitrary base object.
- One works in the new, relative universe as it were the 'classical' one, and then interprets the obtained results from the point of view of the original universe. This process is usually called *externalization*.
- Relativity techniques can be thought as general 'change of base techniques', allowing one to choose the universe relatively to which one works according to one's needs.
- The relativity method has been pioneered by Grothendieck, in particular for schemes, in his categorical refoundation of Algebraic Geometry, and have played a key role in his work.
- We aim for a similar set of tools for toposes, that is, for an efficient formalism for doing topos theory over an arbitrary base topos.

Topos theory over an arbitrary base topos

Our new foundations for relative topos theory are based on stacks (and, more generally, fibrations and indexed categories).

The approach of category theorists (Lawvere, Diaconescu, Johnstone, etc.) to this subject is chiefly based on the notions of internal category and of internal site.

The problem with these notions is that they are too rigid to naturally capture relative topos-theoretic phenomena, as well as for making computations and formalizing 'parametric reasoning'.

We shall resort to the more general and technically flexible notion of stack, developing the point of view originally introduced by J. Giraud in his paper *Classifying topos*.

Sheaves on a topological space

- Categorically, a sheaf on a topological space X can be defined as a functor $\mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ which satisfies certain conditions expressible in categorical language entirely in terms of the poset category $\mathcal{O}(X)$ and of the usual notion of covering on it.

- More precisely, a presheaf $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ on a topological space X is a sheaf if and only if the canonical arrow

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$$

given by $s \to (s|_{U_i} \mid i \in I)$ be the equalizer of the two arrows

$$\prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

given by $(s_i \to (s_i|_{U_i \cap U_j}))$ and $(s_i \to (s_j|_{U_i \cap U_j}))$.

This paves the way for a significant categorical generalization of the notion of sheaf, leading to the notion of Grothendieck topos.

Sieves

In order to 'categorify' the notion of sheaf of a topological space, the first step is to introduce an abstract notion of covering (of an object by a family of arrows to it) in a category.

Definition

Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a sieve S in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c such that

$$f \in S \Rightarrow f \circ g \in S$$

whenever this composition makes sense.

Remark

For any covering family $F = \{U_i \subseteq U \mid i \in I\}$, giving a family of elements $s_i \in \mathcal{F}(U_i)$ such that for any $i, j \in I$ $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ is equivalent to giving a family of elements $\{s_W \in \mathcal{F}(W) \mid W \in S_F\}$ such that for any open set $W' \subseteq W$, $s_W|_{W'} = s_{W'}$, where S_F is the sieve generated by F.

If S is a sieve on c and $h : d \to c$ is any arrow to c, then

$$h^*(S) := \{g \mid \mathrm{cod}(g) = d,\ h \circ g \in S\}$$

is a sieve on d.

Grothendieck topologies

Definition

- A Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each object c of $\mathcal{C}$ a collection $J(c)$ of sieves on c in such a way that
  (i) (maximality axiom) the maximal sieve $M_c = \{f \mid \mathrm{cod}(f) = c\}$ is in $J(c)$;
  (ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;
  (iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

  The sieves S which belong to $J(c)$ for some object c of $\mathcal{C}$ are said to be J-covering.
- A site (resp. small site) is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a category (resp. a small category) and J is a Grothendieck topology on $\mathcal{C}$.
- A site $(\mathcal{C}, J)$ is said to be small-generated if $\mathcal{C}$ is locally small and has a small J-dense subcategory (that is, a category $\mathcal{D}$ such that every object of $\mathcal{C}$ admits a J-covering sieve generated by arrows whose domains lie in $\mathcal{D}$, and for every arrow $f : d \to c$ in $\mathcal{C}$ where d lies in $\mathcal{D}$ the family of arrows $g : \mathrm{dom}(g) \to d$ such that $f \circ g$ lies in $\mathcal{D}$ generates a J-covering sieve).

Examples of Grothendieck topologies

- For any (small) category $\mathcal{C}$, the trivial topology on $\mathcal{C}$ is the Grothendieck topology in which the only sieve covering an object c is the maximal sieve M_c.
- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathcal{O}(X)}$ on the poset category $\mathcal{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathcal{O}(X))$,

$$S \in J_{\mathcal{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U \, .$$

- If $\mathcal{C}$ satisfies the right Ore condition i.e. the property that any two arrows $f : d \to c$ and $g : e \to c$ with a common codomain c can be completed to a commutative square

$$\begin{tikzcd} \bullet \arrow[r, dashed] \arrow[d, dashed] & d \arrow[d, "f"] \\ e \arrow[r, "g"] & c \end{tikzcd}$$

then the atomic topology on $\mathcal{C}$ is the topology J_{at} defined by: for a sieve S,

$$S \in J_{at}(c) \text{ if and only if } S \neq \emptyset \, .$$

Examples of Grothendieck topologies

- Given a small category of topological spaces which is closed under finite limits and under taking open subspaces, one may define the open cover topology on it by specifying as basis the collection of open embeddings $\{Y_i \hookrightarrow X \mid i \in I\}$ such that $\bigcup_{i \in I} Y_i = X$.
- The Zariski topology on the opposite of the category $\mathbf{Rng}_{\text{f.g.}}$ of finitely generated commutative rings with unit is defined by: for any cosieve S in $\mathbf{Rng}_{\text{f.g.}}$ on an object A, $S \in Z(A)$ if and only if S contains a finite family $\{\xi_i : A \to A_{f_i} \mid 1 \leq i \leq n\}$ of canonical maps $\xi_i : A \to A_{f_i}$ in $\mathbf{Rng}_{\text{f.g.}}$ where $\{f_1, \ldots, f_n\}$ is a set of elements of A which is not contained in any proper ideal of A.
- Given a (first-order geometric) theory $\mathbb{T}$, one can naturally associate a site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ with it, called its *syntactic site*, which embodies essential aspects of the syntax and proof theory of $\mathbb{T}$.

Sheaves on a site

Definition

- A presheaf on a (small) category $\mathcal{C}$ is a functor $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$.
- Let $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ be a presheaf on $\mathcal{C}$ and S be a sieve on an object c of $\mathcal{C}$.

  A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

  $$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d\,.$$

  An amalgamation for such a family is a single element $x \in P(c)$ such that

  $$P(f)(x) = x_f \quad \text{for all } f \text{ in } S\,.$$

Sheaves on a site

- Given a site $(\mathcal{C}, J)$, a presheaf on $\mathcal{C}$ is a *J*-sheaf if every matching family for any J-covering sieve on any object of $\mathcal{C}$ has a unique amalgamation.
- The J-sheaf condition can be expressed as the requirement that for every J-covering sieve S the canonical arrow

$$P(c) \to \prod_{f \in S} P(\mathrm{dom}(f))$$

given by $x \to (P(f)(x) \mid f \in S)$ should be the equalizer of the two arrows

$$\prod_{f \in S} P(\mathrm{dom}(f)) \to \prod_{\substack{\text{f,g, } f \in S \\ \mathrm{cod}(g)=\mathrm{dom}(f)}} P(\mathrm{dom}(g))$$

given by $(x_f \to (x_{f \circ g}))$ and $(x_f \to (P(g)(x_f)))$.

The category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on the site $(\mathcal{C}, J)$ is the full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.

The notion of Grothendieck topos

Definition

A Grothendieck topos is any category equivalent to the category of sheaves on a site.

The following examples show that toposes can be naturally attached to mathematical notions as different as (small) categories, topological spaces, or groups:

Examples

- For any (small) category $\mathcal{C}$, $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is the category of sheaves $\mathbf{Sh}(\mathcal{C}, T)$ where T is the trivial topology on $\mathcal{C}$.
- For any topological space X, $\mathbf{Sh}(\mathcal{O}(X), J_{\mathcal{O}(X)})$ is equivalent to the usual category $\mathbf{Sh}(X)$ of sheaves on X.
- For any (topological) group G, the category $BG = \mathbf{Cont}(G)$ of continuous actions of G on discrete sets is a Grothendieck topos (equivalent, as we shall see, to the category $\mathbf{Sh}(\mathbf{Cont}_{\text{t}}(G), J_{\text{at}})$ of sheaves on the full subcategory $\mathbf{Cont}_{\text{t}}(G)$ on the non-empty transitive actions with respect to the atomic topology).

Basic properties of Grothendieck toposes

Grothendieck toposes satisfy all the categorical properties that one might hope for:

Theorem

Let $(\mathcal{C}, J)$ be a (small-generated) site. Then

- *the inclusion* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ *has a left adjoint* $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ *(called the associated sheaf functor), which preserves finite limits.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has all (small) limits, which are preserved by the inclusion functor* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$*; in particular, limits are computed pointwise and the terminal object* $1_{\mathbf{Sh}(\mathcal{C}, J)}$ *of* $\mathbf{Sh}(\mathcal{C}, J)$ *is the functor* $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ *sending each object* $c \in Ob(\mathcal{C})$ *to the singleton* $\{*\}$.
- *The associated sheaf functor* $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ *preserves colimits; in particular,* $\mathbf{Sh}(\mathcal{C}, J)$ *has all (small) colimits.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has exponentials, which are constructed as in the topos* $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has a subobject classifier.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has a separating set of objects (for instance, the one provided by the objects of the form* $l(c)$ *for* $c \in \mathcal{C}$*, where* l *is the canonical functor* $\mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$*).*

Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism:

Definition

(i) Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of a pair of functors $f_* : \mathcal{E} \to \mathcal{F}$ (the direct image of f) and $f^* : \mathcal{F} \to \mathcal{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathcal{E} \to \mathcal{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

(iii) A point of a topos $\mathcal{E}$ is a geometric morphism $\mathbf{Set} \to \mathcal{E}$.

Grothendieck toposes, geometric morphisms and geometric transformations form a 2-category, called **Topos**.

Examples of geometric morphisms

- A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The direct image $\mathbf{Sh}(f)_*$ sends a sheaf $F \in Ob(\mathbf{Sh}(X))$ to the sheaf $\mathbf{Sh}(f)_*(F)$ defined by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for any open subset V of Y. The inverse image $\mathbf{Sh}(f)^*$ acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.

- Every Grothendieck topos $\mathcal{E}$ has a unique geometric morphism $\mathcal{E} \to \mathbf{Set}$. The direct image is the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$, sending an object $e \in \mathcal{E}$ to the set $\mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, e)$, while the inverse image functor $\Delta : \mathbf{Set} \to \mathcal{E}$ sends a set S to the coproduct $\coprod_{s \in S} 1_{\mathcal{E}}$.

- For any (small) site $(\mathcal{C}, J)$, the pair of functors formed by the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and the associated sheaf functor $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ yields a geometric morphism $i : \mathbf{Sh}(\mathcal{C}, J) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

Indexed categories and fibrations

The language in which we shall work for developing relative topos theory is that of indexed categories and fibrations.

- Given a category $\mathcal{C}$, we shall denote by $\mathbf{Ind}_{\mathcal{C}}$ the 2-category of $\mathcal{C}$-indexed categories: it is the 2-category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]_{ps}$ whose 0-cells are the pseudofunctors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, whose 1-cells are the pseudonatural transformations and whose 2-cells are the modifications between them.
- Given a category $\mathcal{C}$, we shall denote by $\mathbf{Fib}_{\mathcal{C}}$ the 2-category of fibrations over $\mathcal{C}$: it is the sub-2-category of $\mathbf{CAT}/\mathcal{C}$ whose 0-cells are the (Street) fibrations $p : \mathcal{D} \to \mathcal{C}$, whose 1-cells are the morphisms of fibrations (with a 'commuting' isomorphism) and whose 2-cells are the natural transformations between them.

  We shall denote by $\mathbf{cFib}_{\mathcal{C}}$ the full sub-2-category of cloven fibrations (i.e. fibrations equipped with a cleavage).

It is well-known that indexed categories and fibrations are in equivalence with each other:

Theorem

For any category $\mathcal{C}$, there is an equivalence of 2-categories between $\mathbf{Ind}_{\mathcal{C}}$ and $\mathbf{cFib}_{\mathcal{C}}$, one half of which is given by the Grothendieck construction and whose other half is given by the functor taking the fibers at the objects of $\mathcal{C}$.

The notion of stack

Definition

Consider a site $(\mathcal{C}, J)$ and a fibration $p : \mathcal{D} \to \mathcal{C}$: then p is a J-prestack (resp. J-stack) if for every J-sieve $m_S : S \rightarrowtail y_{\mathcal{C}}(X)$ the functor

$$- \circ \textstyle\int m_S : \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \to \mathbf{Fib}_{\mathcal{C}}(\int S, \mathcal{D})$$

is full and faithful (resp. an equivalence).

Stacks over a site $(\mathcal{C}, J)$ form a 2-full and faithful subcategory of $\mathbf{Ind}_{\mathcal{C}}$, which we will denote by $\mathbf{St}(\mathcal{C}, J)$.

The notion of stack on a site is a higher-categorical generalization of that of sheaf on that site:

Proposition

*Consider a site $(\mathcal{C}, J)$ and a presheaf $P : \mathcal{C}^{\mathrm{op}} \to$ **Set**: then P is J-separated (resp. J-sheaf) if and only if the fibration $\int P \to \mathcal{C}$ is a J-prestack (resp. J-stack).*

We can rewrite the condition for a pseudofunctor $\mathcal{C}^{\mathrm{op}} \to$ **Cat** to be a J-prestack (resp. J-stack) in the language of indexed categories, as the requirement that for every sieve $m_S : S \rightarrowtail y_{\mathcal{C}}(X)$ the functor

$$\mathbf{Ind}_{\mathcal{C}}(y_{\mathcal{C}}(X), \mathbb{D}) \xrightarrow{-\circ m_S} \mathbf{Ind}_{\mathcal{C}}(S, \mathbb{D})$$

be full and faithful (resp. an equivalence), where both $y_{\mathcal{C}}(X)$ and S are interpreted as discrete $\mathcal{C}$-indexed categories.

Presentations of toposes

The most classical way for building toposes is through sites (indeed, a Grothendieck topos is, by definition, any category equivalent to the category of sheaves on a small-generated site).

Still, toposes can also be canonically associated with groups (or more generally topological or localic groupoids) or with (first-order geometric) theories or with non-commutative structures such as quantales or quantaloids, etc.

Every topos is associated with infinitely many presentations (in particular, with infinitely many sites of definition), which may belong to different areas of mathematics.

In this course we shall approach toposes from the geometric point of view of their site presentations.

Toposes as 'bridges'

- One can exploit the duality between toposes and their presentations to build 'bridges' across different mathematical theories or contexts.

- More specifically, for any topos-theoretic invariant (i.e. notion which is invariant under categorical equivalence of toposes), one can try to construct a 'bridge' by 'computing' it in terms of different presentations of a given topos.

- Provided that such 'unravelings' are technically feasible, this will result in correspondences between 'concrete' notions pertaining to the different presentations.

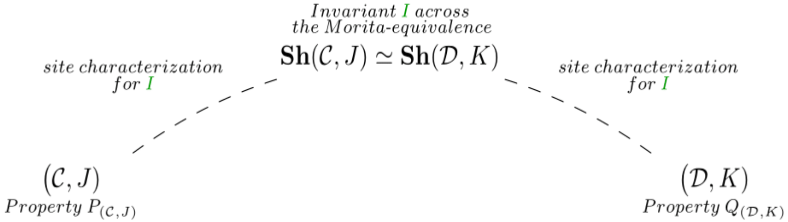
- The effectiveness of the 'bridge' technique actually relies on the natural structural relationship between a topos and its presentations.

Toposes as 'bridges'

- These 'bridges' allow effective and often deep transfers of notions, ideas and results across the theories.
- Note that toposes disappear in the end, though they have been instrumental for performing the 'translation'.
- In fact, 'bridges' have proved useful not only for connecting different theories with each other, but also for working inside a given mathematical theory and investigating it from a multiplicity of different points of view.
- The level of mathematical depth of a 'bridge' may vary enormously from case to case, as it depends on the degree of sophistication of the invariant inducing it, in particular in relation to the given presentations, as well as on the complexity of the given equivalence of toposes. Still, even simple invariants applied to easy-to-establish equivalences can lead to surprising, deep results.

The 'bridge' technique

- Decks of 'bridges': Morita-equivalences (that is, equivalences between different presentations of a given topos, or more generally morphisms or other kinds of relations between toposes)
- Arches of 'bridges': Characterizations for topos-theoretic invariants in terms of different presentations of toposes



For example, this 'bridge' yields a logical equivalence between the 'concrete' properties $P_{(\mathcal{C},J)}$ and $Q_{(\mathcal{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the topo level.

Morphisms and comorphisms of sites

Geometric morphisms can be naturally induced by functors between sites satisfying appropriate properties:

Definition

- A morphism of sites $(\mathcal{C}, J) \to (\mathcal{C}, J')$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ such that there is a geometric morphism $u : \mathbf{Sh}(\mathcal{C}', J') \to \mathbf{Sh}(\mathcal{C}, J)$ making the following square commutative:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "l"'] & \mathcal{C}' \arrow[d, "l'"] \\ \mathbf{Sh}(\mathcal{C}, J) \arrow[r, "u^*"] & \mathbf{Sh}(\mathcal{C}', J'); \end{tikzcd}$$

- A comorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ is a functor $\pi : \mathcal{D} \to \mathcal{C}$ which has the covering-lifting property (in the sense that for any $d \in \mathcal{D}$ and any J-covering sieve S on $\pi(d)$ there is a K-covering sieve R on d such that $\pi(R) \subseteq S$).

Theorem

- *Every morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ induces a geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.*
- *Every comorphism of sites $\pi : (\mathcal{D}, K) \to (\mathcal{C}, J)$ induces a geometric morphism $C_\pi : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.*

Morphisms of sites

Theorem

Let $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ be small-generated sites. Then, given a functor $F : \mathcal{C} \to \mathcal{C}'$, the following conditions are equivalent:

(i) *F is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$;*

(ii) *F satisfies the following properties:*

(1) *F sends every J-covering family in $\mathcal{C}$ into a J'-covering family in $\mathcal{C}'$.*

(2) *Every object c' of $\mathcal{C}'$ admits a J'-covering family*

$$c'_i \longrightarrow c', \quad i \in I,$$

by objects c'_i of $\mathcal{C}'$ which have morphisms

$$c'_i \longrightarrow F(c_i)$$

to the images under F of objects c_i of $\mathcal{C}$.

Morphisms of sites

(3) For any objects c_1, c_2 of $\mathcal{C}$ and any pair of morphisms of $\mathcal{C}'$

$$f'_1 : c' \longrightarrow F(c_1)\,, \quad f'_2 : c' \longrightarrow F(c_2)\,,$$

there exists a J'-covering family

$$g'_i : c'_i \longrightarrow c'\,, \quad i \in I\,,$$

and a family of pairs of morphisms of $\mathcal{C}$

$$f^i_1 : b_i \longrightarrow c_1\,, \quad f^i_2 : b_i \to c_2\,, \quad i \in I\,,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : c'_i \longrightarrow F(b_i)\,, \quad i \in I\,,$$

making the following squares commutative:

$$\begin{tikzcd} c'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & c' \arrow[d, "f'_1"] \\ F(b_i) \arrow[r, "F(f^i_1)"] & F(c_1) \end{tikzcd} \qquad \begin{tikzcd} c'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & c' \arrow[d, "f'_2"] \\ F(b_i) \arrow[r, "F(f^i_2)"] & F(c_2) \end{tikzcd}$$

Morphisms of sites

(4) For any pair of arrows $f_1, f_2 : c \rightrightarrows d$ of $\mathcal{C}$ and any arrow of $\mathcal{C}'$

$$f' : b' \longrightarrow F(c)$$

satisfying

$$F(f_1) \circ f' = F(f_2) \circ f' ,$$

there exist a J'-covering family

$$g'_i : b'_i \longrightarrow b' , \quad i \in I ,$$

and a family of morphisms of $\mathcal{C}$

$$h_i : b_i \longrightarrow c , \quad i \in I ,$$

satisfying

$$f_1 \circ h_i = f_2 \circ h_i , \quad \forall i \in I ,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : b'_i \longrightarrow F(b_i) , \quad i \in I ,$$

making commutative the following squares:

$$\begin{tikzcd} b'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & b' \arrow[d, "f'"] \\ F(b_i) \arrow[r, "F(h_i)"] & F(c) \end{tikzcd}$$

Kan extensions

The direct and image functors of geometric morphisms induced by morphisms or comorphisms of sites can be naturally described in terms of Kan extensions.

Recall that, given a functor $f : \mathcal{C} \to \mathcal{D}$,

- the right Kan extension $\mathrm{ran}_{f^{\mathrm{op}}}$ along f^{op}, which is right adjoint to the functor $f^* : [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, is given by the following formula:

$$\mathrm{ran}_{f^{\mathrm{op}}}(F)(b) = \varprojlim_{\phi : fa \to b} F(a),$$

where the limit is taken over the opposite of the comma category $(f \downarrow b)$.

- the left Kan extension $\mathrm{lan}_{f^{\mathrm{op}}}$ along f^{op}, which is left adjoint to f^*, is given by the following formula:

$$\mathrm{lan}_{f^{\mathrm{op}}}(F)(b) = \varinjlim_{\phi : b \to fa} F(a),$$

where the colimit is taken over the opposite of the comma category $(b \downarrow f)$.

Geometric morphisms and Kan extensions

Proposition

(i) *Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Then*

- *the direct image $\mathbf{Sh}(F)_*$ of the geometric morphism*

$$\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

induced by F is given by the restriction to sheaves of F^*;

- *the inverse image $\mathbf{Sh}(F)^*$ of $\mathbf{Sh}(F)$ is given by*

$$a_K \circ \mathrm{lan}_{F^{\mathrm{op}}} \circ i_J,$$

where $\mathrm{lan}_{F^{\mathrm{op}}}$ is the left Kan extension and i_J is the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

(ii) *Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of small-generated sites. Then*

- *the direct image $(C_F)_*$ of the geometric morphism*

$$C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

induced by F is given by the restriction to sheaves of the right Kan extension $\mathrm{ran}_{F^{\mathrm{op}}}$;

- *the inverse image $(C_F)^*$ of C_F is given by*

$$a_J \circ F^* \circ i_K,$$

where i_K is the inclusion $\mathbf{Sh}(\mathcal{D}, K) \hookrightarrow [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$.

Unifying morphisms and comorphisms of sites

We shall unify the notions of morphism and comorphisms of sites by interpreting them as two fundamentally different ways of describing morphisms of toposes which correspond to each other under a 'bridge'.

More specifically, morphisms of sites provide an 'algebraic' perspective on morphisms of toposes, while comorphisms of sites provide a 'geometric' perspective on them.

The key idea is to replace the given sites of definition with Morita-equivalent ones in such a way that the given morphism (resp. comorphism) of sites acquires a left (resp. right) adjoint, not necessarily in the classical categorial sense but in the weaker topos-theoretic sense of the associated comma categories having equivalent associated toposes.

From morphisms to comorphisms of sites

Theorem

Given a morphism $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ of small-generated sites, let

- *$(1_{\mathcal{D}} \downarrow F)$ be the 'comma category' whose objects are the triplets $(d, c, \alpha : d \to F(c))$*
- *i_F be the functor $\mathcal{C} \to (1_{\mathcal{D}} \downarrow F)$ sending any object c of $\mathcal{C}$ to the triplet $(F(c), c, 1_{F(c)})$,*
- *$\pi_{\mathcal{C}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{C}$ and $\pi_{\mathcal{D}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{D}$ the canonical projection functors, and*
- *$\tilde{K}$ be the Grothendieck topology on $(1_{\mathcal{D}} \downarrow F)$ whose covering sieves are those whose image under $\pi_{\mathcal{D}}$ is K-covering.*

Then:

(i) *$\pi_{\mathcal{C}} \dashv i_F$, $\pi_{\mathcal{D}} \circ i_F = F$, i_F is a morphism of sites $(\mathcal{C}, J) \to ((1_{\mathcal{D}} \downarrow F), \tilde{K})$ and $c_F := \pi_{\mathcal{C}}$ is a comorphism of sites $((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{C}, J)$.*

From morphisms to comorphisms of sites

(ii) $\pi_{\mathcal{D}} : ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{D}, K)$ is both a morphism of sites and a comorphism of sites inducing equivalences

$$C_{\pi_{\mathcal{D}}} : \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to \mathbf{Sh}(\mathcal{D}, K)$$

and

$$\mathbf{Sh}(\pi_{\mathcal{D}}) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K})$$

which are quasi-inverse to each other and make the following triangle commute:

$$\begin{tikzcd}
\mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \arrow[rr, shift left, "C_{\pi_{\mathcal{D}}}"] \arrow[rr, phantom, "\sim"] \arrow[dr, "C_{\pi_{\mathcal{C}}} \cong \mathbf{Sh}(i_F)"'] & & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left, "\mathbf{Sh}(\pi_{\mathcal{D}})"] \arrow[dl, "\mathbf{Sh}(F)"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

For any geometric morphism $f : \mathcal{F} \to \mathcal{E}$, f^* is a morphism of sites $(\mathcal{E}, J_{\mathcal{E}}^{\text{can}}) \to (\mathcal{F}, J_{\mathcal{F}}^{\text{can}})$ such that $f = \mathbf{Sh}(f^*)$. We thus obtain the following

Corollary

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then the canonical projection functor

$$\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$$

is a comorphism of sites $((1_{\mathcal{F}} \downarrow f^*), \widetilde{J_{\mathcal{F}}^{\text{can}}}) \to (\mathcal{E}, J_{\mathcal{E}}^{\text{can}})$ such that $f = C_{\pi_{\mathcal{E}}}$.

The canonical stack of a geometric morphism

The functor $\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$ is actually a stack on $\mathcal{E}$, which we call the canonical stack of f: from an indexed point of view, this stack sends any object E of $\mathcal{E}$ to the topos $\mathcal{F}/f^*(E)$ and any arrow $u : E' \to E$ to the pullback functor $u^* : \mathcal{F}/f^*(E) \to \mathcal{F}/f^*(E')$.
We shall call the Grothendieck topology $\widetilde{J_{\mathcal{F}}^{\mathrm{can}}}$ on $(1_{\mathcal{F}} \downarrow f^*)$ the relative topology of f.

By taking f to be the identity, and choosing a site of definition $(\mathcal{C}, J)$ for $\mathcal{E}$, we obtain the canonical stack $\mathcal{S}_{(\mathcal{C},J)}$ on $(\mathcal{C}, J)$, which sends any object c of $\mathcal{C}$ to the topos $\mathbf{Sh}(\mathcal{C}, J)/l(c)$. The above corollary thus specializes to an equivalence

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{S}_{(\mathcal{C},J)}, \widetilde{J^{\mathrm{can}}_{\mathbf{Sh}(\mathcal{C},J)}}),$$

which represents a 'thickening' of the usual representation of a Grothendieck topos as the topos of sheaves over itself with respect to the canonical topology.

From comorphisms to morphisms of sites

With a comorphism of sites $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ we can associate the morphism of sites

$$m_F : (\mathcal{C}, J) \to (\hat{\mathcal{D}}, \hat{K})$$

sending an object c of $\mathcal{C}$ to the presheaf $\mathrm{Hom}_{\mathcal{C}}(F(-), c)$, where $\hat{K}$ is the extension of the Grothendieck topology K along the Yoneda embedding $y_{\mathcal{D}} : \mathcal{D} \to \hat{\mathcal{D}}$.

This morphism of sites induces a geometric morphism $\mathbf{Sh}(m_F)$ making the following triangle commute:

$$\begin{tikzcd}
\mathbf{Sh}(\hat{D}, \hat{K}) \arrow[rr, shift left, "\mathbf{Sh}(y_{\mathcal{D}})"] \arrow[rr, phantom, "\sim"] \arrow[dr, "\mathbf{Sh}(m_F)"'] & & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left, "C_{y_{\mathcal{D}}}"] \arrow[dl, "C_F"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

Bridging morphisms and comorphisms of sites

We shall call a functor which both a morphism and a comorphism of sites a bimorphism of sites.

We have actually shown that the relationship between a morphism F (resp. comorphism G) of sites and the associated comorphism c_F (resp. morphism m_F) of sites is captured by the equivalence

$$\mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \simeq \mathbf{Sh}((c_F \downarrow 1_{\mathcal{D}}), \overline{\tilde{K}})$$

(resp.

$$\mathbf{Sh}((G \downarrow 1_{\mathcal{C}}), \overline{K}) \simeq \mathbf{Sh}((1_{\hat{\mathcal{D}}} \downarrow m_G), \widetilde{\hat{K}}))$$

of toposes over $\mathbf{Sh}(\mathcal{C}, J)$ induced by bimorphism of sites w_F (resp. z_G) over $\mathcal{C}$.

In fact, F and c_F (resp. G and m_G) are not adjoint to each other in a concrete sense (that is, at the level of sites); nonetheless, they become 'abstractly' adjoint in the world of toposes since toposes naturally attached to such categories are equivalent.

These correspondences actually yield a dual adjunction between a category of morphisms of sites from a given site and a category of comorphisms of sites towards that site.

Continuous functors

Another important class of functors between sites is that of continuous ones:

Definition (Grothendieck)

Given sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, a functor $A : \mathcal{C} \to \mathcal{D}$ is said to be (J, K)-continuous, or simply, continuous, if the functor

$$D_A := (- \circ A^{\text{op}}) : [\mathcal{D}^{\text{op}}, \mathbf{Set}] \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$$

restricts to a functor $\mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.

The property of continuity of a functor can be interpreted as a form of cofinality; in fact, we have shown that it can be explicitly characterized in terms of relative cofinality conditions.

Fibrations as comorphisms of sites

Recall that, given a functor $A : \mathcal{C} \to \mathcal{D}$ and a Grothendieck topology K in $\mathcal{D}$, there is a smallest Grothendieck topology M^A_K on $\mathcal{C}$ which makes A a comorphism of sites to $(\mathcal{D}, K)$.

Proposition

If A is a fibration, the topology M^A_K admits the following simple description: a sieve R is M^A_K-covering if and only if the collection of cartesian arrows in R is sent by A to a K-covering family.

We shall call M^A_K the Giraud topology induced by K, in honour of Jean Giraud, who used it for constructing the classifying topos $\mathbf{Sh}(\mathcal{C}, M^A_K)$ of a stack A on $(\mathcal{D}, K)$.

Proposition

For any Grothendieck topology K on $\mathcal{D}$, every morphism of fibrations $(A : \mathcal{C} \to \mathcal{D}) \to (A' : \mathcal{C}' \to \mathcal{D})$ yields a continuous comorphism of sites $(\mathcal{C}, M^A_K) \to (\mathcal{C}', M^{A'}_K)$.

In particular, a fibration $A : \mathcal{C} \to \mathcal{D}$ yields a continuous comorphism of sites $(\mathcal{C}, M^A_K) \to (\mathcal{D}, K)$ for any Grothendieck topology K on $\mathcal{D}$.

Giraud topologies

The study of the Giraud topology can provide insights on the given fibration. As a basic example of this, under the assumption that J is subcanonical, the property of being a prestack can be checked directly by analysing the Giraud topology:

Proposition (O.C. and R.Z.)

Consider a subcanonical site $(\mathcal{C}, J)$ and a cloven fibration $p : \mathcal{D} \to \mathcal{C}$: then p is a prestack if and only if the Giraud topology M_J^p is subcanonical.

We actually have a Giraud topology functor

$$\mathfrak{G} : \mathbf{Cat}/\mathcal{C} \to \mathbf{Com}/(\mathcal{C}, J),$$

mapping $[p : \mathcal{E} \to \mathcal{C}]$ to $p : (\mathcal{E}, M_J^p) \to (\mathcal{C}, J)$.

By the above results, this functor actually takes values in the subcategory of continuous comorphisms of sites.

Stacks for relative topos theory

The role of stacks in our approach to relative topos theory is two-fold:

- On the one hand, the notion of stack represents a higher-order categorical generalization of the notion of sheaf. Accordingly, categories of stacks on a site represent higher-categorical analogues of Grothendieck toposes. One can thus expect to be able to lift a number of notions and constructions pertaining to sheaves (resp. Grothendieck toposes) to stacks (resp. categories of stacks on a site).
- On the other hand, stacks on a site $(\mathcal{C}, J)$ generalize internal categories in the topos $\mathbf{Sh}(\mathcal{C}, J)$. Since (ordinary) categories can be endowed with Grothendieck topologies, so stacks on a site can also be endowed with suitable analogues of Grothendieck topologies. This leads to the notion of *site relative to a base topos*, which is crucial for developing relative topos theory.

Remark

Every stack is equivalent to a split stack, and hence to an internal category, but most stacks naturally arising in the mathematical practice are not split (think, for instance, of the canonical stack of a topos).

The big picture

Our theory is based on a network of 2-adjunctions (for any small site $(\mathcal{C}, J)$):

$$
\begin{tikzcd}
\mathbf{Ind}_{\mathcal{C}} \arrow[r, shift left=2, "\Lambda"] \arrow[r, phantom, "\perp"] \arrow[d, shift left=2, "s_J"] \arrow[d, phantom, "\vdash"] & \mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J)^{co} \arrow[l, shift left=2, "\Gamma"] \\
\mathbf{St}(\mathcal{C}, J) \arrow[u, hook, shift left=2] \arrow[r, shift left=2, "\Lambda'"] \arrow[r, phantom, "\perp"] \arrow[d, shift left=2, "E \circ \Lambda'"] \arrow[d, phantom, "\vdash"] & \mathbf{EssTopos}/\mathbf{Sh}(\mathcal{C}, J)^{co} \arrow[u, hook] \arrow[l, shift left=2, "\Gamma'"] \arrow[dl, shift right=2, "E"'] \\
\mathbf{Sh}(\mathcal{C}, J) \arrow[u, hook, shift left=2] \arrow[ur, shift right=2, "L"'] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}
$$

In this diagram s_J denotes the stackification functor, **Topos** the category of Grothendieck toposes and geometric morphisms and **EssTopos** the full subcategory on the essential geometric morphisms.

- The functor E sends an essential geometric morphism $f : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$ to the object $f_!(1_{\mathscr{E}})$ (where $f_!$ is the left adjoint to the inverse image f^* of f).
- The functor L sends an object P of $\mathbf{Sh}(\mathcal{C}, J)$ to the canonical local homeomorphism $\mathbf{Sh}(\mathcal{C}, J)/P \to \mathbf{Sh}(\mathcal{C}, J)$.

Pseudo-Kan extensions

Proposition

Denote by $\mathbf{Ind}^s_{\mathcal{C}}$ the sub-2-category of $\mathbf{Ind}_{\mathcal{C}}$ of pseudofunctors with values in **Cat** *(i.e. 'small' $\mathcal{C}$-indexed categories). Consider any functor $F : \mathcal{C} \to \mathcal{D}$ and the direct image 2-functor*

$$F^* : \mathbf{Ind}^s_{\mathcal{D}} \to \mathbf{Ind}^s_{\mathcal{C}}$$

which acts by precomposition with F^{op}. The 2-functor F^* has both a left and a right 2-adjoint, denoted respectively by $\mathrm{Lan}_{F^{\mathrm{op}}}$ and $\mathrm{Ran}_{F^{\mathrm{op}}}$, which act as follows:

- *for any D in $\mathcal{D}$ denote by $\pi^D_F : (D{\downarrow}F) \to \mathcal{C}$ the canonical projection functor: then for $\mathbb{E} : \mathcal{C}^{\mathrm{op}} \to$ **Cat**, its image $\mathrm{Lan}_{F^{\mathrm{op}}}(\mathbb{E}) : \mathcal{D}^{\mathrm{op}} \to$ **Cat** is defined componentwise as*

$$\mathrm{Lan}_{F^{\mathrm{op}}}(\mathbb{E})(D) := \mathrm{colim}_{ps}\left((D{\downarrow}F)^{\mathrm{op}} \xrightarrow{(\pi^D_F)^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathbb{E}} \mathbf{Cat}\right)$$

- *for any D in $\mathcal{D}$ denote by $\pi'^D_F : (F{\downarrow}D) \to \mathcal{C}$ the canonical projection functor: then for $\mathbb{E} : \mathcal{C}^{\mathrm{op}} \to$ **Cat**, its image $\mathrm{Ran}_{F^{\mathrm{op}}}(\mathbb{E}) : \mathcal{D}^{\mathrm{op}} \to$ **Cat** is defined componentwise as*

$$\mathrm{Ran}_{F^{\mathrm{op}}}(\mathbb{E})(D) := \lim_{ps}\left((F{\downarrow}D)^{\mathrm{op}} \xrightarrow{(\pi'^D_F)^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathbb{E}} \mathbf{Cat}\right)$$

Direct and inverse images of stacks

Proposition (O.C. and R.Z.)

Consider two sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ and a functor $F : \mathcal{C} \to \mathcal{D}$.

- *Then F is (J, K)-continuous functor if and only if $F^* : \mathbf{Ind}_{\mathcal{D}} \to \mathbf{Ind}_{\mathcal{C}}$ restricts to a 2-functor $\mathbf{St}(\mathcal{D}, K) \to \mathbf{St}(\mathcal{C}, J)$.*
- *If F is a morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$, or more generally a (J, K)-continuous functor, it induces a 2-adjunction*

$$\begin{tikzcd} \mathbf{St}^s(\mathcal{C}, J) \arrow[r, bend left, "\mathbf{St}(F)^*"] \arrow[r, phantom, "\perp"] & \mathbf{St}^s(\mathcal{D}, K) \arrow[l, bend left, "\mathbf{St}(F)_*"] \end{tikzcd} ,$$

whose pair we shall refer to simply by $\mathbf{St}(F)$.
- *The 2-functor $\mathbf{St}(F)_*$ is called the direct image of stacks along F and acts as the precomposition*

$$F^* := (- \circ F^{\mathrm{op}}) : \mathbf{Ind}_{\mathcal{D}} \to \mathbf{Ind}_{\mathcal{C}};$$

In terms of fibrations, a stack $q : \mathcal{E} \to \mathcal{D}$ is mapped by $\mathbf{St}(F)_*$ to its strict pseudopullback $p : \mathcal{P} \to \mathcal{C}$ along F.

Direct and inverse images of stacks

- The left adjoint $\mathbf{St}(F)^*$ is the inverse image of stacks along F and acts as the composite

$$\mathbf{St}^s(\mathcal{C}, J) \xrightarrow{i_J} \mathbf{Ind}^s_{\mathcal{C}} \xrightarrow{\mathrm{Lan}_{F^{\mathrm{op}}}} \mathbf{Ind}^s_{\mathcal{D}} \xrightarrow{s_K} \mathbf{St}^s(\mathcal{D}, K),$$

where s_K denotes the stackification functor. In terms of fibrations, a stack $p : \mathcal{P} \to \mathcal{C}$ is mapped by $\mathbf{St}(F)^*$ to the stackification of its inverse image $\mathrm{Lan}_{F^{\mathrm{op}}}([p])$ along F, which can be computed as a localization as follows. Consider the fibration of generalized elements

$$(1_{\mathcal{D}} \downarrow (F \circ p)) \xrightarrow{r} \mathcal{D}$$

of the functor $F \circ p$, whose objects are arrows $[d : D \to (F \circ p)(U)]$ of $\mathcal{D}$, and whose morphisms

$$(e, \alpha) : [d' : D' \to (F \circ p)(V)] \to [d : D \to (F \circ p)(U)]$$

are indexed by an arrow $e : D' \to D$ in $\mathcal{D}$ and an arrow $\alpha : V \to U$ in $\mathcal{P}$ such that $(F \circ p)(\alpha) \circ d' = d \circ e$. Consider the class of arrows

$$S := \{(e, \alpha) : [d'] \to [d] \mid (e, \alpha)\ r\text{-vertical}, \alpha \text{ cartesian in } \mathcal{P}\} :$$

then

$$\mathrm{Lan}_{F^{\mathrm{op}}}([p]) \simeq (1_{\mathcal{D}} \downarrow (F \circ p))[S^{-1}] .$$

Direct and inverse images of stacks

In a similar way to morphisms of sites, comorphisms of sites also induce an adjunction between categories of stacks:

Proposition (O.C. and R.Z.)

Consider a comorphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$: it induces a 2-adjunction

$$\begin{tikzcd} \mathbf{St}^s(\mathcal{D}, K) \arrow[r, bend left, "(C_F^{\mathbf{St}})^*"] \arrow[r, phantom, "\perp"] & \mathbf{St}^s(\mathcal{C}, J) \arrow[l, bend left, "(C_F^{\mathbf{St}})_*"] \end{tikzcd} ,$$

whose pair we shall refer to by $C_F^{\mathbf{St}}$.

- *The right adjoint $(C_F^{\mathbf{St}})_*$ acts by restriction of the right pseudo-Kan extension $\mathrm{Ran}_{F^{\mathrm{op}}}$ to stacks;*
- *The left adjoint $(C_F^{\mathbf{St}})^*$ acts as the composite 2-functor*

$$\mathbf{St}^s(\mathcal{D}, K) \xrightarrow{i_K} \mathbf{Ind}^s_{\mathcal{D}} \xrightarrow{F^*} \mathbf{Ind}^s_{\mathcal{C}} \xrightarrow{s_J} \mathbf{St}^s(\mathcal{C}, J),$$

  where $F^* := (- \circ F^{\mathrm{op}})$.
- *If F is also continuous the $C_F^{\mathbf{St}}$ also has a left adjoint $(C_F^{\mathbf{St}})_!$ given by the composite 2-functor*

$$\mathbf{St}^s(\mathcal{C}, J) \xrightarrow{i_J} \mathbf{Ind}^s_{\mathcal{C}} \xrightarrow{\mathrm{Lan}_{F^{\mathrm{op}}}} \mathbf{Ind}^s_{\mathcal{D}} \xrightarrow{s_K} \mathbf{St}^s(\mathcal{D}, K) .$$

Relative 'presheaf toposes'

Given a $\mathcal{C}$-indexed category $\mathbb{D}$, we denote by $\mathcal{G}(\mathbb{D})$ the fibration on $\mathcal{C}$ associated with it (through the Grothendieck construction) and by $p_{\mathbb{D}}$ the canonical projection functor $\mathcal{G}(\mathbb{D}) \to \mathcal{C}$.

Proposition (O.C. and R.Z.)

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathbb{D}$ a $\mathcal{C}$-indexed category and $\mathbb{D}^V$ be the opposite indexed category of $\mathbb{D}$ (defined by setting, for each $c \in \mathcal{C}$, $\mathbb{D}^V(c) = \mathbb{D}(c)^{\text{op}}$). Then we have a natural equivalence

$$\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}}) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}^V, \mathcal{S}_{(\mathcal{C},J)}) .$$

This proposition shows that, if $\mathbb{D}$ is a stack, the classifying topos $\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}})$ of $\mathbb{D}$, which we call the Giraud topos of $\mathbb{D}$, can indeed be seen as the "topos of relative presheaves on $\mathbb{D}$".

Giraud toposes as weighted colimits

We have shown that, for any $\mathbb{D}$, the Giraud topos $C_{p_\mathbb{D}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_\mathbb{D}}) \to \mathbf{Sh}(\mathcal{C}, J)$ can be naturally seen as a weighted colimit of a diagram of étale toposes over $\mathbf{Sh}(\mathcal{C}, J)$:

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}/X, J_X) \arrow[d, bend right=50, "\lambda_{(X,V)}"'] \arrow[d, bend left=50, "\lambda_{(X,U)}"] \arrow[d, phantom, "\xLeftarrow{\lambda_{(X,a)}}"] \arrow[dr, phantom, "\Rightarrow \cong"] & & \mathbf{Sh}(\mathcal{C}/Y, J_X) \arrow[ll, "C_{\Sigma_y}"'] \arrow[dll, bend left, "\lambda_{(Y,(\mathbb{D}(y)(U)))}"] \\
\mathbf{Sh}(\mathcal{G}(\mathbb{D}, M_J^{p_\mathbb{D}}) & &
\end{tikzcd}$$

where $y : Y \to X$ and $a : U \to V$ are arrows respectively in $\mathcal{C}$ and in $\mathbb{D}(X)$, the legs $\lambda_{(X,U)} : \mathbf{Sh}(\mathcal{C}/X, J_X) \to \mathbf{Sh}(\mathcal{G}(\mathbb{D}, M_J^{p_\mathbb{D}})$ of the cocone are the morphisms $C_{\xi_{(X,U)}}$ induced by the morphisms of fibrations $\xi_{(X,U)} : \mathcal{C}/X \to \mathcal{D}$ over $\mathcal{C}$ given by the fibered Yoneda lemma, and the functor $\Sigma_y : \mathcal{C}/Y \to \mathcal{C}/X$ are given by composition with y.

The fundamental adjunction

The universal property of the above weighted colimit yields a fundamental 2-adjunction between cloven fibrations over $\mathcal{C}$ and toposes over $\mathbf{Sh}(\mathcal{C}, J)$:

Theorem (O.C and R.Z.)

For any small-generated site $(\mathcal{C}, J)$, the two pseudofunctors

$$\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{cFib}_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}/(\mathcal{C}, J) \xrightarrow{C_{(-)}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J),$$

$$\Big[[p : \mathcal{D} \to \mathcal{C}] \xrightarrow{(F,\phi)} [q : \mathcal{E} \to \mathcal{C}]\Big] \mapsto \Big[[\mathrm{Gir}_J(p)] \xrightarrow{(C_F, C_\phi)} [\mathrm{Gir}_J(q)]\Big],$$

and

$$\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Ind}_{\mathcal{C}} \simeq \mathbf{cFib}_{\mathcal{C}},$$

$$[E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)] \mapsto \big[\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), [E]) : \mathcal{C}^{\mathrm{op}} \to \mathbf{CAT}\big]$$

are the two components of a 2-adjunction

$$\begin{tikzcd} \mathbf{cFib}_{\mathcal{C}} \arrow[r, bend left, "\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}"] \arrow[r, phantom, "\perp"] & \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \arrow[l, bend left, "\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}"] \end{tikzcd}$$

Remark

Since $\mathrm{Gir}_J(p) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathcal{D}^V, \mathcal{S}_{(\mathcal{C},J)})$, the canonical stack $\mathcal{S}_{(\mathcal{C},J)}$ has a similar behavior to that of a dualizing object for the adjunction $\Lambda \dashv \Gamma$.

The discrete setting

The restriction of our fundamental adjunction in the setting of presheaves (that is, discrete fibrations) will yield a generalization to the context of arbitrary sites of the classical adjunction

$$\begin{tikzcd} \mathbf{Psh}(X) \arrow[r, bend left, "\Lambda"] \arrow[r, phantom, "\perp"] & \mathbf{Top}/X \arrow[l, bend left, "\Gamma"] \end{tikzcd} .$$

between presheaves on a topological space X and bundles over it.

This adjunction and its applications to the theory of sheaves (notably leading to fibrational descriptions of the sheafification functor, as well as of direct and inverse images of sheaves) will be discussed in the forthcoming lecture by Riccardo Zanfa.

Relative sheaf toposes

As any Grothendieck topos is a subtopos of a presheaf topos, so any relative topos should be a subtopos of a relative presheaf topos. This motivates the following

Definition

Let $(\mathcal{C}, J)$ be a small-generated site. A site relative to $(\mathcal{C}, J)$ is a pair consisting of a $\mathcal{C}$-indexed category $\mathbb{D}$ and a Grothendieck topology K on $\mathcal{G}(\mathbb{D})$ which contains the Giraud topology $M_J^{p_{\mathbb{D}}}$.

The topos of sheaves on such a relative site $(\mathbb{D}, K)$ is defined to be the geometric morphism

$$C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), K) \to \mathbf{Sh}(\mathcal{C}, J)$$

induced by the comorphism of sites $p_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$.

Remark

Not every Grothendieck topology on K can be generated starting by horizontal or vertical data (that is, by sieves generated by cartesian arrows or entirely lying in some fiber), but many important relative topologies naturally arising in practice are of this form.

Examples of relative topologies

- The relative topology on the canonical stack of a geometric morphism (which allows one to represent *any* relative topos as the topos of sheaves on a relative site).
- The Giraud topology is an example of a relative topology generated by horizontal data.
- The total topology of a fibered site, in the sense of Grothendieck, is generated by vertical data.

We have shown that, for a wide class of relative topologies generated by horizontal and vertical data, one can describe bases for them consisting of multicompositions of horizontal families with vertical families.

A question of Grothendieck

As recently brought to the public attention by Colin McLarty, Grothendieck expressed, in his 1973 Buffalo lectures, the aspiration of viewing any object of a topos geometrically as an étale space over the terminal object:

> *The intuition is the following: viewing objects of a topos as being something like étale spaces over the final object of the topos, and the induced topos over an object as just the object itself. That is I think the way one should handle the situation.*
> *It's a funny situation because in strict terms, you see, the language which I want to push through doesn't make sense. But of course there are a number of mathematical statements which substantiate it.*

Given his conception of *gros* and *petit* toposes, we can more broadly interpret his wish as that for a framework allowing one to think geometrically about any topos, that is, as it were a 'petit' topos related to a 'gros' topos by a local retraction.

Local morphisms

Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be local if f_* has a fully faithful right adjoint.

Theorem (O.C.)

Let $F : \mathcal{D} \to \mathcal{C}$ be a bimorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$. Then:

(i) *The geometric morphism $C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ is essential, and*

$$(C_F)_! \cong \mathbf{Sh}(F)^* \dashv \mathbf{Sh}(F)_* \cong (C_F)^* = D_F := (- \circ F^{\mathrm{op}}) \dashv (C_F)_*$$

(ii) *The morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ is local if and only if C_F is an inclusion, that is, if and only if F is K-faithful and K-full. In this case, the morphisms C_F and $\mathbf{Sh}(F)$ realize the topos $\mathbf{Sh}(\mathcal{D}, K)$ as a (coadjoint) retract of $\mathbf{Sh}(\mathcal{C}, J)$ in $\mathbf{Topos}$.*

Gros and *petit* toposes

Pairs of *gros* and *petit* toposes are important for several reasons. Morally, a *petit* topos is thought of as a generalized space, while a *gros* topos is conceived as a category of spaces.

In fact, one advantage of *gros* toposes is that they are associated with sites which tend to have better categorical properties than those of the site presenting the *petit* topos.

Still, *gros* and *petit* toposes in a given pair are homotopically equivalent (as they are related by a local retraction), whence they share the same cohomological invariants.

The above result can be notably applied to construct pairs of *gros* and *petit* toposes starting from a (K-)full and (K-)faithful bimorphism of sites

$$(\mathcal{D}, K) \to (\mathcal{T}/T_{\mathcal{D}}, E_{T_{\mathcal{D}}}),$$

where $\mathcal{T}$ is a category endowed with a Grothendieck topology E, $T_{\mathcal{D}}$ is an object of $\mathcal{T}$ and $E_{T_{\mathcal{D}}}$ is the Grothendieck topology induced on $(\mathcal{T}/T_{\mathcal{D}})$ by E.

Every Grothendieck topos is a 'small topos'

We define a Grothendieck topology $J^{\text{ét}}$ on **Topos**, which we call the étale cover topology, by postulating that a sieve on a topos $\mathcal{E}$ is $J^{\text{ét}}$-covering if and only if it contains a family $\{\mathcal{E}/A_i \to \mathcal{E} \mid i \in I\}$ of canonical local homeomorphisms such that the family of arrows $\{!_{A_i} : A_i \to 1_{\mathcal{E}} \mid i \in I\}$ is epimorphic in $\mathcal{E}$.

The functor L is a J-full and J-faithful bimorphism of sites

$$(\mathcal{C}, J) \to (\mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J), J^{\text{ét}}_{\mathbf{Sh}(\mathcal{C},J)}) \, .$$

So, by the above result, the 'petit' topos $\mathbf{Sh}(\mathcal{C}, J)$ identifies with a coadjoint retract of the 'big' topos $\mathbf{Sh}(\mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J), J^{\text{ét}}_{\mathbf{Sh}(\mathcal{C},J)}) \simeq \mathbf{Sh}(\mathbf{Topos}, J^{\text{ét}})/l(\mathbf{Sh}(\mathcal{C}, J))$ (in a suitable Grothendieck universe) via the local morphism $\mathbf{Sh}(L)$ and the essential inclusion C_L.

This shows that *every* Grothendieck topos can be naturally regarded as a 'petit' topos embedded in an associated 'gros' topos, and that this embedding allows one to view any object of the original topos as an étale morphism to the terminal object in the associated 'gros' topos, thus providing a solution to Grothendieck's problem.

Future developments

Our notion of relative site will play a key role in our future development the theory of *relative toposes*.

We expect the development of this theory to parallel that of the classical theory; indeed, by using a general stack semantics, we plan to introduce, in a canonical, not *ad hoc* way, natural generalizations to the relative setting of the classical notions of morphism and comorphism of sites, flat functors, separating sets for a topos, denseness conditions etc.

This will notably lead us to relative versions, in the language of stacks (or, more generally, of indexed categories), of Giraud's and Diaconescu's theorems, as well as to a theory of classifying toposes of (higher-order) relative geometric theories.

Towards relative geometric logic

Indeed, the geometric approach to relative toposes which we have developed so far has a logical counterpart, which we may call relative geometric logic.

In its classical formulation, geometric logic does not have parameters embedded in its formalism; still, it is possible to introduce them without changing its degree of expressivity.

In a relative setting, parameters are fundamental if one wants to reason geometrically and use fibrational techniques. In fact, the semantics of stacks involves parameters in an essential way.

It turns out that the logical framework corresponding to relative toposes is a kind of fibrational, higher-order parametric logic in which it is possible to express a great number of higher-order constructions by using the parameters belonging to the base topos.

For further reading

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